PRINCIPAL VERTEX OPERATOR REPRESENTATIONS FOR TOROIDAL LIE ALGEBRAS

Yuly Billig *

Department of Mathematics and Statistics, University of New Brunswick, Fredericton, N.B., E3B 5A3, Canada

We introduce the principal vertex operator representations for the toroidal Lie algebras generalizing the construction for the affine Kac-Moody algebras. We also represent the derivations of the toroidal algebras and introduce analogues of the Sugawara operators.

* E-mail address: ybillig@unb.ca

I. INTRODUCTION.

Vertex operators discovered by physicists in string theory have turned out to be important objects in mathematics. One can use vertex operators to construct various realizations of the irreducible highest weight representations for affine Kac-Moody algebras. Two of these, the principal and homogeneous realizations, are of particular interest. The principal vertex operator construction for the affine algebra $A_1^{(1)}$ allows one to construct soliton solutions of the Korteweg - de Vries hierarchy of partial differential equations. On the other hand, the homogeneous realization is linked to the fundamental nonlinear Schrödinger equation [1] .

S.Eswara Rao and R.V. Moody [2] studied the homogeneous vertex operator construction for toroidal Lie algebras. The present paper is devoted to the principal realization. Here we construct the principal vertex operator representation for the toroidal Lie algebra $\hat{\mathfrak{g}}$ which is a universal central extension of $\tilde{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, \ldots, t_n^{\pm}]$, in which $\dot{\mathfrak{g}}$ is a simply-laced simple finite-dimensional Lie algebra over $\mathbb{C}$. This generalizes the principal vertex operator realization of the basic representations of affine Lie algebras [3,4] .

We add the Lie algebra $\mathcal{D}$ of vector fields on a torus to the toroidal Lie algebra $\hat{\mathfrak{g}}$ to form a larger algebra $\mathfrak{g}$. This is necessary in order to have a sufficiently large principal Heisenberg subalgebra. To construct a representation of $\mathfrak{g}$ we consider the standard module F for the principal Heisenberg subalgebra and then extend the action to all of $\mathfrak{g}$ by means of the vertex operators. The module F is irreducible as a module over $\mathfrak{g}$, and it is reducible over $\hat{\mathfrak{g}}$. It will be interesting to see how this representation fits in the framework of the Verma modules over the toroidal Lie algebras [5] .

In a subsequent article [6] we construct an extension of the KdV hierarchy using the representations introduced in the present paper.

The vertex operator representations for the toroidal Lie algebras constructed here may also be useful for quantum field theories in space-time of more than two dimensions [7,8] .

This paper is organized as follows. In Section II we recall the construction of the toroidal Lie algebra and in Section III we sketch the principal vertex operator realization of the basic representations of affine Lie algebras. In Section IV we present the principal Heisenberg subalgebra of $\mathfrak{g}$ and define its standard module. We also introduce there vertex operators that will allow us to extend this representation to $\mathfrak{g}$. At the end of the section we state the main theorem of the paper. The proof of the main theorem occupies Section V. In the final section we construct the analogues of the Sugawara operators.

II. DEFINITIONS AND NOTATIONS.

Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional complex Lie algebra of type A_ℓ, D_ℓ or E_ℓ (i.e., simply-laced). The Killing form $(\cdot|\cdot)$ is non-degenerate on the Cartan subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$ and induces the map $\nu : \dot{\mathfrak{h}} \to \dot{\mathfrak{h}}^*$. Let $\dot{\Delta}$ be the root system of $\dot{\mathfrak{g}}$. Since $\dot{\mathfrak{g}}$ is simply-laced then $(\alpha|\alpha) = 2$ for all nonzero roots $\alpha \in \dot{\Delta}$.

To construct a toroidal algebra let us consider a tensor product of $\dot{\mathfrak{g}}$ with the algebra of Laurent polynomials in $n+1$ variable:

$$\tilde{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}].$$

We define the toroidal Lie algebra corresponding to $\dot{\mathfrak{g}}$ as the universal central extension of $\tilde{\mathfrak{g}}$. The explicit construction of this extension [9,10] can be given as follows. Let $\dot{\mathcal{K}}$ be an $(n+1)$-dimensional space with the basis $\{K_0, K_1, \ldots, K_n\}$. Consider a derivation $d_p = t_p \frac{d}{dt_p}$ of the algebra $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$ which can be naturally extended on the tensor product

$$\tilde{\mathcal{K}} = \dot{\mathcal{K}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}].$$

The space of the universal central extension of $\tilde{\mathfrak{g}}$ is

$$\mathcal{K} = \tilde{\mathcal{K}}/d\tilde{\mathcal{K}},$$

where

$$d\tilde{\mathcal{K}} = \left\{ \sum_{p=0}^{n} K_p \otimes d_p(f) \big| f \in \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \right\} \subset \tilde{\mathcal{K}}.$$

We will denote the image of $K_p \otimes t_0^{r_0}\mathbf{t^r}$ in $\mathcal{K}$ by $t_0^{r_0}\mathbf{t^r}K_p$, where $\mathbf{r} = (r_1, \ldots r_n)$ and $\mathbf{t^r} = t_1^{r_1} \ldots t_n^{r_n}$. Note that $\mathcal{K}$ has the defining relations

$$r_0 t_0^{r_0}\mathbf{t^r}K_0 + r_1 t_0^{r_0}\mathbf{t^r}K_1 + \ldots + r_n t_0^{r_0}\mathbf{t^r}K_n = 0.$$

The toroidal algebra is the Lie algebra

$$\hat{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K}$$

with the bracket

$$[g_1 \otimes f_1(t_0 \ldots t_n), g_2 \otimes f_2(t_0 \ldots t_n)] = [g_1, g_2] \otimes (f_1 f_2) + (g_1|g_2) \sum_{p=0}^{n} d_p(f_1) f_2 K_p \qquad (2.1)$$

and

$$[\hat{\mathfrak{g}}, \mathcal{K}] = 0. \tag{2.2}$$

Finally, we shall add certain derivations to $\hat{\mathfrak{g}}$. Specifically, let $\mathcal{D}$ be the Lie algebra of derivations of $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$:

$$\mathcal{D} = \left\{ \sum_{p=0}^{n} f_p(t_0, \ldots, t_n) d_p \middle| f_0, \ldots, f_n \in \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \right\}.$$

It is a general fact, that a derivation acting on a Lie algebra can be lifted in a unique way to a derivation of the universal central extension of this Lie algebra [11]. Thus the natural action of $\mathcal{D}$ on $\tilde{\mathfrak{g}}$

$$f_1(t_0, \ldots, t_n) d_p \left(g \otimes f_2(t_0, \ldots, t_n) \right) = g \otimes f_1 d_p(f_2) \tag{2.3}$$

has a unique extension to $\hat{\mathfrak{g}}$. We shall denote the lift of $f(t_0, \ldots, t_n) d_p$ by $f(t_0, \ldots, t_n) D_p$. Its action on the subspace $\tilde{\mathfrak{g}}$ is unchanged, while the action on $\mathcal{K}$ is given by the formula [2]:

$$f_1 D_a(f_2 K_b) = f_1 d_a(f_2) K_b + \delta_{ab} \sum_{p=0}^{n} f_2 d_p(f_1) K_p. \tag{2.4}$$

Consider a subalgebra $\mathcal{D}_+$ in $\mathcal{D}$:

$$\mathcal{D}_+ = \left\{ \sum_{p=1}^{n} f_p(t_0, \ldots, t_n) D_p \middle| f_1, \ldots, f_n \in \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \right\}.$$

We will be working with the algebra $\mathfrak{g}$ which is a deformation of the semidirect product of $\hat{\mathfrak{g}}$ with $\mathcal{D}_+$:

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K} \oplus \mathcal{D}_+.$$

Here multiplication in $\hat{\mathfrak{g}}$ is given by (2.1) and (2.2), the action of $\mathcal{D}_+$ on $\tilde{\mathfrak{g}}$ is given by (2.3) and on $\mathcal{K}$ by (2.4). The Lie bracket in $\mathcal{D}_+$ is the following:

$$[t_0^{r_0} \mathbf{t}^{\mathbf{r}} D_a, t_0^{m_0} \mathbf{t}^{\mathbf{m}} D_b] = m_a t_0^{r_0+m_0} \mathbf{t}^{\mathbf{r}+\mathbf{m}} D_b - r_b t_0^{r_0+m_0} \mathbf{t}^{\mathbf{r}+\mathbf{m}} D_a - m_a r_b \Big\{ \sum_{p=0}^{n} r_p t_0^{r_0+m_0} \mathbf{t}^{\mathbf{r}+\mathbf{m}} K_p \Big\}.$$

The last term in this formula is a 2-cocycle with values in a non-trivial $\mathcal{D}$-module $\mathcal{K}$. This abelian extension of the Lie algebra of vector fields on a torus is a generalization of the Virasoro algebra and was introduced by Rao and Moody [2] .

III. PRINCIPAL REALIZATION OF THE BASIC REPRESENTATION OF AFFINE LIE ALGEBRA.

When $n = 0$ the algebra $\mathfrak{g}$ constructed above yields the derived affine Kac-Moody algebra. We set $n = 0$ throughout this section. In this case $\mathcal{K}$ is one-dimensional and is spanned by K_0, while $\mathcal{D}_+$ is trivial, so

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}K_0.$$

Lepowsky and Wilson [3] constructed the principal realization of the basic representation of affine Kac-Moody algebra $A_1^{(1)}$, while the construction for a general affine algebra was given by Kac, Kazhdan, Lepowsky and Wilson [4]. Let us recall this construction (see also Ref.1 for details). Its starting point is the principal Heisenberg subalgebra in $\mathfrak{g}$.

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be simple roots and θ be the highest positive root in $\dot{\Delta}$. We define the height for a root $\beta = \sum_{i=1}^{\ell} k_i \alpha_i$ as

$$\mathrm{ht}(\beta) = \sum_{i=1}^{\ell} k_i.$$

Note that $\mathrm{ht}(\theta) = h - 1$, where h is a Coxeter number of $\dot{\mathfrak{g}}$ [12] .

Consider the principle gradation of $\tilde{\mathfrak{g}}$ defined by

$$\deg(\dot{\mathfrak{g}}^{\alpha_1} \otimes 1) = \ldots = \deg(\dot{\mathfrak{g}}^{\alpha_\ell} \otimes 1) = \deg(\dot{\mathfrak{g}}^{-\theta} \otimes t_0) = 1,$$

$$\deg(\dot{\mathfrak{g}}^{-\alpha_1} \otimes 1) = \ldots = \deg(\dot{\mathfrak{g}}^{-\alpha_\ell} \otimes 1) = \deg(\dot{\mathfrak{g}}^{\theta} \otimes t_0^{-1}) = -1.$$

We choose nonzero root vectors $e_0 \in \dot{\mathfrak{g}}^{-\theta} \otimes t_0, e_1 \in \dot{\mathfrak{g}}^{\alpha_1} \otimes 1, \ldots, e_\ell \in \dot{\mathfrak{g}}^{\alpha_\ell}$ and form the element $\overline{e} = \sum_{i=0}^{\ell} e_i \in \tilde{\mathfrak{g}}$. Since $\overline{e}$ is of degree 1 then its centralizer $\tilde{\mathfrak{s}}$ in $\tilde{g}$ is homogeneous with respect to the gradation.

Next, consider a projection

$$\pi : \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \to \dot{\mathfrak{g}} \quad \text{by} \quad t_0 \mapsto 1.$$

Note that $\deg(\dot{\mathfrak{g}}^{-\theta} \otimes t_0) = 1$ and $\deg(\dot{\mathfrak{g}}^{-\theta} \otimes 1) = ht(-\theta) = -h + 1$, while $\pi(\dot{\mathfrak{g}}^{-\theta} \otimes t_0) = \pi(\dot{\mathfrak{g}}^{-\theta} \otimes 1)$, and thus the principal $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$ induces the $\mathbb{Z}_h$-gradation of $\dot{\mathfrak{g}}$:

$$\dot{\mathfrak{g}} = \sum_{j \in \mathbb{Z}_h} \dot{\mathfrak{g}}_j.$$

The element $\pi(\overline{e})$ is regular in $\dot{\mathfrak{g}}$, hence its centralizer $\dot{\mathfrak{s}} = \pi(\tilde{\mathfrak{s}})$ in $\dot{\mathfrak{g}}$ is a Cartan subalgebra, which implies that both $\dot{\mathfrak{s}}$ and $\tilde{\mathfrak{s}} = \dot{\mathfrak{s}} \otimes \mathbb{C}[t_0, t_0^{-1}]$ are abelian.

The preimage $\mathfrak{s} = \dot{\mathfrak{s}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}K_0$ of $\tilde{\mathfrak{s}}$ in $\mathfrak{g}$ is an infinite-dimensional Heisenberg algebra which is called the principal Heisenberg subalgebra of $\mathfrak{g}$. This subalgebra is the cornerstone of the construction of the vertex operator representation of $\mathfrak{g}$. The scheme of this construction is the following: one starts with the standard irreducible representation of $\mathfrak{s}$ and then magically one can lift this representation to the whole of $\mathfrak{g}$. Moreover the action of $\mathfrak{g}$ is prescribed and given by vertex operators. Finally one has only to check in one way or another that this construction works.

The irreducible representation of $\mathfrak{g}$ constructed in this way is of the highest weight Λ_0, where the linear functional Λ_0 is given (we consider the simply-laced case) by

$$\Lambda_0(\nu^{-1}(\alpha_i) \otimes t_0^0) = 0, \quad \Lambda_0(K_0) = 1.$$

This highest weight representation is called the basic representation of $\mathfrak{g}$.

The Cartan subalgebra $\dot{\mathfrak{s}}$ is homogeneous with respect to the principle $\mathbb{Z}_h$-gradation:

$$\dot{\mathfrak{s}} = \sum_{j \in \mathbb{Z}_h} \dot{\mathfrak{s}}_j.$$

Since $(\dot{\mathfrak{g}}_i|\dot{\mathfrak{g}}_j) = 0$ unless $i + j = 0 \pmod h$ and $(\cdot|\cdot)$ is nondegenerate on $\dot{\mathfrak{s}}$ then one can choose a basis in $\dot{\mathfrak{s}}$: $\{T_1, T_2, \ldots, T_\ell\}$ such that $T_i \in \dot{\mathfrak{g}}_{m_i}$, where $1 \leq m_1 \leq m_2 \leq \ldots \leq m_\ell < h$ and

$$(T_i|T_{\ell+1-j}) = h\delta_{ij}.$$

The numbers $\{m_1, \ldots, m_\ell\}$ are called the exponents of $\dot{\mathfrak{g}}$ [12] . Note that $\dot{\mathfrak{s}} \cap \dot{\mathfrak{h}} = (0)$, and therefore zero is not an exponent. Also, $m_{\ell+1-i} = h - m_i$.

The principal Heisenberg subalgebra $\mathfrak{s}$ is spanned by $T_i \otimes t_0^j$, $j \in \mathbb{Z}, i = 1, \ldots, \ell$ and K_0. We can now write the multiplication in $\mathfrak{s}$ explicitly:

$$\left[T_{i_1} \otimes t_0^{j_1}, T_{\ell+1-i_2} \otimes t_0^{j_2}\right] = hj_1\delta_{i_1 i_2}\delta_{j_1,-j_2}.$$

It turns out that it is more convenient to work with a slightly different realization of $\mathfrak{g}$ based on the $\mathbb{Z}_h$-gradation of $\dot{\mathfrak{g}}$. Consider

$$\mathfrak{g}_s = \sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \oplus \mathbb{C}K_0,$$

with the Lie bracket

$$[g_1 \otimes s^i, g_2 \otimes s^j] = [g_1, g_2] \otimes s^{i+j} + \frac{i}{h}(g_1|g_2)\delta_{i,-j}K_0,$$

$$[K_0, \mathfrak{g}_s] = 0.$$

The proof of the following lemma is straightforward, so we omit it.

Lemma 1. *The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_s$ are isomorphic and the isomorphism*

$$\psi : \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}K_0 \to \sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \oplus \mathbb{C}K_0$$

is given by

$$\psi(e_\alpha \otimes t^i) = e_\alpha \otimes s^{\mathrm{ht}(\alpha)+ih},$$

$$\psi(\nu^{-1}(\alpha) \otimes t^i) = \nu^{-1}(\alpha) \otimes s^{ih} + \delta_{i,0}\frac{\mathrm{ht}(\alpha)}{h}K_0,$$

$$\psi(K_0) = K_0.$$

We shall identify $\mathfrak{g}$ and $\mathfrak{g}_s$ using this isomorphism.

In order to describe a basis in the principal Heisenberg subalgebra $\psi(\mathfrak{s})$ we need to introduce the sequence $\{b_i\}_{i\in\mathbb{Z}}$ such that $b_{i+j\ell} = m_i + jh$ for $j \in \mathbb{Z}, i = 1, \dots, \ell$. Then $\psi(\mathfrak{s})$ is spanned by $\left\{T_i \otimes s^{b_i}, K_0\right\}, i \in \mathbb{Z}$. Note that $b_{1-i} = -b_i$.

Let $\dot{\Delta}_s$ be the root system of $\dot{\mathfrak{g}}$ with respect to the Cartan subalgebra $\dot{\mathfrak{s}}$. For $\alpha \in \dot{\Delta}_s$ choose a root element

$$A^\alpha = \sum_{j\in\mathbb{Z}_h} A_j^\alpha.$$

Since $\pi(\bar{e})$ is a regular element of the Cartan subalgebra then $[\pi(\bar{e}), A^\alpha] = \lambda^\alpha A^\alpha$ with $\lambda^\alpha \neq 0$. Hence $A_j^\alpha = \left(\frac{\mathrm{ad}\pi(\bar{e})}{\lambda^\alpha}\right)^j A_0^\alpha$ and all the components A_j^α are nonzero. We let the indices i, j in T_i and A_j^α run over $\mathbb{Z}$ by setting $T_{i_1} = T_{i_2}$ if $i_1 \equiv i_2 (\mathrm{mod}\ \ell)$ and $A_{j_1}^\alpha = A_{j_2}^\alpha$ if $j_1 \equiv j_2 (\mathrm{mod}\ h)$.

Define constants $\lambda_i^\alpha = \alpha(T_i)$. Then $[T_i, A^\alpha] = \lambda_i^\alpha A^\alpha$ and $[T_i, A_j^\alpha] = \lambda_i^\alpha A_{j+m_i}^\alpha$.

Consider an automorphism σ of $\dot{\mathfrak{g}}$ such that

$$\sigma = \zeta^j \mathrm{Id} \quad \text{on} \ \ \dot{\mathfrak{g}}_j,$$

where $\zeta \in \mathbb{C}$ is a primitive h-root of 1. For a root element $A^\alpha = \sum_{j=1}^{h} A_j^\alpha$ we have

$$[T_i, \sigma(A^\alpha)] = \sigma([\sigma^{-1}(T_i), A^\alpha]) = \sigma([\zeta^{-m_i}T_i, A^\alpha]) = \zeta^{-m_i}\lambda_i^\alpha \sigma(A^\alpha).$$

Thus $\sigma(A^\alpha) = \sum_{j=1}^{h} \zeta^j A_j^\alpha$ is also a root element and σ induces an automorphism of $\dot{\Delta}_s$. Since all A_j^α are nonzero then the length of each orbit of σ in $\dot{\Delta}_s$ is h.

We may choose ℓ roots $\beta_1, \ldots, \beta_\ell \in \dot{\Delta}_s$ such that $A_0^{\beta_1}, \ldots, A_0^{\beta_\ell}$ span $\dot{\mathfrak{g}}_0$. Note that if two root elements belong to the same orbit under the action of σ then their zero componens are proportional. Therefore every orbit of σ contains exactly one of $\beta_1, \ldots, \beta_\ell$. Consequently, we may choose $\left\{\sigma^j(A^{\beta_i})\right\}, i = 1, \ldots, \ell, j = 1, \ldots, h$ as our family of the root elements. The set $\left\{A_j^{\beta_i} \otimes s^j, T_j \otimes s^{b_j}, K_0\right\}, i = 1, \ldots, \ell, j \in \mathbb{Z}$ forms a basis of $\mathfrak{g}_s$.

Remark. The properties of the automorphism σ were studied by Kostant [12] . He proved, in particular, that σ acts on the root system $\dot{\Delta}_s$ as a Coxeter transformation. Remarkably, the complex coefficients λ_i^α could be interpreted as the orthogonal projections of the roots on a real plane invariant under the Coxeter transformation. These projections were introduced by Coxeter in order to visualize the regular polytopes of higher dimensions. In Chapters 12, 13 of Ref.13 and in section 12.5 of Ref.14 the case of the root system of type H_4 is treated, which also gives the answer for E_8.

Now we shall construct the standard representation (φ, F) of $\mathfrak{s}$. The space of this representation (called the Fock space) is the polynomial algebra in the infinitely many variables:

$$F = \mathbb{C}[x_1, x_2, x_3, \ldots].$$

For $i \geq 1$, $T_{1-i} \otimes s^{-b_i}$ is represented by a multiplication operator:

$$\varphi(T_{1-i} \otimes s^{-b_i}) = b_i x_i,$$

$T_i \otimes s^{b_i}$ is represented by a differentiation operator:

$$\varphi(T_i \otimes s^{b_i}) = \frac{\partial}{\partial x_i}$$

and $\varphi(K_0) = \mathrm{Id}$.

Indeed, the only relation to be checked is

$$[\varphi(T_i \otimes s^{b_i}), \varphi(T_{1-j} \otimes s^{b_{1-j}}) = \frac{1}{h}(T_i | T_{1-j}) b_i \delta_{b_i b_j} \varphi(K_0).$$

But as it can be easily seen, both expressions are equal to $b_i \delta_{ij}$ Id.

One can lift this representation of $\mathfrak{s}$ to the whole $\mathfrak{g}$ using vertex operators. We consider the space $\mathfrak{g}[[z, z^{-1}]]$ of formal Laurent series in a variable z with coefficients in $\mathfrak{g}$ and the space $\mathrm{End}(F)[[z, z^{-1}]]$ of series with coefficients in $\mathrm{End}(F)$ (see Chapter 2 in Ref.15 for details). The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[[z, z^{-1}]]$ is well-defined.

Every representation $\varphi : \mathfrak{g} \to \mathrm{End}(F)$ defines a homomorphism $\varphi : \mathfrak{g}[[z, z^{-1}]] \to \mathrm{End}(F)[[z, z^{-1}]]$.

The following theorem shows how to lift the standard representation of the principal Heisenberg subalgebra to $\mathfrak{g}$:

Theorem 2 (Ref.1). *There exists a representation*

$$\varphi : \mathfrak{g} \to \mathrm{End}(\mathbb{C}[x_1, x_2, \ldots])$$

such that for $i \in \mathbb{N}$

$$\varphi(T_{1-i} \otimes s^{-b_i}) = b_i x_i,$$

$$\varphi(T_i \otimes s^{b_i}) = \frac{\partial}{\partial x_i},$$

$$\varphi(K_0) = \mathrm{Id},$$

$$\sum_{j \in \mathbb{Z}} \varphi(A_j^\alpha \otimes s^j) z^{-j} = A^\alpha(z), \text{ where}$$

$$A^\alpha(z) = \Lambda_0(A_0^\alpha \otimes s^0) \exp\left(\sum_{i=1}^{\infty} \lambda_i^\alpha z^{b_i} x_i\right) \exp\left(-\sum_{i=1}^{\infty} \lambda_{1-i}^\alpha \frac{z^{-b_i}}{b_i} \frac{\partial}{\partial x_i}\right).$$

Since A_j^α together with $T_i \otimes s^{b_i}$ and K_0 span $\mathfrak{g}$ then the above formulas completely determine this representation.

We will use extensively the following Laurent series:

$$\delta(z) = \sum_{j \in \mathbb{Z}} z^j \quad \text{and} \quad D\delta(z) = \sum_{j \in \mathbb{Z}} j z^j.$$

The first of these is the formal analogue of the delta function.

For an element $B = \sum_{j \in \mathbb{Z}_h} B_j \in \dot{\mathfrak{g}}$ denote by $B(z)$ the formal series $\sum_{j \in \mathbb{Z}} \varphi(B_j \otimes s^j) z^{-j} \in \mathrm{End}(F)[[z, z^{-1}]]$.

Corollary 3. *Let* $A = \sum_{j=1}^{h} A_j, B = \sum_{j=1}^{h} B_j \in \dot{\mathfrak{g}}$ *and let* $C^j = [A_j, B] \in \dot{\mathfrak{g}}$. *Then*

$$[A(z_1), B(z_2)] = \sum_{j=1}^{h} \big(C^j(z_2) + \frac{j}{h}(A_j|B_{-j})\big)\big(\frac{z_2}{z_1}\big)^j \delta\big(\big(\frac{z_2}{z_1}\big)^h\big) + \sum_{j=1}^{h} (A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j D\delta\big(\big(\frac{z_2}{z_1}\big)^h\big).$$

Proof of the Corollary.

$$[A(z_1), B(z_2)] = \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} [\varphi(A_j \otimes s^j), \varphi(B_i \otimes s^i)] z_1^{-j} z_2^{-i} =$$

$$= \sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \left\{ \varphi([A_j, B_i]\otimes s^{j+i}) + \frac{1}{h}(A_j|B_i)j\delta_{j,-i}\varphi(K_0)\right\} z_1^{-j}z_2^{-i} =$$

$$= \sum_{j\in\mathbb{Z}}\sum_{k=i+j\in\mathbb{Z}} \varphi([A_j, B_{k-j}]\otimes s^k)z_1^{-j}z_2^{j-k} + \frac{1}{h}\sum_{j\in\mathbb{Z}}(A_j|B_{-j})j\big(\frac{z_2}{z_1}\big)^j =$$

$$= \sum_{j\in\mathbb{Z}} C^j(z_2)\big(\frac{z_2}{z_1}\big)^j + \frac{1}{h}\sum_{j_1=1}^{h}\sum_{\substack{j_2\in\mathbb{Z}\\ j=j_1+hj_2}} (A_{j_1}|B_{-j_1})(j_1+hj_2)\big(\frac{z_2}{z_1}\big)^{j_1+hj_2} =$$

$$= \sum_{j_1=1}^{h}\big(C^{j_1}(z_2) + \frac{j_1}{h}(A_{j_1}|B_{-j_1})\big)\big(\frac{z_2}{z_1}\big)^{j_1}\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) + \sum_{j_1=1}^{h}(A_{j_1}|B_{-j_1})\big(\frac{z_2}{z_1}\big)^{j_1}D\delta\big(\big(\frac{z_2}{z_1}\big)^h\big).$$

IV. CONSTRUCTION OF THE VERTEX OPERATOR REPRESENTATION FOR THE TOROIDAL LIE ALGEBRA.

In this section we construct a vertex operator representation for the toroidal Lie algebra that generalizes the principal realization of the basic representation of affine Lie algebra. Again it will be convenient to replace $\mathfrak{g}$ with

$$\mathfrak{g}_s = \sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j\mathbb{C}[t_1^{\pm},\dots,t_n^{\pm}] \oplus \mathcal{K} \oplus \mathcal{D}_+,$$

with the Lie bracket

$$[g_1\otimes f_1(s,\mathbf{t}), g_2\otimes f_2(s,\mathbf{t})] = [g_1,g_2]\otimes(f_1f_2) + (g_1|g_2)\left\{\frac{1}{h}(s\frac{d}{ds}f_1)f_2K_0 + \sum_{p=1}^{n} d_p(f_1)f_2K_p\right\},$$

$$[f_1(s,\mathbf{t})D_p, g\otimes f_2(s,\mathbf{t})] = g\otimes f_1(d_pf_2)$$

while the multiplication in $\mathcal{D}_+$ and its action on $\mathcal{K}$ are the same as in $\mathfrak{g}$. Here and below we make identifications $s^{hr_0}\mathbf{t}^{\mathbf{r}}K_a = t_0^{r_0}\mathbf{t}^{\mathbf{r}}K_a$ and $s^{hr_0}\mathbf{t}^{\mathbf{r}}D_b = t_0^{r_0}\mathbf{t}^{\mathbf{r}}D_b$, $a = 0,1,\dots,n, b = 1,\dots,n$.

The following lemma is an immediate generalization of Lemma 1:

Lemma 4. *The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_s$ are isomorphic and the isomorphism is given by*

$$\psi(e_\alpha\otimes t_0^{r_0}\mathbf{t}^{\mathbf{r}}) = e_\alpha\otimes s^{\mathrm{ht}(\alpha)+hr_0}\mathbf{t}^{\mathbf{r}},$$

$$\psi(\nu^{-1}(\alpha)\otimes t_0^{r_0}\mathbf{t}^{\mathbf{r}}) = \nu^{-1}(\alpha)\otimes s^{hr_0}\mathbf{t}^{\mathbf{r}} + \frac{\mathrm{ht}(\alpha)}{h}s^{hr_0}\mathbf{t}^{\mathbf{r}}K_0,$$

$$\psi = \text{ Id } \text{ on } \mathcal{K} \oplus \mathcal{D}_+.$$

We shall identify $\mathfrak{g}$ and $\mathfrak{g}_s$ using this isomorphism.

The subalgebra $\mathfrak{s}$ with the basis $\left\{T_i \otimes s^{b_i}, s^{ih} D_p, s^{ih} K_p, K_0\right\}$, $i \in \mathbb{Z}$, $p = 1, \ldots, n$ is the principal (degenerate) Heisenberg subalgebra of $\mathfrak{g}$.

Indeed, K_0 is its central element and the multiplication in $\mathfrak{s}$ is given by

$$[T_i \otimes s^{b_i}, T_{1-j} \otimes s^{-b_j}] = b_i \delta_{ij} K_0,$$

$$[s^{ih} D_a, s^{jh} K_b] = \delta_{ab} i s^{(i+j)h} K_0 = i \delta_{ab} \delta_{i,-j} K_0,$$

$$[s^{jh} D_p, T_i \otimes s^{b_i}] = 0, [s^{ih} D_a, s^{jh} D_b] = 0,$$

$$[T_i \otimes s^{b_i}, s^{jh} K_p] = 0, [s^{ih} K_a, s^{jh} K_b] = 0,$$

where $i, j \in \mathbb{Z}, a, b, p = 1, \ldots, n$. This subalgebra is degenerate as a Heisenberg algebra since $[D_p, K_p] = 0$.

The Heisenberg algebra $\mathfrak{s}$ can be represented on the space

$$F = \mathbb{C}[q_p^{\pm}, x_i, u_{pi}, v_{pi}]_{i \in \mathbb{N}}^{p=1,\ldots,n}$$

by differentiation and multiplication operators:

$$\varphi(T_i \otimes s^{b_i}) = \frac{\partial}{\partial x_i} \ , \ \ \varphi(T_{1-i} \otimes s^{-b_i}) = b_i x_i,$$

$$\varphi(s^{ih} D_p) = \frac{\partial}{\partial u_{pi}} \ , \ \ \varphi(s^{-ih} D_p) = i v_{pi},$$

$$\varphi(s^{ih} K_p) = \frac{\partial}{\partial v_{pi}} \ , \ \ \varphi(s^{-ih} K_p) = i u_{pi},$$

$$\varphi(D_p) = q_p \frac{\partial}{\partial q_p} \ , \ \ \varphi(K_p) = 0,$$

$$\varphi(K_0) = \text{Id},$$

where $i \geq 1$ and $p = 1, \ldots, n$. Our goal is to extend this representation of $\mathfrak{s}$ to $\mathfrak{g}$. Consider the following elements of $\mathfrak{g}[[z, z^{-1}]]$:

$$\sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} K_0 z^{-jh} \quad \text{and} \sum_{j \in \mathbb{Z}} A_j^{\alpha} \otimes s^j \mathbf{t}^{\mathbf{r}} z^{-j}.$$

Note that

$$[T_i \otimes s^{b_i}, \sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} K_0 z^{-jh}] = 0,$$

$$[s^{ih}K_p, \sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t}^{\mathbf{r}}K_0 z^{-jh}] = 0,$$

$$[s^{ih}D_p, \sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t}^{\mathbf{r}}K_0 z^{-jh}] = \sum_{j\in\mathbb{Z}} r_p s^{(i+j)h}\mathbf{t}^{\mathbf{r}}K_0 z^{-jh} =$$

$$= \sum_{k=i+j\in\mathbb{Z}} r_p s^{kh}\mathbf{t}^{\mathbf{r}}K_0 z^{-kh+ih} = r_p z^{ih}\sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t}^{\mathbf{r}}K_0 z^{-jh}.$$

The theory of vertex operators suggests (see Lemma 14.5 in Ref.1) that $\sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t}^{\mathbf{r}}K_0 z^{-jh}$ should be represented by

$$K_0(z,\mathbf{r}) = \mathbf{q}^{\mathbf{r}}\exp\left(\sum_{p=1}^{n} r_p \sum_{j\geq 1} z^{jh}u_{pj}\right)\exp\left(-\sum_{p=1}^{n} r_p \sum_{j\geq 1}\frac{z^{-jh}}{j}\frac{\partial}{\partial v_{pj}}\right).$$

Here $\mathbf{q}^{\mathbf{r}} = q_1^{r_1}\dots q_n^{r_n}$.

In a similar way the commutator relations

$$[T_i\otimes s^{b_i}, \sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j}] = \sum_{j\in\mathbb{Z}} \lambda_i^\alpha A_{j+b_i}^\alpha\otimes s^{j+b_i}\mathbf{t}^{\mathbf{r}}z^{-j} =$$

$$= \lambda_i^\alpha \sum_{k=j+b_i\in\mathbb{Z}} A_k^\alpha\otimes s^k\mathbf{t}^{\mathbf{r}}z^{-k+b_i} = \lambda_i^\alpha z^{b_i}\sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j},$$

$$[s^{ih}K_p, \sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j}] = 0,$$

$$[s^{ih}D_p, \sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j}] = \sum_{j\in\mathbb{Z}} r_p A_j^\alpha\otimes s^{j+ih}\mathbf{t}^{\mathbf{r}}z^{-j} =$$

$$= r_p \sum_{k=j+ih\in\mathbb{Z}} A_j^\alpha\otimes s^k\mathbf{t}^{\mathbf{r}}z^{-k+ih} = r_p z^{ih}\sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j}$$

suggest that $\sum_{j\in\mathbb{Z}} A_j^\alpha\otimes s^j\mathbf{t}^{\mathbf{r}}z^{-j}$ should be represented by

$$A^\alpha(z,\mathbf{r}) =$$

$$= \mathbf{q}^{\mathbf{r}}\Lambda_0(A_0^\alpha\otimes s^0)\exp\left(\sum_{i=1}^{\infty}\lambda_i^\alpha z^{b_i}x_i\right)\exp\left(-\sum_{i=1}^{\infty}\lambda_{1-i}^\alpha\frac{z^{-b_i}}{b_i}\frac{\partial}{\partial x_i}\right)\times$$

$$\times\exp\left(\sum_{p=1}^{n} r_p\sum_{j\in\mathbb{Z}} z^{jh}u_{pj}\right)\exp\left(-\sum_{p=1}^{n} r_p\sum_{j\in\mathbb{Z}}\frac{z^{-jh}}{j}\frac{\partial}{\partial v_{pj}}\right) =$$

$$= A^{\alpha}(z) K_0(z, \mathbf{r}).$$

The last formula hints that for $B \in \dot{\mathfrak{g}}$ we may try to represent $\sum_{j \in \mathbb{Z}} B_j \otimes s^j \mathbf{t}^{\mathbf{r}} z^{-j}$ by

$$B(z, \mathbf{r}) = B(z) K_0(z, \mathbf{r}).$$

Note that $B(z)$ is known from the affine case (Theorem 2).

In fact, we shall see that the same approach is valid even for $\mathcal{K}$ and $\mathcal{D}_+$, but for these we should first discuss the concept of the normal ordering.

Let $X(z) = \sum_{i \in \mathbb{Z}} X_i z^i, Y(z) = \sum_{j \in \mathbb{Z}} Y_j z^j \in \mathrm{End}(F)[[z, z^{-1}]]$. We say that the product $X(z)Y(z)$ exists if for every $k \in \mathbb{Z}$ and for every $v \in F$ the sum

$$\sum_{i \in \mathbb{Z}} X_i Y_{k-i} v$$

has finitely many nonzero terms. If this is the case then

$$X(z)Y(z) = \sum_{k \in \mathbb{Z}} \big(\sum_{i \in \mathbb{Z}} X_i Y_{k-i} \big) z^k.$$

It is possible that $X(z)Y(z)$ exists while $Y(z)X(z)$ does not. For example, this happens for

$$X(z) = \sum_{i \geq 1} x_i z^i \quad \text{and} \quad Y(z) = \sum_{i \geq 1} \frac{\partial}{\partial x_i} z^{-i}.$$

To improve the situation we define the normal ordering for the product of operators on F. For a differential operator $P \in \mathrm{End}(F)$ the normal ordering $:x_i P:$ is defined as $x_i P$, while the normal ordering of the product $:\frac{\partial}{\partial x_i} P:$ is defined as $P \frac{\partial}{\partial x_i}$. Note that for $X(z)$ and $Y(z)$ from the above example $:X(z)Y(z): = :Y(z)X(z): = X(z)Y(z)$.

From the action of the principle Heisenberg subalgebra on F we see that the series $\sum_{j \in \mathbb{Z}} s^{jh} K_p z^{-jh}$ and $\sum_{j \in \mathbb{Z}} s^{jh} D_p z^{-jh}$ are represented by

$$K_p(z) = \sum_{i \geq 1} i u_{pi} z^{ih} + \sum_{i \geq 1} \frac{\partial}{\partial v_{pi}} z^{-ih}$$

and

$$D_p(z) = \sum_{i \geq 1} i v_{pi} z^{ih} + q_p \frac{\partial}{\partial q_p} + \sum_{i \geq 1} \frac{\partial}{\partial u_{pi}} z^{-ih}.$$

Using the analogy with $A^{\alpha}(z, \mathbf{r})$ we shall represent $\sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} K_p z^{-jh}$ by

$$K_p(z, \mathbf{r}) = K_p(z) K_0(z, \mathbf{r})$$

and $\sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t}^{\mathbf{r}} D_p z^{-jh}$ by

$$D_p(z,\mathbf{r}) = :D_p(z)K_0(z,\mathbf{r}): \ .$$

Note that we use the normal ordering in the case of $D_p(z,\mathbf{r})$ in order for the product to exist.

We summarize this discussion in our main theorem:

Theorem 5. *There exists a representation φ of the toroidal Lie algebra $\mathfrak{g}_s$ on the space*

$$F = \mathbb{C}[q_p^{\pm}, x_i, u_{pi}, v_{pi}]_{i\in\mathbb{N}}^{p=1,\dots,n}$$

such that for $i=1,\dots,\ell,\ p=1,\dots,n,\ \alpha\in\dot{\Delta}_s, \mathbf{r}\in\mathbb{Z}^n$

$$\sum_{j\in\mathbb{Z}} \varphi(s^{jh}\mathbf{t}^{\mathbf{r}}K_0)z^{-jh} = K_0(z,\mathbf{r}) =$$

$$= \mathbf{q}^{\mathbf{r}}\exp\left(\sum_{p=1}^{n} r_p \sum_{j\geq 1} z^{jh}u_{pj}\right)\exp\left(-\sum_{p=1}^{n} r_p\sum_{j\geq 1}\frac{z^{-jh}}{j}\frac{\partial}{\partial v_{pj}}\right), \tag{4.1}$$

$$\sum_{j\in\mathbb{Z}} \varphi(T_i\otimes s^{m_i+jh}\mathbf{t}^{\mathbf{r}})z^{-m_i-jh} = T_i(z,\mathbf{r}) =$$

$$= \left\{\sum_{j\geq 1}(jh-m_i)z^{jh-m_i}x_{j\ell+1-i} + \sum_{j\geq 0} z^{-jh-m_i}\frac{\partial}{\partial x_{j\ell+i}}\right\}K_0(z,\mathbf{r}), \tag{4.2}$$

$$\sum_{j\in\mathbb{Z}} \varphi(A_j^{\alpha}\otimes s^j\mathbf{t}^{\mathbf{r}})z^{-j} = A^{\alpha}(z,\mathbf{r}) =$$

$$\Lambda_0(A_0^{\alpha}\otimes s^0)\exp\left(\sum_{i=1}^{\infty}\lambda_i^{\alpha}z^{b_i}x_i\right)\exp\left(-\sum_{i=1}^{\infty}\lambda_{1-i}^{\alpha}\frac{z^{-b_i}}{b_i}\frac{\partial}{\partial x_i}\right)K_0(z,\mathbf{r}), \tag{4.3}$$

$$\sum_{j\in\mathbb{Z}} \varphi(s^{jh}\mathbf{t}^{\mathbf{r}}K_p)z^{-jh} = K_p(z,\mathbf{r}) = \left\{\sum_{i\geq 1} iu_{pi}z^{ih} + \sum_{i\geq 1}\frac{\partial}{\partial v_{pi}}z^{-ih}\right\}K_0(z,\mathbf{r}), \tag{4.4}$$

$$\sum_{j\in\mathbb{Z}} \varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_p)z^{-jh} = D_p(z,\mathbf{r}) = :\left\{\sum_{i\geq 1} iv_{pi}z^{ih} + q_p\frac{\partial}{\partial q_p} + \sum_{i\geq 1}\frac{\partial}{\partial u_{pi}}z^{-ih}\right\}K_0(z,\mathbf{r}): . \tag{4.5}$$

Remark. By choosing a specialization $q_p \in \mathbb{C}\backslash\{0\}$, $p = 1, \dots, n$, we obtain a representation of the toroidal algebra $\hat{\mathfrak{g}}$ on the Fock space

$$\hat{F} = \mathbb{C}[x_i, u_{pi}, v_{pi}]_{i\in\mathbb{N}}^{p=1,\dots,n}.$$

V. PROOF OF THE MAIN THEOREM.

First of all, let us check that formulas (4.1) - (4.5) define a linear map $\varphi : \mathfrak{g} \to \mathrm{End}(F)$. The linear dependencies between the momenta at the left hand side are of the form

$$\varphi(A_j^{\sigma^k(\alpha)} \otimes s^j \mathbf{t}^{\mathbf{r}}) = \zeta^{kj} \varphi(A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{r}})$$

and

$$j\varphi(s^{jh}\mathbf{t}^{\mathbf{r}} K_0) + \sum_{p=1}^n r_p \varphi(s^{jh}\mathbf{t}^{\mathbf{r}} K_p) = 0.$$

These dependencies extend to the following relations for the corresponding series:

$$\sum_{j\in\mathbb{Z}} \varphi(A_j^{\sigma^k(\alpha)} \otimes s^j \mathbf{t}^{\mathbf{r}}) z^{-j} = \sum_{j\in\mathbb{Z}} \varphi(A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{r}})(\zeta^{-k} z)^{-j}$$

and

$$-\frac{1}{h} D_z \sum_{j\in\mathbb{Z}} \varphi(s^{jh}\mathbf{t}^{\mathbf{r}} K_0) z^{-jh} + \sum_{p=1}^n r_p \sum_{j\in\mathbb{Z}} \varphi(s^{jh}\mathbf{t}^{\mathbf{r}} K_p) z^{-jh} = 0,$$

where $D_z = z\frac{\partial}{\partial z}$. We have to show that the same relations hold for the right hand sides of (4.1) - (4.5). Indeed, noting that $\lambda_i^{\sigma^k(\alpha)} = \zeta^{-km_i}\lambda_i^\alpha$ we obtain

$$A^{\sigma^k(\alpha)}(z,\mathbf{r}) = \Lambda_0(A_0^{\sigma^k(\alpha)}) \exp\Big(\sum_{i=1}^\infty \lambda_i^\alpha \zeta^{-km_i} z^{b_i} x_i\Big) \exp\Big(-\sum_{i=1}^\infty \lambda_{1-i}^\alpha \zeta^{km_i} \frac{z^{-b_i}}{b_i} \frac{\partial}{\partial x_i}\Big) K_0(z,\mathbf{r}) =$$

$$= \Lambda_0(A_0^\alpha) \exp\Big(\sum_{i=1}^\infty \lambda_i^\alpha (\zeta^{-k} z)^{b_i} x_i\Big) \exp\Big(-\sum_{i=1}^\infty \lambda_{1-i}^\alpha \frac{(\zeta^{-k} z)^{-b_i}}{b_i} \frac{\partial}{\partial x_i}\Big) K_0(\zeta^{-k} z,\mathbf{r}) = A^\alpha(\zeta^{-k} z,\mathbf{r}).$$

The verification of the second relation is also straightforward:

$$D_z K_0(z,\mathbf{r}) = D_z \mathbf{q}^{\mathbf{r}} \exp\left(\sum_{p=1}^n r_p \sum_{j\geq 1} z^{jh} u_{pj}\right) \exp\left(-\sum_{p=1}^n r_p \sum_{j\geq 1} \frac{z^{-jh}}{j} \frac{\partial}{\partial v_{pj}}\right) =$$

$$\sum_{p=1}^{n} r_p \left\{ \sum_{j\geq 1} z^{jh} jhu_{pj} + \sum_{j\geq 1} z^{-jh} h \frac{\partial}{\partial v_{pj}} \right\} K_0(z,\mathbf{r}) = h \sum_{p=1}^{n} r_p K_p(z,\mathbf{r}).$$

Observing that the momenta of the series in the left hand sides of (4.1) - (4.5) span $\mathfrak{g}$ we conclude that the linear map $\varphi : \mathfrak{g} \to \mathrm{End}(F)$ is well-defined. We need to show now that φ is a homomorphism of Lie algebras.

The following Lemma and its Corollary which we state without proof are very useful for the computations with formal series.

Lemma 6 (cf. Proposition 2.2.2 in Ref.15). *Let*

$$\delta(z) = \sum_{k\in\mathbb{Z}} z^k \quad \text{and} \quad D\delta(z) = D_z\delta(z) = \sum_{k\in\mathbb{Z}} kz^k.$$

If the products in the left hand sides exist then the following equalities hold:

$$(i) \qquad X(z_1)\delta(\frac{z_2}{z_1}) = X(z_2)\delta(\frac{z_2}{z_1}),$$

$$(ii) \qquad X(z_1)D\delta(\frac{z_2}{z_1}) = X(z_2)D\delta(\frac{z_2}{z_1}) + D_{z_2}(X(z_2))\delta(\frac{z_2}{z_1}),$$

Corollary 7. *Let $X(z) = Y(z^h)$. The following equalities hold provided the products in the left hand sides exist:*

$$(i) \qquad X(z_1)\delta\left((\frac{z_2}{z_1})^h\right) = X(z_2)\delta\left((\frac{z_2}{z_1})^h\right),$$

$$(ii) \qquad X(z_1)D\delta\left((\frac{z_2}{z_1})^h\right) = X(z_2)D\delta\left((\frac{z_2}{z_1})^h\right) + \frac{1}{h}D_{z_2}(X(z_2))\delta\left((\frac{z_2}{z_1})^h\right),$$

Note that $A^\alpha(z,\mathbf{r}) = A^\alpha(z)K_0(z,\mathbf{r})$ and $T_i(z,\mathbf{r}) = T_i(z)K_0(z,\mathbf{r})$, hence for every $B \in \dot{\mathfrak{g}}$ we have $B(z,\mathbf{r}) = B(z)K_0(z,\mathbf{r})$.

In order to verify the commutator relations between the elements of $\sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j\mathbb{C}[t_1^{\pm},\ldots,t_n^{\pm}]$ we have to show that for every $A, B \in \dot{\mathfrak{g}}$

$$\sum_{j,i\in\mathbb{Z}} \varphi(\left[A_j \otimes s^j\mathbf{t}^{\mathbf{r}}, B_i \otimes s^i\mathbf{t}^{\mathbf{m}}\right])z_1^{-j}z_2^{-i} = \sum_{j,i\in\mathbb{Z}} \left[\varphi(A_j \otimes s^j\mathbf{t}^{\mathbf{r}}), \varphi(B_i \otimes s^i\mathbf{t}^{\mathbf{m}})\right] z_1^{-j}z_2^{-i}. \tag{5.1}$$

Indeed, let $C^j = [A_j, B]$. Then

$$\sum_{j,i\in\mathbb{Z}} \varphi(\big[A_j \otimes s^j \mathbf{t}^{\mathbf{r}}, B_i \otimes s^i \mathbf{t}^{\mathbf{m}}\big]) z_1^{-j} z_2^{-i} =$$

$$= \sum_{j,i\in\mathbb{Z}} \varphi([A_j, B_i] \otimes s^{j+i} \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_1^{-j} z_2^{-i}$$

$$+ \sum_{j,i\in\mathbb{Z}} (A_j|B_i) \varphi(s^{j+i} \mathbf{t}^{\mathbf{r}+\mathbf{m}} (\frac{j}{h} K_0 + \sum_{p=1}^{n} r_p K_p)) =$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k=j+i\in\mathbb{Z}} \varphi([A_j, B_{k-j}] \otimes s^k \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_1^{-j} z_2^{j-k}$$

$$+ \sum_{j_1=1}^{h} \sum_{\substack{j_2\in\mathbb{Z} \\ j=j_1+hj_2}} \sum_{\substack{i_2\in\mathbb{Z} \\ i=-j_1+hi_2}} (A_{j_1}|B_{-j_1}) \frac{j_1 + hj_2}{h} \varphi(s^{h(j_2+i_2)} \mathbf{t}^{\mathbf{r}+\mathbf{m}} K_0) z_1^{-j_1-hj_2} z_2^{j_1-hi_2}$$

$$+ \sum_{j_1=1}^{h} \sum_{\substack{j_2\in\mathbb{Z} \\ j=j_1+hj_2}} \sum_{\substack{i_2\in\mathbb{Z} \\ i=-j_1+hi_2}} (A_{j_1}|B_{-j_1}) \sum_{p=1}^{n} r_p \varphi(s^{h(j_2+i_2)} \mathbf{t}^{\mathbf{r}+\mathbf{m}} K_p) z_1^{-j_1-hj_2} z_2^{j_1-hi_2} =$$

$$= \sum_{j_1=1}^{h} \sum_{j_2\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \varphi(C_k^{j_1} \otimes s^k \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_2^{-k} \big(\frac{z_2}{z_1}\big)^{j_1} \big(\frac{z_2}{z_1}\big)^{hj_2}$$

$$+ \sum_{j_1=1}^{h} \sum_{j_2\in\mathbb{Z}} \sum_{k=j_2+i_2\in\mathbb{Z}} (A_{j_1}|B_{-j_1}) \frac{j_1 + hj_2}{h} \varphi(s^{hk} \mathbf{t}^{\mathbf{r}+\mathbf{m}} K_0) z_1^{-j_1-hj_2} z_2^{j_1+hj_2-hk}$$

$$+ \sum_{j_1=1}^{h} \sum_{j_2\in\mathbb{Z}} \sum_{k=j_2+i_2\in\mathbb{Z}} (A_{j_1}|B_{-j_1}) \sum_{p=1}^{n} r_p \varphi(s^{hk} \mathbf{t}^{\mathbf{r}+\mathbf{m}} K_p) z_1^{-j_1-hj_2} z_2^{j_1+hj_2-hk} =$$

$$= \sum_{j_1=1}^{h} C^{j_1}(z_2, \mathbf{r}+\mathbf{m}) \big(\frac{z_2}{z_1}\big)^{j_1} \delta\big(\big(\frac{z_2}{z_1}\big)^{h}\big)$$

$$+ \frac{1}{h} \sum_{j_1=1}^{h} j_1 (A_{j_1}|B_{-j_1}) K_0(z_2, \mathbf{r}+\mathbf{m}) \big(\frac{z_2}{z_1}\big)^{j_1} \delta\big(\big(\frac{z_2}{z_1}\big)^{h}\big)$$

$$+ \sum_{p=1}^{n} r_p \sum_{j_1=1}^{h} (A_{j_1}|B_{-j_1}) K_p(z_2, \mathbf{r}+\mathbf{m}) \big(\frac{z_2}{z_1}\big)^{j_1} \delta\big(\big(\frac{z_2}{z_1}\big)^{h}\big)$$

$$+ \sum_{j_1=1}^{h} (A_{j_1}|B_{-j_1}) K_0(z_2, \mathbf{r}+\mathbf{m}) \big(\frac{z_2}{z_1}\big)^{j_1} D\delta\big(\big(\frac{z_2}{z_1}\big)^{h}\big).$$

For the right hand side of (5.1) we use Corollary 3, Corollary 7 and the fact that

$K_0(z,\mathbf{r})K_0(z,\mathbf{m}) = K_0(z,\mathbf{r}+\mathbf{m})$:

$$\sum_{j,i\in\mathbb{Z}} \left[\varphi(A_j\otimes s^j\mathbf{t}^{\mathbf{r}}),\varphi(B_i\otimes s^i\mathbf{t}^{\mathbf{m}})\right] z_1^{-j}z_2^{-i} = [A(z_1,\mathbf{r}),B(z_2,\mathbf{m})] =$$

$$= [A(z_1)K_0(z_1,\mathbf{r}),B(z_2)K_0(z_2,\mathbf{m})] = [A(z_1),B(z_2)]\,K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) =$$

$$\begin{aligned}
&= \sum_{j=1}^{h} C^j(z_2)\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\\
&+ \frac{1}{h}\sum_{j=1}^{h} j(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\\
&+ \sum_{j=1}^{h}(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j D\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) =\\
&= \sum_{j=1}^{h} C^j(z_2)\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m})\\
&+ \frac{1}{h}\sum_{j=1}^{h} j(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m})\\
&+ \sum_{j=1}^{h}(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j D\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m})\\
&+ \sum_{j=1}^{h}(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j \delta(\big(\frac{z_2}{z_1}\big)^h)\frac{1}{h}D_{z_2}(K_0(z_2,\mathbf{r}))K_0(z_2,\mathbf{m}) =
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{h} C^j(z_2)\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r}+\mathbf{m})\\
&+ \frac{1}{h}\sum_{j=1}^{h} j(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r}+\mathbf{m})\\
&+ \sum_{j=1}^{h}(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j D\delta(\big(\frac{z_2}{z_1}\big)^h)K_0(z_2,\mathbf{r}+\mathbf{m})\\
&+ \sum_{j=1}^{h}(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j \delta(\big(\frac{z_2}{z_1}\big)^h)\sum_{p=1}^{n} r_pK_p(z_2)K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m}) =
\end{aligned}$$

$$= \sum_{j=1}^{h} C^j(z_2)\big(\frac{z_2}{z_1}\big)^j \delta\big(\big(\frac{z_2}{z_1}\big)^h\big) K_0(z_2, \mathbf{r}+\mathbf{m})$$

$$+ \frac{1}{h}\sum_{j=1}^{h} j(A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j \delta\big(\big(\frac{z_2}{z_1}\big)^h\big) K_0(z_2, \mathbf{r}+\mathbf{m})$$

$$+ \sum_{p=1}^{n} r_p \sum_{j=1}^{h} (A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j \delta\big(\big(\frac{z_2}{z_1}\big)^h\big) K_p(z_2, \mathbf{r}+\mathbf{m})$$

$$+ \sum_{j=1}^{h} (A_j|B_{-j})\big(\frac{z_2}{z_1}\big)^j D\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) K_0(z_2, \mathbf{r}+\mathbf{m}).$$

Thus (5.1) holds.

It is easy to see that operators $\varphi(s^{jh}t^{\mathbf{r}}K_p), p = 0, 1, \ldots, n$ commute with $\varphi\big(\sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K}\big)$. This follows from the obvious equalities:

$$[K_p(z_1, \mathbf{r}), A^{\alpha}(z_2, \mathbf{m})] = [K_p(z_1, \mathbf{r}), T_i(z_2, \mathbf{m})] = 0,$$

$$[K_a(z_1, \mathbf{r}), K_b(z_2, \mathbf{m})] = 0, \quad p, a, b = 0, 1, \ldots, n.$$

In order to check the action of $\mathcal{D}_+$ on $\sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$ it is sufficient to verify the following equality for the generating series:

$$\sum_{j,i\in\mathbb{Z}} \varphi\big(\big[s^{jh}\mathbf{t}^{\mathbf{r}}D_p, s^i\mathbf{t}^{\mathbf{m}}A_i^{\alpha}\big]\big) z_1^{-jh} z_2^{-i} = \sum_{j,i\in\mathbb{Z}} \big[\varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_p), \varphi(A_i^{\alpha}\otimes s^i\mathbf{t}^{\mathbf{m}})\big]\, z_1^{-jh} z_2^{-i}. \quad (5.2)$$

The corresponding equalities for $T_i \otimes s^{b_i}\mathbf{t}^{\mathbf{m}}$ and $s^{ih}\mathbf{t}^{\mathbf{m}}K_p$ will follow since $s^{jh}\mathbf{t}^{\mathbf{r}}D_p$ acts as a derivation and $s^i\mathbf{t}^{\mathbf{m}}A_i^{\alpha}$ generate $\sum_{j\in\mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K}$.

Let us prove that (5.2) holds.

$$\sum_{j,i\in\mathbb{Z}} \varphi\big(\big[s^{jh}\mathbf{t}^{\mathbf{r}}D_p, s^i\mathbf{t}^{\mathbf{m}}A_i^{\alpha}\big]\big) z_1^{-jh} z_2^{-i} = \sum_{j,i\in\mathbb{Z}} m_p\varphi(s^{i+jh}\mathbf{t}^{\mathbf{r}+\mathbf{m}}A_i^{\alpha}) z_1^{-jh} z_2^{-i} =$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k=i+jh\in\mathbb{Z}} m_p\varphi(s^k\mathbf{t}^{\mathbf{r}+\mathbf{m}}A_k^{\alpha}) z_1^{-jh} z_2^{-k+jh} = m_p A^{\alpha}(z_2, \mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big).$$

Next, we will show that the right hand side of (5.2) reduces to the same expression. In the following computations it will be convenient to denote $\varphi(s^{jh}D_p) = \frac{\partial}{\partial u_{pj}}$ by $\gamma_p(j)$, $\varphi(s^{-jh}D_p) = jv_{pj}$ by $\gamma_p(-j)$ and $\varphi(D_p) = q_p\frac{\partial}{\partial q_p}$ by $\gamma_p(0)$. Note that $[\gamma_p(j), K_0(z, \mathbf{m})] = m_p z^{jh} K_0(z, \mathbf{m})$ for $j \in \mathbb{Z}$.

$$\sum_{j,i\in\mathbb{Z}} \left[\varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_p), \varphi(A_i^\alpha\otimes s^i\mathbf{t}^{\mathbf{m}})\right] z_1^{-jh}z_2^{-i} = [D_p(z_1,\mathbf{r}), A^\alpha(z_2,\mathbf{m})] =$$

$$= \left[:\left\{\sum_{j\in\mathbb{Z}} \gamma_p(j)z_1^{-jh}\right\} K_0(z_1,\mathbf{r}): \quad , \quad A^\alpha(z_2)K_0(z_2,\mathbf{m})\right] =$$

$$= A^\alpha(z_2)\left[:\left\{\sum_{j\in\mathbb{Z}} \gamma_p(j)z_1^{-jh}\right\} K_0(z_1,\mathbf{r}): \quad , \quad K_0(z_2,\mathbf{m})\right] =$$

$$= A^\alpha(z_2)\left[\left\{\sum_{j<0} \gamma_p(j)z_1^{-jh}\right\} K_0(z_1,\mathbf{r}) \quad , \quad K_0(z_2,\mathbf{m})\right]$$

$$+A^\alpha(z_2)\left[K_0(z_1,\mathbf{r})\left\{\sum_{j\geq 0} \gamma_p(j)z_1^{-jh}\right\} \quad , \quad K_0(z_2,\mathbf{m})\right] =$$

$$= A^\alpha(z_2)\left[\sum_{j<0} \gamma_p(j)z_1^{-jh} \quad , \quad K_0(z_2,\mathbf{m})\right] K_0(z_1,\mathbf{r})$$

$$+A^\alpha(z_2)K_0(z_1,\mathbf{r})\left[\sum_{j\geq 0} \gamma_p(j)z_1^{-jh} \quad , \quad K_0(z_2,\mathbf{m})\right] =$$

$$= A^\alpha(z_2)\sum_{j<0} z_1^{-jh}z_2^{jh}m_pK_0(z_2,\mathbf{m})K_0(z_1,\mathbf{r})$$

$$+A^\alpha(z_2)K_0(z_1,\mathbf{r})\sum_{j\geq 0} z_1^{-jh}z_2^{jh}m_pK_0(z_2,\mathbf{m}) =$$

$$= m_pA^\alpha(z_2)K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) =$$

$$= m_pA^\alpha(z_2)K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) =$$

$$= m_pA^\alpha(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big).$$

From the above computation we can see that

$$[D_p(z_1,\mathbf{r}), K_0(z_2,\mathbf{m})] = m_pK_0(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big).$$

To complete the proof of Theorem 5 we need to compute the commutators $\left[\varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_a), \varphi(s^{ih}\mathbf{t}^{\mathbf{m}}D_b)\right]$. We will check the following equality for the generating series:

$$\sum_{j,i\in\mathbb{Z}} \varphi(\left[s^{jh}\mathbf{t}^{\mathbf{r}}D_a \quad , \quad s^{ih}\mathbf{t}^{\mathbf{m}}D_b\right])z_1^{-jh}z_2^{-ih} = \sum_{j,i\in\mathbb{Z}} \left[\varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_a) \quad , \quad \varphi(s^{ih}\mathbf{t}^{\mathbf{m}}D_b)\right] z_1^{-jh}z_2^{-ih}. \tag{5.3}$$

To compute the left hand side we use multiplication in $\mathfrak{g}$:

$$\sum_{j,i\in\mathbb{Z}} \varphi(\left[s^{jh}\mathbf{t}^{\mathbf{r}}D_a, s^{ih}\mathbf{t}^{\mathbf{m}}D_b\right])z_1^{-jh}z_2^{-ih} =$$

$$= \sum_{j,i\in\mathbb{Z}} \varphi(m_a s^{(j+i)h}\mathbf{t}^{\mathbf{r}+\mathbf{m}}D_b - r_b s^{(j+i)h}\mathbf{t}^{\mathbf{r}+\mathbf{m}}D_a)z_1^{-jh}z_2^{-ih}$$
$$- m_a r_b \varphi(s^{(j+i)h}\mathbf{t}^{\mathbf{r}+\mathbf{m}}\left\{jK_0 + \sum_{p=1}^{n} r_p K_p\right\})z_1^{-jh}z_2^{-ih} =$$
$$= \sum_{j\in\mathbb{Z}}\sum_{k=j+i\in\mathbb{Z}} \big(m_a\varphi(s^{kh}\mathbf{t}^{\mathbf{r}+\mathbf{m}}D_b) - r_b\varphi(s^{kh}\mathbf{t}^{\mathbf{r}+\mathbf{m}}D_a)\big)z_1^{-jh}z_2^{-kh+jh}$$
$$- \sum_{j\in\mathbb{Z}}\sum_{k=j+i\in\mathbb{Z}} m_a r_b \varphi(s^{kh}\mathbf{t}^{\mathbf{r}+\mathbf{m}}\left\{jK_0 + \sum_{p=1}^{n} r_p K_p\right\}))z_1^{-jh}z_2^{-kh+jh} =$$
$$= \big(m_a D_b(z_2, \mathbf{r}+\mathbf{m}) - r_b D_a(z_2, \mathbf{r}+\mathbf{m})\big)\delta((\frac{z_2}{z_1})^h)$$
$$- m_a r_b \sum_{p=1}^{n} r_p K_p(z_2, \mathbf{r}+\mathbf{m})\delta((\frac{z_2}{z_1})^h) - m_a r_b K_0(z_2, \mathbf{r}+\mathbf{m})D\delta((\frac{z_2}{z_1})^h).$$

Observe that

$$[D_a(z,\mathbf{r}) \quad , \quad \gamma_b(i)] = \left[:\left\{\sum_{j\in\mathbb{Z}} \gamma_a(j)z^{-jh}\right\} K_0(z,\mathbf{r}): \quad , \quad \gamma_b(i)\right] =$$

$$= \left[\sum_{j<0} \gamma_a(j)z^{-jh}K_0(z,\mathbf{r}) \quad , \quad \gamma_b(i)\right] + \left[K_0(z,\mathbf{r})\sum_{j\geq 0}\gamma_a(j)z^{-jh} \quad , \quad \gamma_b(i)\right] =$$
$$= \sum_{j<0}\gamma_a(j)z^{-jh}\left[K_0(z,\mathbf{r}) \quad , \quad \gamma_b(i)\right] + \left[K_0(z,\mathbf{r}) \quad , \quad \gamma_b(i)\right]\sum_{j\geq 0}\gamma_a(j)z^{-jh} =$$
$$= -r_b z^{ih}\sum_{j<0}\gamma_a(j)z^{-jh}K_0(z,\mathbf{r}) - r_b z^{ih}K_0(z,\mathbf{r})\sum_{j\geq 0}\gamma_a(j)z^{-jh} =$$
$$= -r_b z^{ih} D_a(z,\mathbf{r}).$$

We use the last equality to compute the right hand side of (5.3):

$$\sum_{j,i\in\mathbb{Z}} \left[\varphi(s^{jh}\mathbf{t}^{\mathbf{r}}D_a),\varphi(s^{ih}\mathbf{t}^{\mathbf{m}}D_b)\right] z_1^{-jh}z_2^{-ih} = [D_a(z_1,\mathbf{r}),D_b(z_2,\mathbf{m})] =$$

$$= \left[D_a(z_1,\mathbf{r}),\sum_{i<0}\gamma_b(i)z_2^{-ih}K_0(z_2,\mathbf{m})\right] + \left[D_a(z_1,\mathbf{r}),K_0(z_2,\mathbf{m})\sum_{i\geq 0}\gamma_b(i)z_2^{-ih}\right] =$$

$$= \sum_{i<0}\left[D_a(z_1,\mathbf{r}),\gamma_b(i)\right]z_2^{-ih}K_0(z_2,\mathbf{m}) + \sum_{i<0}\gamma_b(i)z_2^{-ih}\left[D_a(z_1,\mathbf{r}),K_0(z_2,\mathbf{m})\right]$$

$$+ \left[D_a(z_1,\mathbf{r}),K_0(z_2,\mathbf{m})\right]\sum_{i\geq 0}\gamma_b(i)z_2^{-ih} + K_0(z_2,\mathbf{m})\sum_{i\geq 0}\left[D_a(z_1,\mathbf{r}),\gamma_b(i)\right]z_2^{-ih} =$$

$$= -r_b\sum_{i<0}D_a(z_1,\mathbf{r})z_1^{ih}z_2^{-ih}K_0(z_2,\mathbf{m}) + \sum_{i<0}\gamma_b(i)z_2^{-ih}m_aK_0(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)$$

$$+ m_aK_0(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\sum_{i\geq 0}\gamma_b(i)z_2^{-ih} - r_bK_0(z_2,\mathbf{m})\sum_{i\geq 0}D_a(z_1,\mathbf{r})z_1^{ih}z_2^{-ih} =$$

$$\begin{aligned}
&= m_aD_b(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\\
&- r_b\sum_{i<0}\big(\frac{z_1}{z_2}\big)^{ih}\sum_{j<0}\gamma_a(j)z_1^{-jh}K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\\
&- r_b\sum_{i<0}\big(\frac{z_1}{z_2}\big)^{ih}K_0(z_1,\mathbf{r})\sum_{j\geq 0}\gamma_a(j)z_1^{-jh}K_0(z_2,\mathbf{m})\\
&- r_bK_0(z_2,\mathbf{m})\sum_{i\geq 0}\sum_{j<0}\gamma_a(j)z_1^{-jh}K_0(z_1,\mathbf{r})\big(\frac{z_1}{z_2}\big)^{ih}\\
&- r_bK_0(z_2,\mathbf{m})\sum_{i\geq 0}K_0(z_1,\mathbf{r})\big(\frac{z_1}{z_2}\big)^{ih}\sum_{j\geq 0}\gamma_a(j)z_1^{-jh} =\\
&= m_aD_b(z_2,\mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\\
&- r_b\sum_{i<0}\sum_{j<0}\big(\frac{z_1}{z_2}\big)^{ih}z_1^{-jh}\gamma_a(j)K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\\
&- r_b\sum_{i<0}\sum_{j\geq 0}\big(\frac{z_1}{z_2}\big)^{ih}z_1^{-jh}K_0(z_1,\mathbf{r})\left[\gamma_a(j),K_0(z_2,\mathbf{m})\right]\\
&- r_b\sum_{i<0}\sum_{j\geq 0}\big(\frac{z_1}{z_2}\big)^{ih}z_1^{-jh}K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})\gamma_a(j)\\
&- r_b\sum_{i\geq 0}\sum_{j<0}z_1^{-jh}\big(\frac{z_1}{z_2}\big)^{ih}\left[K_0(z_2,\mathbf{m}),\gamma_a(j)\right]K_0(z_1,\mathbf{r})\\
&- r_b\sum_{i\geq 0}\sum_{j<0}z_1^{-jh}\big(\frac{z_1}{z_2}\big)^{ih}\gamma_a(j)K_0(z_2,\mathbf{m})K_0(z_1,\mathbf{r})\\
&- r_b\sum_{i\geq 0}\sum_{j\geq 0}\big(\frac{z_1}{z_2}\big)^{ih}z_1^{-jh}K_0(z_2,\mathbf{m})K_0(z_1,\mathbf{r})\gamma_a(j) =
\end{aligned}$$

$$= m_a D_b(z_2, \mathbf{r}+\mathbf{m})\delta((\frac{z_2}{z_1})^h)$$

$$- r_b\delta((\frac{z_2}{z_1})^h) : \left\{ \sum_{j\in\mathbb{Z}} \gamma_a(j) z_2^{-jh} \right\} K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m}):$$

$$- m_a r_b \sum_{i>0}\sum_{j\geq 0} (\frac{z_2}{z_1})^{(i+j)h} K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})$$

$$+ m_a r_b \sum_{i\leq 0}\sum_{j<0} (\frac{z_2}{z_1})^{(i+j)h} K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) =$$

$$= m_a D_b(z_2,\mathbf{r}+\mathbf{m})\delta((\frac{z_2}{z_1})^h) - r_b D_a(z_2,\mathbf{r}+\mathbf{m})\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b \sum_{k=i+j>0} \sum_{j=0}^{k-1} (\frac{z_2}{z_1})^{kh} K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m})$$

$$+ m_a r_b \sum_{k=i+j<0} \sum_{j=k}^{-1} (\frac{z_2}{z_1})^{kh} K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) =$$

$$= \big(m_a D_b(z_2,\mathbf{r}+\mathbf{m}) - r_b D_a(z_2,\mathbf{r}+\mathbf{m})\big)\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b \sum_{k\in\mathbb{Z}} k(\frac{z_2}{z_1})^{kh} K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) =$$

$$= \big(m_a D_b(z_2,\mathbf{r}+\mathbf{m}) - r_b D_a(z_2,\mathbf{r}+\mathbf{m})\big)\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b K_0(z_1,\mathbf{r})K_0(z_2,\mathbf{m}) D\delta((\frac{z_2}{z_1})^h) =$$

$$= \big(m_a D_b(z_2,\mathbf{r}+\mathbf{m}) - r_b D_a(z_2,\mathbf{r}+\mathbf{m})\big)\delta((\frac{z_2}{z_1})^h)$$

$$- \frac{1}{h} m_a r_b D_{z_2}(K_0(z_2,\mathbf{r}))K_0(z_2,\mathbf{m})\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b K_0(z_2,\mathbf{r})K_0(z_2,\mathbf{m}) D\delta((\frac{z_2}{z_1})^h) =$$

$$= \big(m_a D_b(z_2,\mathbf{r}+\mathbf{m}) - r_b D_a(z_2,\mathbf{r}+\mathbf{m})\big)\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b \sum_{p=1}^{n} r_p K_p(z_2,\mathbf{r}+\mathbf{m})\delta((\frac{z_2}{z_1})^h)$$

$$- m_a r_b K_0(z_2,\mathbf{r}+\mathbf{m}) D\delta((\frac{z_2}{z_1})^h).$$

This establishes (5.3) and completes the proof of Theorem 5.

VI. SUGAWARA OPERATORS.

In the representation we constructed the operators from $\mathcal{D}_+$ act as derivations on the algebra $\hat{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K}$. It is possible to extend this action on $\mathcal{D}$, that is to represent $t_0^j \mathbf{t}^{\mathbf{r}} D_0$ by operators on F. For the affine case these are the Sugawara operators.

We will work with the toroidal algebra realized as $\sum\limits_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K}$. For this realization

$$\psi(t_0^j \mathbf{t}^{\mathbf{r}} D_0) = \frac{1}{h} s^{jh} \mathbf{t}^{\mathbf{r}} D_s - \frac{1}{h} \nu^{-1}(\rho) \otimes s^{jh} \mathbf{t}^{\mathbf{r}}, \tag{6.1}$$

where $\rho \in \dot{\mathfrak{h}}$ with $(\rho|\alpha_i) = 1$ for every simple root $\alpha_i \in \dot{\Delta}$ and $D_s = s\frac{d}{ds}$.

Since for $\alpha \in \Delta$ the elements $A^\alpha \otimes t_0^i \mathbf{t}^{\mathbf{m}}$ generate $\hat{\mathfrak{g}}$ then we have to check that

$$\psi(\left[t_0^j \mathbf{t}^{\mathbf{r}} D_0, A^\alpha \otimes t_0^i \mathbf{t}^{\mathbf{m}}\right]) = \left[\psi(t_0^j \mathbf{t}^{\mathbf{r}} D_0), \psi(A^\alpha \otimes t_0^i \mathbf{t}^{\mathbf{m}})\right].$$

Note that $A^\alpha \in \dot{\mathfrak{g}}_{\mathrm{ht}(\alpha)}$ and $(\rho|\alpha) = \mathrm{ht}(\alpha)$.

We have

$$\psi(\left[t_0^j \mathbf{t}^{\mathbf{r}} D_0, A^\alpha \otimes t_0^i \mathbf{t}^{\mathbf{m}}\right]) = i\psi(A^\alpha \otimes t_0^{i+j} \mathbf{t}^{\mathbf{r}+\mathbf{m}}) = iA^\alpha \otimes s^{\mathrm{ht}(\alpha)+h(i+j)} \mathbf{t}^{\mathbf{r}+\mathbf{m}},$$

while

$$\left[\psi(t_0^j \mathbf{t}^{\mathbf{r}} D_0), \psi(A^\alpha \otimes t_0^i \mathbf{t}^{\mathbf{m}})\right] =$$

$$= \left[\frac{1}{h} s^{jh} \mathbf{t}^{\mathbf{r}} D_s - \frac{1}{h} \nu^{-1}(\rho) \otimes s^{jh} \mathbf{t}^{\mathbf{r}}, A^\alpha \otimes s^{\mathrm{ht}(\alpha)+ih} \mathbf{t}^{\mathbf{m}}\right] =$$

$$= \frac{1}{h}(\mathrm{ht}(\alpha) + ih - (\rho|\alpha)) A^\alpha \otimes s^{\mathrm{ht}(\alpha)+h(i+j)} \mathbf{t}^{\mathbf{r}+\mathbf{m}} =$$

$$= iA^\alpha \otimes s^{\mathrm{ht}(\alpha)+h(i+j)} \mathbf{t}^{\mathbf{r}+\mathbf{m}}.$$

This establishes (6.1). Tt is then sufficient to construct operators on F corresponding to $s^{jh} \mathbf{t}^{\mathbf{r}} D_s$.

It will be convenient to denote in this section $\varphi(T_i \otimes s^{b_i})$ by $\tau(i)$, $\varphi(s^{jh} D_p)$ by $\gamma_p(j)$ and $\varphi(s^{jh} K_p)$ by $\kappa_p(j)$.

In these notations

$$K_0(z, \mathbf{r}) = \mathbf{q}^{\mathbf{r}} \exp\left(\sum_{p=1}^n r_p \sum_{j \geq 1} \frac{\kappa_p(-j)}{j} z^{jh}\right) \exp\left(-\sum_{p=1}^n r_p \sum_{j \geq 1} \frac{\kappa_p(j)}{j} z^{-jh}\right) \quad \text{and}$$

$$A^\alpha(z) = \Lambda_0(A_0^\alpha \otimes s^0) \exp\left(\sum_{i \geq 1} \lambda_i^\alpha \frac{\tau(1-i)}{b_i} z^{b_i}\right) \exp\left(-\sum_{i \geq 1} \lambda_{1-i}^\alpha \frac{\tau(i)}{b_i} z^{-b_i}\right).$$

Nontrivial commutators in the principal Heisenberg subalgebra are

$$[\tau(i), \tau(1-i)] = b_i \quad \text{and} \quad [\gamma_p(j), \kappa_p(-j)] = j.$$

The action of the Heisenberg subalgebra on the vertex operators is determined by

$$[\tau(i), A^\alpha(z)] = \lambda_i^\alpha z^{b_i} A^\alpha(z),$$

$$[\gamma_p(j), K_0(z, \mathbf{r})] = r_p z^{jh} K_0(z, \mathbf{r}).$$

Consider the following operators on the space F:

$$L_\tau(j) = \frac{1}{2} \sum_{i \in \mathbb{Z}} {:}\tau(1-i)\tau(i+j\ell){:} \quad ,$$

$$L_{\gamma\kappa}(j) = \sum_{p=1}^{n} \sum_{i \in \mathbb{Z}} {:}\gamma_p(-i)\kappa_p(i+j){:} \quad .$$

These are the analogues of the Sugawara operators.

Construct the formal generating series for these:

$$D_s(z) = -\sum_{j \in \mathbb{Z}} \big(L_\tau(j) + h L_{\gamma\kappa}(j) \big) z^{-jh}.$$

Proposition 8. *Formula (4.5) together with*

$$\sum_{i \in \mathbb{Z}} \varphi(s^{ih} \mathbf{t}^{\mathbf{r}} D_s) z^{-ih} = D_s(z, \mathbf{r}) = {:}D_s(z) K_0(z, \mathbf{r}){:}$$

defines the representation of the Lie algebra $\mathcal{D}$ on the space

$$\varphi(\hat{\mathfrak{g}}) = \varphi(\sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[t_1^\pm, \ldots, t_n^\pm] \oplus \mathcal{K}).$$

Proof. We have an action of $\mathcal{D}$ on $\hat{\mathfrak{g}}$ which is a unique extension of its natural action on $\tilde{\mathfrak{g}}$. Thus we need to prove that $\varphi([D, B]) = [\varphi(D), \varphi(B)]$ for every $D \in \mathcal{D}$ and $B \in \hat{\mathfrak{g}}$. For subalgebra $\mathcal{D}_+$ this was proved in the course of Theorem 4, hence we need to consider only $D = s^{ih} \mathbf{t}^{\mathbf{r}} D_s$. Since $s^{ih} \mathbf{t}^{\mathbf{r}} D_s$ act on $\hat{\mathfrak{g}}$ as derivations and $\hat{\mathfrak{g}}$ is generated as an algebra by elements $A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}}$ then it is sufficient to prove that

$$\varphi(\left[s^{ih} \mathbf{t}^{\mathbf{r}} D_s, A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}} \right]) = \left[\varphi(s^{ih} \mathbf{t}^{\mathbf{r}} D_s), \varphi(A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}}) \right].$$

Again we shall replace this with an equivalent identity for the corresponding series:

$$\varphi\big([\sum_{i\in\mathbb{Z}} s^{ih}\mathbf{t}^{\mathbf{r}} D_s z_1^{-ih}, \sum_{j\in\mathbb{Z}} A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}} z_2^{-j}]\big) = [\sum_{i\in\mathbb{Z}} \varphi(s^{ih}\mathbf{t}^{\mathbf{r}} D_s) z_1^{-ih}, \sum_{j\in\mathbb{Z}} \varphi(A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}}) z_2^{-j}]. \tag{6.2}$$

We have

$$\varphi\big([\sum_{i\in\mathbb{Z}} s^{ih}\mathbf{t}^{\mathbf{r}} D_s z_1^{-ih}, \sum_{j\in\mathbb{Z}} A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}} z_2^{-j}]\big) =$$

$$\begin{aligned}
&= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} j\varphi(A_j^\alpha \otimes s^{j+ih}\mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_1^{-ih} z_2^{-j} = \\
&= \sum_{i\in\mathbb{Z}}\sum_{k=j+ih\in\mathbb{Z}} (k-ih)\varphi(A_k^\alpha \otimes s^k \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_1^{-ih} z_2^{-k+ih} = \\
&= \sum_{k\in\mathbb{Z}} k\varphi(A_k^\alpha \otimes s^k \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_2^{-k} \delta\big(\big(\frac{z_2}{z_1}\big)^h\big) \\
&- h\sum_{k\in\mathbb{Z}} \varphi(A_k^\alpha \otimes s^k \mathbf{t}^{\mathbf{r}+\mathbf{m}}) z_2^{-k} D\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) = \\
&= -\big(D_{z_2} A^\alpha(z_2, \mathbf{r}+\mathbf{m})\big)\delta\big(\big(\frac{z_2}{z_1}\big)^h\big) \\
&- A^\alpha(z_2, \mathbf{r}+\mathbf{m})\big(D_{z_2}\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\big) = \\
&= -D_{z_2}\big(A^\alpha(z_2, \mathbf{r}+\mathbf{m})\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\big),
\end{aligned}$$

where $D_{z_2} = z_2 \frac{\partial}{\partial z_2}$.

In order to compute the right hand side of (6.2), we will need two lemmas.

Lemma 9.

(i) $$[L_\tau(j), A^\alpha(z)] = z^{jh}(D_z + jh)A^\alpha(z),$$

(ii) $$[L_\tau(z_1), A^\alpha(z_2)] = D_{z_2}\big(A^\alpha(z_2)\delta\big(\big(\frac{z_2}{z_1}\big)^h\big)\big).$$

Proof. We will prove (i) assuming that $j \geq 0$. The case $j < 0$ is analogous.

$$[L_\tau(j), A^\alpha(z)] = \frac{1}{2}\left[\sum_{i\in\mathbb{Z}} :\tau(1-i)\tau(i+j\ell):, A^\alpha(z)\right] =$$

$$= \frac{1}{2}\left[\sum_{i>0} \tau(1-i)\tau(i+j\ell), A^\alpha(z)\right] + \frac{1}{2}\left[\sum_{i\leq 0} \tau(i+j\ell)\tau(1-i), A^\alpha(z)\right] =$$

$$= \frac{1}{2}\sum_{i>0} \tau(1-i)\,[\tau(i+j\ell), A^\alpha(z)] + \frac{1}{2}\sum_{i>0} [\tau(1-i), A^\alpha(z)]\,\tau(i+j\ell)$$

$$+\frac{1}{2}\sum_{i\le 0} \tau(i+j\ell)\,[\tau(1-i), A^\alpha(z)] + \frac{1}{2}\sum_{i\le 0} [\tau(i+j\ell), A^\alpha(z)]\,\tau(1-i) =$$

$$= \frac{1}{2}\sum_{i>0} \tau(1-i)\lambda_i^\alpha z^{b_i+jh} A^\alpha(z) + \frac{1}{2}\sum_{i>0} \lambda_{1-i}^\alpha z^{-b_i} A^\alpha(z)\tau(i+j\ell)$$

$$+\frac{1}{2}\sum_{i\le 0} \tau(i+j\ell)\lambda_{1-i}^\alpha z^{-b_i} A^\alpha(z) + \frac{1}{2}\sum_{i\le 0} \lambda_i^\alpha z^{b_i+jh} A^\alpha(z)\tau(1-i) =$$

$$= \frac{1}{2}\sum_{i>0} \tau(1-i)\lambda_i^\alpha z^{b_i+jh} A^\alpha(z) + \frac{1}{2}\sum_{i>0} \lambda_{1-i}^\alpha z^{-b_i} A^\alpha(z)\tau(i+j\ell)$$

$$+\frac{1}{2}\sum_{i\le -j\ell} \tau(i+j\ell)\lambda_{1-i}^\alpha z^{-b_i} A^\alpha(z) + \frac{1}{2}\sum_{-j\ell<i\le 0} \lambda_{1-i}^\alpha z^{-b_i}\,[\tau(i+j\ell), A^\alpha(z)]$$

$$+\frac{1}{2}\sum_{-j\ell<i\le 0} \lambda_{1-i}^\alpha z^{-b_i} A^\alpha(z)\tau(i+j\ell) + \frac{1}{2}\sum_{i\le 0} \lambda_i^\alpha z^{b_i+jh} A^\alpha(z)\tau(1-i) =$$

$$= \frac{1}{2}\sum_{k=i>0} \tau(1-k)\lambda_k^\alpha z^{b_k+jh} A^\alpha(z) + \frac{1}{2}\sum_{k=i+j\ell>j\ell} \lambda_{1-k}^\alpha z^{-b_k+jh} A^\alpha(z)\tau(k)$$

$$+\frac{1}{2}\sum_{k=1-i-j\ell>0} \lambda_k^\alpha z^{b_k+jh}\tau(1-k) A^\alpha(z) + \frac{1}{2}\sum_{-j\ell<i\le 0} \lambda_{1-i}^\alpha z^{-b_i}\lambda_i^\alpha z^{b_i+jh} A^\alpha(z)$$

$$+\frac{1}{2}\sum_{0<k=i+j\ell\le j\ell} \lambda_{1-k}^\alpha z^{-b_k+jh} A^\alpha(z)\tau(k) + \frac{1}{2}\sum_{k=1-i>0} \lambda_{1-k}^\alpha z^{-b_k+jh} A^\alpha(z)\tau(k) =$$

$$= z^{jh} D_z A^\alpha(z) + \frac{j}{2} z^{jh} \big(\sum_{k=1}^{\ell} \lambda_k^\alpha \lambda_{1-k}^\alpha\big) A^\alpha(z) =$$

$$= z^{jh}(D_z + hj)A^\alpha(z).$$

At the last step we used the equality

$$\frac{1}{2}\sum_{k=1}^{\ell} \lambda_k^\alpha\lambda_{1-k}^\alpha = \frac{1}{2}\sum_{k=1}^{\ell} (T_k|\nu^{-1}(\alpha))(T_{1-k}|\nu^{-1}(\alpha)) =$$

$$\frac{h}{2}(\nu^{-1}(\alpha)|\nu^{-1}(\alpha)) = \frac{h}{2}(\alpha|\alpha) = h,$$

which follows from the fact that $\{\frac{1}{h}T_{1-k}\}$ is the dual basis for $\{T_k\}$.

Part (ii) is an immediate consequence of (i):

$$[L_\tau(z_1), A^\alpha(z_2)] = \sum_{j\in\mathbb{Z}} z_1^{-jh}\,[L_\tau(j), A^\alpha(z_2)] =$$

$$= \sum_{j\in\mathbb{Z}} z_1^{-jh} z_2^{jh}(D_{z_2} + hj)A^\alpha(z_2) = \big(D_{z_2}A^\alpha(z_2)\big)\delta((\frac{z_2}{z_1})^h) + A^\alpha(z_2)\big(D_{z_2}\delta((\frac{z_2}{z_1})^h)\big) =$$

$$= D_{z_2}\big(A^\alpha(z_2)\delta((\frac{z_2}{z_1})^h)\big).$$

Lemma 10.

$$[:L_{\gamma\kappa}(z_1)K_0(z_1,\mathbf{r}):, K_0(z_2,\mathbf{m})] = \frac{1}{h}\delta((\frac{z_2}{z_1})^h)D_{z_2}(K_0(z_2,\mathbf{m}))K_0(z_1,\mathbf{r}).$$

Proof.

$$[:L_{\gamma\kappa}(z_1)K_0(z_1,\mathbf{r}):, K_0(z_2,\mathbf{m})] =$$

$$= \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n}\sum_{i>0}[\gamma_p(-i)\kappa_p(i+j)K_0(z_1,\mathbf{r}), K_0(z_2,\mathbf{m})]$$

$$+ \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n}\sum_{i\le 0}[\kappa_p(i+j)K_0(z_1,\mathbf{r})\gamma_p(-i), K_0(z_2,\mathbf{m})] =$$

$$= \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n}\sum_{i>0}[\gamma_p(-i), K_0(z_2,\mathbf{m})]\,\kappa_p(i+j)K_0(z_1,\mathbf{r})$$

$$+ \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n}\sum_{i\le 0}\kappa_p(i+j)K_0(z_1,\mathbf{r})\,[\gamma_p(-i), K_0(z_2,\mathbf{m})] =$$

$$= \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n} m_p\sum_{i>0} z_2^{-ih}K_0(z_2,\mathbf{m})\kappa_p(i+j)K_0(z_1,\mathbf{r})$$

$$+ \sum_{j\in\mathbb{Z}} z_1^{-jh}\sum_{p=1}^{n} m_p\sum_{i\le 0}\kappa_p(i+j)K_0(z_1,\mathbf{r})z_2^{-ih}K_0(z_2,\mathbf{m}) =$$

$$= \sum_{j\in\mathbb{Z}}\ \sum_{k=i+j\in\mathbb{Z}}\ \sum_{p=1}^{n} m_p z_1^{-jh}z_2^{-kh+jh}\kappa_p(k)K_0(z_2,\mathbf{m})K_0(z_1,\mathbf{r}) =$$

$$= \frac{1}{h}\delta((\frac{z_2}{z_1})^h)D_{z_2}(K_0(z_2,\mathbf{m}))K_0(z_1,\mathbf{r}).$$

Using these two lemmas we can handle the right hand side of (6.2).

$$\left[\sum_{i\in\mathbb{Z}} \varphi(s^{ih}\mathbf{t}^{\mathbf{r}} D_s) z_1^{-ih} \quad , \quad \sum_{j\in\mathbb{Z}} \varphi(A_j^\alpha \otimes s^j \mathbf{t}^{\mathbf{m}}) z_2^{-j}\right] =$$

$$= [D_s(z_1, \mathbf{r}) \quad , \quad A^\alpha(z_2, \mathbf{m})] =$$

$$\begin{aligned}
&= [:D_s(z_1) K_0(z_1, \mathbf{r}):, A^\alpha(z_2) K_0(z_2, \mathbf{m})] = \\
&= -[L_\tau(z_1) K_0(z_1, \mathbf{r}), A^\alpha(z_2) K_0(z_2, \mathbf{m})] \\
&\quad - h\, [:L_{\gamma\kappa}(z_1) K_0(z_1, \mathbf{r}):, A^\alpha(z_2) K_0(z_2, \mathbf{m})] = \\
&\quad - [L_\tau(z_1), A^\alpha(z_2)]\, K_0(z_1, \mathbf{r}) K_0(z_2, \mathbf{m}) \\
&\quad - h A^\alpha(z_2)\, [:L_{\gamma\kappa}(z_1) K_0(z_1, \mathbf{r}):, K_0(z_2, \mathbf{m})] = \\
&= -D_{z_2}\big(A^\alpha(z_2) \delta((\frac{z_2}{z_1})^h)\big) K_0(z_2, \mathbf{m}) K_0(z_1, \mathbf{r}) \\
&\quad - A^\alpha(z_2) \delta((\frac{z_2}{z_1})^h) D_{z_2}(K_0(z_2, \mathbf{m})) K_0(z_1, \mathbf{r}) = \\
&= -D_{z_2}\big(A^\alpha(z_2) \delta((\frac{z_2}{z_1})^h) K_0(z_2, \mathbf{m}) K_0(z_1, \mathbf{r})\big) = \\
&= -D_{z_2}\big(A^\alpha(z_2) \delta((\frac{z_2}{z_1})^h) K_0(z_2, \mathbf{m}) K_0(z_2, \mathbf{r})\big) = \\
&= -D_{z_2}\big(A^\alpha(z_2, \mathbf{r} + \mathbf{m}) \delta((\frac{z_2}{z_1})^h)\big).
\end{aligned}$$

ACKNOWLEDGMENT.
This work is supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

[1] V.G. Kac, *Infinite-dimensional Lie algebras,* Cambridge University Press, 1990, 3rd ed.

[2] S. Eswara Rao, R.V. Moody, *Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra,* Comm. Math. Phys. 159, 239-264 (1994).

[3] J. Lepowsky, R.L. Wilson, *Construction of affine Lie algebra $A_1^{(1)}$,* Comm. Math. Phys. 62, 43-53 (1978).

[4] V.G. Kac, D.A. Kazhdan, J. Lepowsky, R.L. Wilson, *Realization of the basic representation of the Euclidean Lie algebras,* Adv. Math. 42, 83-112 (1981).

[5] S. Berman, B. Cox, *Enveloping algebras and representations of toroidal Lie algebras,* Pacific J. Math. 165, 239-267 (1994).

[6] Y. Billig, *An extension of the KdV hierarchy arising from a representation of a toroidal Lie algebra*, preprint, solv-int/9706008.

[7] T. Inami, H. Kanno, T. Ueno, C.-S. Xiong, *Two-toroidal Lie algebra as current algebra of four-dimensional Kähler WZW model,* Phys. Lett. B, 399, 97-104 (1997).

[8] T. Inami, H. Kanno, T. Ueno, *Higher-dimensional WZW model on Kähler manifold and toroidal Lie algebra,* Mod. Phys. Lett. A, 12, 2757-2764 (1997).

[9] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra,* J. Pure Appl. Algebra 34, 265-275 (1985).

[10] R.V. Moody, S. Eswara Rao, T. Yokonuma, *Toroidal Lie algebras and vertex representations,* Geom. Ded. 35, 283-307 (1990).

[11] G.M. Benkart, R.V. Moody, *Derivations, central extensions, and affine Lie algebras,* Algebras Groups Geom. 3, 456-492 (1986).

[12] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group,* Amer. J. Math. 81, 973-1032 (1959).

[13] H.S.M. Coxeter, *Regular polytopes,* Methuen, London, 1948.

[14] H.S.M. Coxeter, *Regular complex polytopes,* Cambridge University Press, 1974.

[15] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster,* Academic Press, Boston, 1989.