Algebraic combinatorics, semigroup representations and random walks on hyperplane chambers after Ken Brown

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Tsetlin Libary

- You have a bookshelf with n books, labelled $\{1, \ldots, n\}$.
- Each time you finish with a book you replace it at the beginning of the shelf.
- After a long period of time, you expect that the books you frequently use will be located at the front of your shelf, while less often used books will make their way to the back of the shelf.
- Let's consider a mathematical model.

- States: permutations of $\{1, \ldots, n\}$
- Evolution of the system: with probability p_i book i is removed from the shelf and placed at the front
- Goal: understand the asymptotic behaviour

More precisely, we consider the Markov transition matrix M.

- M is an $n! \times n!$ matrix indexed by the permutations of $\{1, \ldots, n\}$.
- The entry $M_{\sigma,\tau}$ gives the probability of going from σ to τ in a single move.
- $M^n_{\sigma,\tau}$ gives the probability of going from σ to τ in exactly n steps.
- Natural questions include can M be diagonalized and what is the spectrum of M?

Inverse Riffle Shuffle

- In the usual riffle shuffle of a deck of n cards one chooses a point to cut the deck. The two parts of the deck are then interleaved preserving the relative orders of the parts.
- The inverse riffle shuffle chooses a particular set of cards to bring to the front. The relative order of these cards is kept as it was.
- The inverse riffle shuffle is hard to perform, but has the same dynamics.
- The model:
  - States: permutations of $\{1, \ldots, n\}$
  - A probability distribution (typically binomial) is placed on $2^{\{1,\ldots,n\}}$.
  - One chooses a subset of the cards to move to the front according to this distribution.
  - The Tsetlin library is the case where the support of the probability measure is on the singletons.

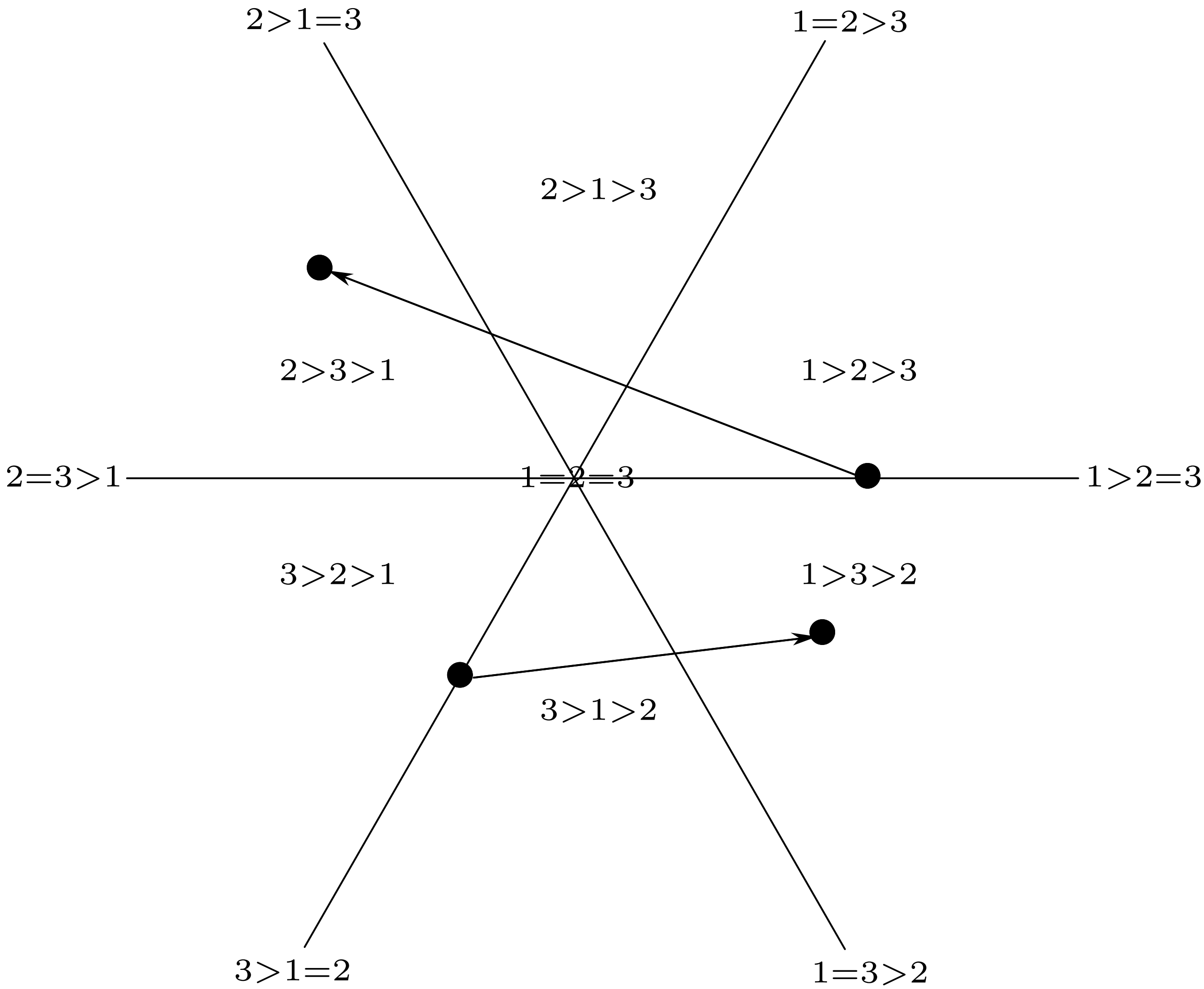
Hyperplane chamber walks

- Bidigare, Hanlon and Rockmere had the idea of modelling the above Markov chains as random walks on the reflection arrangement (=Coxeter complex) associated to the symmetric group.
- This was explored further by Diaconis and Brown.
- The chambers are in correspondence with permutations.
- There is a semigroup structure on the faces of any hyperplane arrangement, due to Tits.
- The above Markov chains are then random walks on the Tits hyperplane face semigroup.

Braid arrangement

- The hyperplanes:
  $H_{ij} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$,
  $1 \le i \neq j \le n$.
- Tits multiplication of faces $\mathcal{F}_1, \mathcal{F}_2$: $\mathcal{F}_1 * F_2$ is the face entered upon walking a distance of ϵ on the line segment from an interior point of $\mathcal{F}_1$ to an interior point of $\mathcal{F}_2$.
- The chambers form a left ideal: $\mathcal{F}_1 * C$ is a chamber whenever C is a chamber. In fact the chambers form a minimal left ideal.
- The Tits semigroup satisfies the identities:
  $x^2 = x$ and $xyx = xy$.

Tits multiplication in the Coxeter complex of S_3



Calculating:

$$(1 > 2 = 3) * (2 > 3 > 1) = 1 > 2 > 3$$

$$(3 > 1 = 2) * (1 > 3 > 2) = 3 > 1 > 2$$

- In general the face $i > 1 = \cdots = \widehat{i} = \cdots = n$ acts on chambers by moving i to the front.
- So the Tsetlin library is a semigroup random walk on the minimal left ideal of the Tits semigroup.
- The inverse riffle shuffle can also be implemented this way. For instance $1 = 2 = 3 > 4 = \cdots = n$ acts on chambers by moving $1, 2, 3$ to the front, but keeping the relative order.

Semigroup random walks

- Let S be a finite semigroup.
- Let L be a minimal left ideal.
- Let $\pi = \sum_{s \in S} p_s s$ be a probability on S.
- The Markov transition matrix is the $L \times L$ matrix M with
  - $M_{s,t}$ the probability that if an element x of S is randomly chosen according to π, then $xs = t$.
- Goal: Calculate the spectrum of M.
- The random walk is independent of the choice of minimal left ideal as all are isomorphic via right multiplication in S.

Diaconis-Brown trick

- The key to calculating the spectrum is an algebraic reinterpretation of M.
- We view π as an element of $\mathbb{C}S$.
- $\mathbb{C}L$ is a left ideal in $\mathbb{C}S$.
- Let $\rho : S \to M_{|L|}(\mathbb{C})$ be the associated matrix representation, with basis L.
- Then $M = \rho(\pi)^T$.
- So we want the spectrum of $\rho(\pi)$.
- Goal: Use representation theory to get a better basis.

Diaconis' approach for finite abelian groups

- G a finite abelian group
- $\pi = \sum_{g \in G} p_g g$ a probability measure on G
- $\chi_1, \ldots, \chi_n$ the characters of G
- The representation of G on $\mathbb{C}G$ can be put in diagonal form

$$\begin{pmatrix} \chi_1 & & \\ & \ddots & \\ & & \chi_n \end{pmatrix}$$

- So M has an eigenvalue λ_{χ_i} for each character χ_i, occurring with multiplicity one.
- $\lambda_{\chi_i} = \sum_{g \in G} p_g \chi_i(g)$

Brown's approach for idempotent semigroups

- Brown showed that for idempotent semigroups S, the representation of S on $\mathbb{C}L$ can be placed in upper triangular form with the characters on the diagonal.
- So there is then an eigenvalue for each character.
- He used Solomon's results on Möbius inversion and representation theory of semilattices to calculate the characters and their multiplicities.
- For hyperplane face semigroups the associated semilattice is the support lattice of the arrangement.
- If S satisfies the additional identity $xyx = xy$ then Brown showed M is diagonalizable.

Triangularizable semigroups

A finite semigroup is called *triangularizable* if it admits a faithful representation over $\mathbb{C}$ by upper triangular matrices.

Theorem 1 (AMSV). *A semigroup S is triangularizable iff*

1. *Each group subsemigroup is abelian;*
2. *Each von Neumann regular element*[a] *satisfies an identity of the; form $x^m = x$ and products of $\mathcal{D}$-equivalent idempotents are idempotent.*

- We also showed that each irreducible representation of a triangularizable semigroup has degree one: i.e. is a character.
- I studied random walks on triangularizable semigroups.
- The eigenvalue result generalizes easily; the multiplicity result required new ideas.

[a] An element s of a semigroup is von Neumann regular if there exists t with $sts = s$.

- Let S be triangularizable, L be a minimal left ideal and $\pi = \sum_{s \in S} p_s s$ be a probability measure.
- The representation of S on L can be put in the form
$$\begin{pmatrix} \chi_1 & * & * \\ & \ddots & * \\ & & \chi_n \end{pmatrix}$$
with the χ_i characters of S appearing with multiplicities.
- There is an eigenvalue λ_{χ_i} for each character χ_i given by $\lambda_{\chi_i} = \sum_{s \in S} p_s \chi_i(s)$.
- I can explicitly calculate the characters using techniques of Rhodes and Zalcstein.
- I can also calculate the multiplicities.
- This combines the orthogonality relations from group theory and the combinatorial tool of Möbius inversion from Solomon's theory.

Main new results

- Developed a new approach to semigroup representation theory.
- It allows for the calculation of multiplicities of irreducible constituents for a large class of semigroups.
- The multiplicity results for eigenvalues for random walks come from the more general theory.
- Key point: a semigroup S is triangularizable iff it admits a homomorphism $\varphi : S \to T$ with T a commutative inverse semigroup such that $\overline{\varphi} : \mathbb{C}S \to \mathbb{C}T$ is the semisimple quotient.