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SUMS OF $4k$ SQUARES: A POLYNOMIAL APPROACH

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Abstract

Let k and n be positive integers. Let $s_k(n)$ denote the number of representations of n as the sum of k squares. Ramanujan [17], [18, p. 159] gave without proof a formula for $s_k(n)$ when k is even. Mordell [15] used modular forms to give the first proof of Ramanujan's formula. In 2001 Cooper [6] used Ramanujan's ${}_1\psi_1$ summation formula and Jacobian elliptic functions and their derivatives to give a proof. It is our purpose to show that when k is a multiple of 4, Ramanujan's formula can be proved in an entirely elementary way using the properties of a certain class of polynomials. The values of $s_k(n)$ are determined explicitly for $k = 4,8,12,16,20,24,28,32,36,40,44$ and 48.

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1 Introduction

Let $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$, $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{Z} = \{0,\pm1,\pm2,\pm3,\ldots\}$. For $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $s_k(n)$ denote the number of representations of n as the sum of k squares, that is

$$s_k(n) = \text{card}\left\{(x_1,\ldots,x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \cdots + x_k^2\right\}, \tag{1.1}$$

so that $s_k(0) = 1$. Recently Cooper [6] has proved Ramanujan's formula for $s_k(n)$ $(n \in \mathbb{N})$ when $k \equiv 0 \pmod 2$. He makes use of Ramanujan's ${}_1\psi_1$ summation formula as well as derivatives of Jacobian elliptic functions. We show that in the case $k \equiv 0 \pmod 4$ the proof of the formula for $s_k(n)$ can be accomplished in a very elementary manner using only simple properties of polynomials. The starting point of our proof is a formula for the Eisenstein series $E_k(q)$ $(k \geq 2)$ (see (2.2)) in terms of $E_2(q)$ and $E_3(q)$ (see (2.3)) in a form stated by Ramanujan [17, eq. (26)], [18, p. 141]. An elementary proof of this formula has been given by Berndt [2].

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2 The Polynomials $e_{k,r}(x)$, $k(\in \mathbb{N}) \geq 2$, $r \in \{1,2,4\}$

For $a \in \mathbb{N}$ and $n \in \mathbb{N}$ we set

$$\sigma_a(n) := \sum_{d|n} d^a, \tag{2.1}$$

where d runs through the positive integers dividing n. If $n \notin \mathbb{N}$ we set $\sigma_a(n) = 0$. We also set $\sigma(n) := \sigma_1(n)$. Let $\mathbb{C}$ denote the field of complex numbers. The classical Eisenstein series $E_k(q)$ is defined by

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \ k \in \mathbb{N}, \ q \in \mathbb{C}, \ |q| < 1, \tag{2.2}$$

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330$, $B_{22} = 854513/138$, $B_{24} = -236364091/2730, \ldots$ and $B_{2k+1} = 0$ ($k \in \mathbb{N}$) are the Bernoulli numbers. Clearly $E_k(0) = 1$. Let $\mathbb{Q}$ denote the field of rational numbers. It is a classical result known to Weierstrass, and probably also to Eisenstein, that all E_k for $k \geq 2$ are in the ring $\mathbb{Q}[E_2, E_3]$, see for example [1, p. 12]. For $k \geq 2$ we have

$$E_k(q) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) E_2(q)^t E_3(q)^u \tag{2.3}$$

for some rational numbers $a_k(t,u)$ depending upon t, u and k but not on q. An elementary proof of (2.3) has been given by Berndt [2, pp. 96-97]. The $a_k(t,u)$ are uniquely determined by (2.3). Taking $q = 0$ in (2.3), we obtain

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) = 1, \ k \in \mathbb{N}, \ k \geq 2. \tag{2.4}$$

We can now define our polynomials $e_{k,r}(x) \in \mathbb{Q}[x]$ for $k \in \mathbb{N}$, $k \geq 2$ and $r \in \{1,2,4\}$. We set

$$e_{k,1}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u)(1+14x+x^2)^t (1-33x-33x^2+x^3)^u, \tag{2.5}$$

$$e_{k,2}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u)(1-x+x^2)^t \left(1-\frac{3}{2}x-\frac{3}{2}x^2+x^3\right)^u, \tag{2.6}$$

$$e_{k,4}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u)\left(1-x+\frac{1}{16}x^2\right)^t \left(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\right)^u. \tag{2.7}$$

Next we prove the four basic properties of the these polynomials that we shall need.

Proposition 2.1. *For $k \in \mathbb{N}$ and $k \geq 2$*

$$\begin{cases} e_{k,1}(x) = 1 + a_{k,1}x + \cdots + a_{k,k-1}x^{k-1} + x^k, \\ e_{k,2}(x) = 1 + b_{k,1}x + \cdots + b_{k,k-1}x^{k-1} + x^k, \\ e_{k,4}(x) = 1 + c_{k,1}x + \cdots + c_{k,k-1}x^{k-1} + \dfrac{1}{2^{2k}}x^k, \end{cases}$$

where $a_{k,1}, \ldots, a_{k,k-1}, b_{k,1}, \ldots, b_{k,k-1}, c_{k,1}, \ldots, c_{k,k-1} \in \mathbb{Q}$.

Proof. From (2.4)–(2.7) we deduce

$$e_{k,r}(0) = 1,\ k \geq 2,\ r \in \{1,2,4\}. \tag{2.8}$$

From (2.5)–(2.7), we see that the highest power of x in $e_{k,r}(x)$ $(k \geq 2,\ r \in \{1,2,4\})$ is at most $x^{2t+3u} = x^k$, and from (2.4)–(2.7) that the coefficient of x^k in each of $e_{k,1}(x)$ and $e_{k,2}(x)$ is

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) = 1$$

and the coefficient of x^k in $e_{k,4}(x)$ is

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u)\frac{1}{16^t}\frac{1}{64^u} = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} \frac{a_k(t,u)}{2^{4t+6u}} = \frac{1}{2^{2k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) = \frac{1}{2^{2k}}.$$

This completes the proof. □

Proposition 2.2. *Let $k \in \mathbb{N}$. If $k \equiv 0 \pmod 2$ then*

$$e_{k,2}(x) \in \mathbb{Q}[x(1-x)].$$

Proof. As $k \in \mathbb{N}$ and $k \equiv 0 \pmod 2$ we have $k = 2k_1$ with $k_1 \in \mathbb{N}$. By (2.6) we obtain

$$\begin{aligned} e_{k,2}(x) &= \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=2k_1}} a_k(t,u)(1-x+x^2)^t \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)^u \\ &= \sum_{\substack{(t,v) \in \mathbb{N}_0^2 \\ t+3v=k_1}} a(t,2v)(1-x+x^2)^t \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)^{2v} \\ &= \sum_{\substack{(t,v) \in \mathbb{N}_0^2 \\ t+3v=k_1}} a(t,2v)(1-x(1-x))^t \Big(1 - 3x(1-x) \\ &\qquad - \frac{15}{4}(x(1-x))^2 - (x(1-x))^3\Big)^v \in \mathbb{Q}[x(1-x)], \end{aligned}$$

completing the proof. □

Proposition 2.3. *Let $k \in \mathbb{N}$. Then for $k \geq 2$*

$$e_{k,1}(x) + (-1)^k 2^{2k} e_{k,4}(x) \in \mathbb{Q}[x(1-x)].$$

Proof. Let $a(y),\ b(y),\ c(y),\ d(y),\ e(y) \in \mathbb{Q}[y]$. It is clear by the binomial theorem that the terms involving odd powers of $\sqrt{e(y)}$ in

$$\Big(a(y)+b(y)\sqrt{e(y)}\Big)^t\Big(c(y)+d(y)\sqrt{e(y)}\Big)^u + \Big(a(y)-b(y)\sqrt{e(y)}\Big)^t\Big(c(y)-d(y)\sqrt{e(y)}\Big)^u \tag{2.9}$$

contribute 0 for all $t,u \in \mathbb{N}_0$, and so the quantity (2.9) belongs to $\mathbb{Q}[y]$. Taking

$$a(y) = \frac{17}{2} - y,\ b(y) = \frac{15}{2},\ c(y) = -\frac{63}{2} + \frac{63}{2}y,$$
$$d(y) = -\Big(\frac{65}{2} + \frac{1}{2}y\Big),\ e(y) = 1-4y,$$

we deduce that

$$\Big(\frac{17}{2} - y + \frac{15}{2}\sqrt{1-4y}\Big)^t\Big(-\frac{63}{2} + \frac{63}{2}y - \Big(\frac{65}{2} + \frac{1}{2}y\Big)\sqrt{1-4y}\Big)^u + \Big(\frac{17}{2} - y - \frac{15}{2}\sqrt{1-4y}\Big)^t\Big(-\frac{63}{2} + \frac{63}{2}y + \Big(\frac{65}{2} + \frac{1}{2}y\Big)\sqrt{1-4y}\Big)^u \in \mathbb{Q}[y]$$

for all $t,u \in \mathbb{N}_0$. Choose $y = x(1-x)$ so that

$$\sqrt{1-4y} = 1-2x,$$
$$\frac{17}{2} - y + \frac{15}{2}\sqrt{1-4y} = 16-16x+x^2,$$
$$\frac{17}{2} - y - \frac{15}{2}\sqrt{1-4y} = 1+14x+x^2,$$
$$-\frac{63}{2} + \frac{63}{2}y - \Big(\frac{65}{2} + \frac{1}{2}y\Big)\sqrt{1-4y} = -64+96x-30x^2-x^3,$$
$$-\frac{63}{2} + \frac{63}{2}y + \Big(\frac{65}{2} + \frac{1}{2}y\Big)\sqrt{1-4y} = 1-33x-33x^2+x^3.$$

Hence

$$(16-16x+x^2)^t(-64+96x-30x^2-x^3)^u + (1+14x+x^2)^t(1-33x-33x^2+x^3)^u \in \mathbb{Q}[x(1-x)]$$

for all $t,u \in \mathbb{N}_0$. Thus

$$(1+14x+x^2)^t(1-33x-33x^2+x^3)^u + (-1)^u 2^{4t+6u}\Big(1-x+\frac{1}{16}x^2\Big)^t\Big(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\Big)^u \in \mathbb{Q}[x(1-x)]$$

for all $t,u \in \mathbb{N}_0$. Finally, by (2.5) and (2.7), we have

$$e_{k,1}(x) + (-1)^k 2^{2k} e_{k,4}(x) = \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u)(1+14x+x^2)^t(1-33x-33x^2+x^3)^u$$

$$
\begin{aligned}
&+(-1)^k 2^{2k} \sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} a_k(t,u)\Big(1-x+\frac{1}{16}x^2\Big)^t\Big(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\Big)^u\\
&= \sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} a_k(t,u)\Big((1+14x+x^2)^t(1-33x-33x^2+x^3)^u\\
&\qquad +(-1)^u 2^{4t+6u}\Big(1-x+\frac{1}{16}x^2\Big)^t\Big(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\Big)^u\Big)\\
&\in \mathbb{Q}[x(1-x)],
\end{aligned}
$$

as asserted. □

Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \;\; q\in\mathbb{C}, \;\; |q|<1, \tag{2.10}$$

see [2, eq. (1.2.2), p. 6]. Following [2, pp. 119, 120] we set

$$x := 1-\frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z := \varphi^2(q). \tag{2.11}$$

Proposition 2.4. *Define x and z as in* (2.11). *For $k\in\mathbb{N}$, $k\geq 2$ and $r\in\{1,2,4\}$, we have*

$$E_k(q^r) = e_{k,r}(x)z^{2k}.$$

Proof. From [2, pp. 126, 127] or [5, p. 44], we have

$$E_2(q) = (1+14x+x^2)z^4, \tag{2.12}$$
$$E_3(q) = (1-33x-33x^2+x^3)z^6. \tag{2.13}$$

Hence, by (2.3) and (2.5), we have

$$
\begin{aligned}
E_k(q) &= \sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} a_k(t,u)E_2(q)^tE_3(q)^u\\
&= \sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} a_k(t,u)(1+14x+x^2)^t z^{4t}(1-33x-33x^2+x^3)^u z^{6u}\\
&= e_{k,1}(x)z^{2k}.
\end{aligned}
$$

This completes the proof of Proposition 2.4 when $r=1$. The result for $r=2$ follows similarly using

$$E_2(q^2) = (1-x+x^2)z^4, \tag{2.14}$$
$$E_3(q^2) = \Big(1-\frac{3}{2}x-\frac{3}{2}x^2+x^3\Big)z^6, \tag{2.15}$$

see [2, pp. 126, 127] or [5, p. 44], and that for $r=4$ from

$$E_2(q^4)=\Big(1-x+\frac{1}{16}x^2\Big)z^4, \tag{2.16}$$

$$E_3(q^4)=\Big(1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3\Big)z^6, \tag{2.17}$$

see [2, p. 128] or [5, p. 44], completing the proof. □

Using (2.2) in (2.3) and the values of the Bernoulli numbers given at the beginning of this section, we obtain the values of the $a_k(t,u)$ $(t,u\in\mathbb{N}_0,\ 2t+3u=k)$ for $k=2,3,\ldots,12$:

$$\begin{aligned}
&a_2(1,0)=1,\\
&a_3(0,1)=1,\\
&a_4(2,0)=1,\\
&a_5(1,1)=1,\\
&a_6(0,2)=\frac{250}{691},\ a_6(3,0)=\frac{441}{691},\\
&a_7(2,1)=1,\\
&a_8(1,2)=\frac{2000}{3617},\ a_8(4,0)=\frac{1617}{3617},\\
&a_9(0,3)=\frac{5500}{43867},\ a_9(3,1)=\frac{38367}{43867},\\
&a_{10}(2,2)=\frac{121250}{174611},\ a_{10}(5,0)=\frac{53361}{174611},\\
&a_{11}(1,3)=\frac{20500}{77683},\ a_{11}(4,1)=\frac{57183}{77683},\\
&a_{12}(0,4)=\frac{10285000}{236364091},\ a_{12}(3,2)=\frac{176400000}{236364091},\ a_{12}(6,0)=\frac{49679091}{236364091}.
\end{aligned}$$

These values were given by Ramanujan in [17, Table 1]. (Amazingly Ramanujan also calculated $a_{13}(2,3)$, $a_{13}(5,1)$, $a_{14}(1,4)$, $a_{14}(4,2)$, $a_{14}(7,0)$, $a_{15}(0,5)$, $a_{15}(3,3)$ and $a_{15}(6,1)$.) Using these values in (2.5)-(2.7), we obtain the polynomials $e_{2,r}(x)$, $e_{3,r}(x),\ldots,$ $e_{12,r}(x)$ for $r\in\{1,2,4\}$, see TABLE.

3 Ramanujan's Discriminant Function

Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n,\ q\in\mathbb{C},\ |q|<1, \tag{3.1}$$

where $\tau(n)$ is Ramanujan's tau function [17, eq. (92)], [18, p. 151]. It is known that

$$\Delta(q)=\frac{x(1-x)^4z^{12}}{2^4}, \tag{3.2}$$

$$\Delta(q^2)=\frac{x^2(1-x)^2z^{12}}{2^8}, \tag{3.3}$$

$$\Delta(q^4) = \frac{x^4(1-x)z^{12}}{2^{16}}, \tag{3.4}$$

see [5, p. 44]. For $a,b,c \in \mathbb{Q}$ we define

$$\Delta(a,b,c;q) := \Delta(q)^a \Delta(q^2)^b \Delta(q^4)^c \tag{3.5}$$

so that by (3.1) and (3.5) we have

$$\Delta(a,b,c;q) \quad = \quad q^{a+2b+4c} \prod_{n=1}^{\infty} (1-q^n)^{24a}(1-q^{2n})^{24b}(1-q^{4n})^{24c} \tag{3.6}$$

and by (3.1)-(3.4)

$$\Delta(a,b,c;q) \quad = \quad \frac{x^{a+2b+4c}(1-x)^{4a+2b+c}z^{12(a+b+c)}}{2^{4(a+2b+4c)}}. \tag{3.7}$$

We are interested in those $a,b,c \in \mathbb{Q}$ such that

$$a+2b+4c \in \mathbb{N}_0, \;\; 4a+2b+c \in \mathbb{N}_0, \;\; 12(a+b+c) \in 2\mathbb{N}. \tag{3.8}$$

Hence

$$\begin{aligned} &a+2b+4c = r, \; r \in \mathbb{N}_0, \\ &4a+2b+c = s, \; s \in \mathbb{N}_0, \\ &6a+6b+6c = t, \; t \in \mathbb{N}, \end{aligned} \tag{3.9}$$

so that

$$a = \frac{1}{3}(r+2s-t), \;\; b = \frac{1}{6}(-6r-6s+5t), \;\; c = \frac{1}{3}(2r+s-t). \tag{3.10}$$

Thus, by (3.7) and (3.9), we have

$$\Delta(a,b,c;q) \quad = \frac{x^r(1-x)^s z^{2t}}{2^{4r}} \tag{3.11}$$

and by (3.6) and (3.10)

$$\Delta(a,b,c;q) = q^r \prod_{n=1}^{\infty} (1-q^n)^{8(r+2s-t)}(1-q^{2n})^{4(-6r-6s+5t)}(1-q^{4n})^{8(2r+s-t)}. \tag{3.12}$$

As $r \in \mathbb{N}_0$ and $8(r+2s-t),\, 4(-6r-6s+5t),\, 8(2r+s-t) \in \mathbb{Z}$ we can define $\delta(a,b,c;n) \in \mathbb{Z}$ for $n \in \mathbb{N}_0$ by

$$\Delta(a,b,c;q) = \sum_{n=0}^{\infty} \delta(a,b,c;n) q^n. \tag{3.13}$$

Clearly

$$\delta(a,b,c;n) = 0 \quad \text{for} \quad n = 0,1,\ldots,r-1 \tag{3.14}$$

and

$$\delta(a,b,c;n) = 1 \text{ for } n = r. \tag{3.15}$$

Further for $n \in \mathbb{N}$ we have

$$\delta(1,0,0;n) = \tau(n), \tag{3.16}$$
$$\delta(0,1,0;n) = \tau(n/2), \tag{3.17}$$
$$\delta(0,0,1;n) = \tau(n/4). \tag{3.18}$$

An easy calculation shows that

$$\Delta(-q) = -\Delta(q)^{-1}\Delta(q^2)^3\Delta(q^4)^{-1}. \tag{3.19}$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \delta(-1,3,-1;n)(-q)^n &= \Delta(-1,3,-1;-q)\\
&= \Delta(-q)^{-1}\Delta((-q)^2)^3\Delta((-q)^4)^{-1}\\
&= \Delta(-q)^{-1}\Delta(q^2)^3\Delta(q^4)^{-1}\\
&= \Big(-\Delta(q)^{-1}\Delta(q^2)^3\Delta(q^4)^{-1}\Big)^{-1}\Delta(q^2)^3\Delta(q^4)^{-1}\\
&= -\Delta(q)\Delta(q^2)^{-3}\Delta(q^4)\Delta(q^2)^3\Delta(q^4)^{-1}\\
&= -\Delta(q)\\
&= -\sum_{n=1}^{\infty} \tau(n)q^n
\end{aligned}$$

so

$$\delta(-1,3,-1;n) = \begin{cases} 0, & \text{if } n = 0,\\ (-1)^{n-1}\tau(n), & \text{if } n \geq 1. \end{cases} \tag{3.20}$$

Now let $b, c \in \mathbb{Q}$ satisfy (3.8) with $a = 0$, so that in particular $2b + 4c \in \mathbb{N}_0$, $24b \in \mathbb{Z}$ and $24c \in \mathbb{Z}$. Then, by (3.13) and (3.6), we have

$$\sum_{n=0}^{\infty} \delta(0,b,c;n)q^n = q^{2b+4c}\prod_{n=1}^{\infty}(1-q^{2n})^{24b}(1-q^{4n})^{24c}.$$

Hence if $2b + 4c \equiv 0 \pmod 2$ we have

$$\delta(0,b,c;n) = 0 \text{ for } n \equiv 1 \pmod 2, \tag{3.21}$$

and if $2b + 4c \equiv 1 \pmod 2$ we have

$$\delta(0,b,c;n) = 0 \text{ for } n \equiv 0 \pmod 2. \tag{3.22}$$

4 Sums of 4k Squares.

For $k \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} s_{4k}(n) q^n = \varphi(q)^{4k}. \tag{4.1}$$

From (2.11) we have

$$\varphi(q) = z^{1/2}. \tag{4.2}$$

Hence

$$\sum_{n=0}^{\infty} s_{4k}(n) q^n = z^{2k}. \tag{4.3}$$

First we treat the case $k = 1$. From [2, pp. 125, 128] or [5, p. 44], we have

$$E_1(q) - 4E_1(q^4) = -3z^2$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty} s_4(n) q^n &= z^2 \\
&= -\frac{1}{3} E_1(q) + \frac{4}{3} E_1(q^4) \\
&= -\frac{1}{3}\left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n\right) + \frac{4}{3}\left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{4n}\right) \\
&= 1 + 8 \sum_{n=1}^{\infty} \sigma(n) q^n - 32 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{4}\right) q^n.
\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$s_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right). \tag{4.4}$$

Now we turn to the case $k > 1$. We define

$$\begin{aligned}
f_k(x) := 1 - \frac{(-1)^k}{2^{2k}-1}\left(e_{k,1}(x) + (-1)^k 2^{2k} e_{k,4}(x)\right) \\
+ \frac{(1+(-1)^k)}{2^{2k}-1} e_{k,2}(x), \;\; k > 1.
\end{aligned} \tag{4.5}$$

From Proposition 2.1 and (4.5) we deduce

$$f_k(x) = d_{k,1}x + \cdots + d_{k,k-1}x^{k-1}, \; k > 1, \tag{4.6}$$

for some $d_{k,1}, \ldots, d_{k,k-1} \in \mathbb{Q}$. Hence

$$\deg f_k(x) \le k-1, \; k > 1. \tag{4.7}$$

Next, by (4.5) and Propositions 2.2 and 2.3, we see that

$$f_k(x) \in \mathbb{Q}[x(1-x)], \ k>1. \tag{4.8}$$

Thus

$$\deg f_k(x) \equiv 0 \ (\mathrm{mod}\ 2), \ k>1. \tag{4.9}$$

From (4.7) and (4.9) we deduce

$$\begin{cases} \deg f_k(x) \leq k-2, \ k>1, \ k\equiv 0 \ (\mathrm{mod}\ 2), \\ \deg f_k(x) \leq k-1, \ k>1, \ k\equiv 1 \ (\mathrm{mod}\ 2). \end{cases} \tag{4.10}$$

For a real number u let $[u]$ denote the greatest integer less than or equal to u. Then, by (4.6), (4.8) and (4.10), there exist $c(k,m) \in \mathbb{Q}$ such that

$$f_k(x) = \sum_{m=1}^{[\frac{k-1}{2}]} \frac{c(k,m)}{2^{4m}} (x(1-x))^m, \ k>1. \tag{4.11}$$

The $c(k,m)$ are uniquely determined by (4.11). Then, appealing to (2.2), Proposition 2.4, (4.3), (4.5), (4.11), (3.11), (3.13) and (3.14), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(s_{4k}(n)+\frac{(-1)^k 4k}{(2^{2k}-1)B_{2k}}\sigma_{2k-1}(n)-\frac{(1+(-1)^k)4k}{(2^{2k}-1)B_{2k}}\sigma_{2k-1}\Big(\frac{n}{2}\Big)\\
&\qquad+\frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}}\sigma_{2k-1}\Big(\frac{n}{4}\Big)\Big)q^n\\
&=\sum_{n=1}^{\infty}s_{4k}(n)q^n+\frac{(-1)^k}{2^{2k}-1}\Big(\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\Big)\\
&\qquad-\frac{(1+(-1)^k)}{2^{2k}-1}\Big(\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n}\Big)+\frac{2^{2k}}{2^{2k}-1}\Big(\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{4n}\Big)\\
&=\sum_{n=1}^{\infty}s_{4k}(n)q^n+\frac{(-1)^k}{2^{2k}-1}(1-E_k(q))-\frac{(1+(-1)^k)}{2^{2k}-1}(1-E_k(q^2))\\
&\qquad+\frac{2^{2k}}{2^{2k}-1}(1-E_k(q^4))\\
&=\sum_{n=1}^{\infty}s_{4k}(n)q^n+1-\frac{(-1)^k}{2^{2k}-1}E_k(q)+\frac{(1+(-1)^k)}{2^{2k}-1}E_k(q^2)-\frac{2^{2k}}{2^{2k}-1}E_k(q^4)\\
&=\sum_{n=0}^{\infty}s_{4k}(n)q^n-\frac{(-1)^k}{2^{2k}-1}E_k(q)+\frac{(1+(-1)^k)}{2^{2k}-1}E_k(q^2)-\frac{2^{2k}}{2^{2k}-1}E_k(q^4)\\
&=z^{2k}-\frac{(-1)^k}{2^{2k}-1}e_{k,1}(x)z^{2k}+\frac{(1+(-1)^k)}{2^{2k}-1}e_{k,2}(x)z^{2k}-\frac{2^{2k}}{2^{2k}-1}e_{k,4}(x)z^{2k}\\
&=\Big(1-\frac{(-1)^k}{2^{2k}-1}(e_{k,1}(x)+(-1)^k 2^{2k}e_{k,4}(x))+\frac{(1+(-1)^k)}{2^{2k}-1}e_{k,2}(x)\Big)z^{2k}\\
&=f_k(x)z^{2k}
\end{aligned}$$

$$= \sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \frac{x^m(1-x)^m}{2^{4m}} z^{2k}$$

$$= \sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \Delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; q\Big)$$

$$= \sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \sum_{n=0}^{\infty} \delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\Big) q^n$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\Big)\Big) q^n$$

$$= \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\Big)\Big) q^n.$$

We have shown that

$$\sum_{n=1}^{\infty} \Big(s_{4k}(n) + \frac{(-1)^k 4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n) - \frac{(1+(-1)^k)4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\Big(\frac{n}{2}\Big) + \frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\Big(\frac{n}{4}\Big)\Big) q^n$$
$$= \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\Big)\Big) q^n.$$

This result is equivalent to eqs. (146) and (147) (in the case that s is even) in Ramanujan [17], to Theorem 3.4 in Cooper [6], and to eq. (3) in Mordell [15]. Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the following result. (The case $k=1$ is included by (4.4) and the fact that the sum in the theorem is the empty sum $=0$ when $k=1$.)

Theorem 4.1. *Let $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$*

$$s_{4k}(n) = -\frac{(-1)^k 4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n) + \frac{(1+(-1)^k)4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\Big(\frac{n}{2}\Big) - \frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\Big(\frac{n}{4}\Big) + \sum_{m=1}^{[\frac{k-1}{2}]} c(k,m) \delta\Big(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\Big),$$

where the $c(k,m)$ are implicitly determined by (4.11) *and the* $\delta\left(m-\frac{k}{3}, \frac{5}{6}k-2m, m-\frac{k}{3}; n\right)$ *by* (3.13).

Appealing to the TABLE, (4.5) and (4.11), we obtain the values of $c(k,m)$ for $k = 2,\ldots,12$ and $m = 1,2,\ldots,[\frac{k-1}{2}]$. Putting these values into Theorem 4.1 we obtain the following theorem.

Theorem 4.2. *For* $n \in \mathbb{N}$

$$\text{(i)} \quad s_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right),$$

$$\text{(ii)} \quad s_8(n) = 16\sigma_3(n) - 32\sigma_3\left(\frac{n}{2}\right) + 256\sigma_3\left(\frac{n}{4}\right),$$

$$\text{(iii)} \quad s_{12}(n) = 8\sigma_5(n) - 512\sigma_5\left(\frac{n}{4}\right) + 16\,\delta(0,1/2,0;n),$$

$$\text{(iv)} \quad s_{16}(n) = \frac{32}{17}\sigma_7(n) - \frac{64}{17}\sigma_7\left(\frac{n}{2}\right) + \frac{8192}{17}\sigma_7\left(\frac{n}{4}\right) + \frac{512}{17}\,\delta(-1/3,4/3,-1/3;n),$$

$$\text{(v)} \quad s_{20}(n) = \frac{8}{31}\sigma_9(n) - \frac{8192}{31}\sigma_9\left(\frac{n}{4}\right) + \frac{1232}{31}\,\delta(-2/3,13/6,-2/3;n) - \frac{256}{31}\,\delta(1/3,1/6,1/3;n),$$

$$\text{(vi)} \quad s_{24}(n) = \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}\left(\frac{n}{2}\right) + \frac{65536}{691}\sigma_{11}\left(\frac{n}{4}\right) + \frac{33152}{691}\,\delta(-1,3,-1;n) - \frac{65536}{691}\,\delta(0,1,0;n),$$

$$\text{(vii)} \quad s_{28}(n) = \frac{8}{5461}\sigma_{13}(n) - \frac{131072}{5461}\sigma_{13}\left(\frac{n}{4}\right) + \frac{305808}{5461}\,\delta(-4/3,23/6,-4/3;n) - \frac{1594368}{5461}\,\delta(-1/3,11/6,-1/3;n) + \frac{4096}{5461}\,\delta(2/3,-1/6,2/3,n),$$

$$\text{(viii)} \quad s_{32}(n) = \frac{64}{929569}\sigma_{15}(n) - \frac{128}{929569}\sigma_{15}\left(\frac{n}{2}\right) + \frac{4194304}{929569}\sigma_{15}\left(\frac{n}{4}\right) + \frac{59492352}{929569}\,\delta(-5/3,14/3,-5/3;n) - \frac{537526272}{929569}\,\delta(-2/3,8/3,-2/3;n) + \frac{67108864}{929569}\,\delta(1/3,2/3,1/3;n),$$

$$\text{(ix)} \quad s_{36}(n) = \frac{8}{3202291}\sigma_{17}(n) - \frac{2097152}{3202291}\sigma_{17}\left(\frac{n}{4}\right) + \frac{230564944}{3202291}\,\delta(-2,11/2,-2;n) - \frac{2998392576}{3202291}\,\delta(-1,7/2,-1;n) + \frac{2066243584}{3202291}\,\delta(0,3/2,0;n)$$

$$-\frac{65536}{3202291}\,\delta(1,-1/2,1;n),$$

(x) $$s_{40}(n) = \frac{16}{221930581}\sigma_{19}(n) - \frac{32}{221930581}\sigma_{19}\left(\frac{n}{2}\right) + \frac{16777216}{221930581}\sigma_{19}\left(\frac{n}{4}\right)$$
$$+\frac{17754446464}{221930581}\,\delta(-7/3,19/3,-7/3;n)$$
$$-\frac{301833977856}{221930581}\,\delta(-4/3,13/3,-4/3;n)$$
$$+\frac{549814534144}{221930581}\,\delta(-1/3,7/3,-1/3;n)$$
$$-\frac{4294967296}{221930581}\,\delta(2/3,1/3,2/3;n),$$

(xi) $$s_{44}(n) = \frac{8}{4722116521}\sigma_{21}(n) - \frac{33554432}{4722116521}\sigma_{21}\left(\frac{n}{4}\right)$$
$$+\frac{415546253840}{4722116521}\,\delta(-8/3,43/6,-8/3;n)$$
$$-\frac{8726488107520}{4722116521}\,\delta(-5/3,31/6,-5/3;n)$$
$$+\frac{29836318535680}{4722116521}\,\delta(-2/3,19/6,-2/3;n)$$
$$-\frac{2677850439680}{4722116521}\,\delta(1/3,7/6,1/3;n)$$
$$+\frac{1048576}{4722116521}\,\delta(4/3,-5/6,4/3;n),$$

(xii) $$s_{48}(n) = \frac{32}{968383680827}\sigma_{23}(n) - \frac{64}{968383680827}\sigma_{23}\left(\frac{n}{2}\right)$$
$$+\frac{536870912}{968383680827}\sigma_{23}\left(\frac{n}{4}\right)$$
$$+\frac{92964833359360}{968383680827}\,\delta(-3,8,-3;n)$$
$$-\frac{2324121102417920}{968383680827}\,\delta(-2,6,-2;n)$$
$$+\frac{12392311417733120}{968383680827}\,\delta(-1,4,-1;n)$$
$$-\frac{4503638282076160}{968383680827}\,\delta(0,2,0;n)$$
$$+\frac{2199023255552}{968383680827}\,\delta(1,0,1;n).$$

The results in Theorem 4.2 for $s_4(n),\ldots,s_{24}(n)$ were essentially given by Ramanujan

[17, Table VI] and those for $s_4(n),\ldots,s_{48}(n)$ were summarized by Cooper [6, Table 3].

We just give the details of the proof for $s_{20}(n)$. Here $4k=20$ so $k=5$. From the TABLE we have

$$e_{5,1}(x)=1-19x-494x^2-494x^3-19x^4+x^5$$

and

$$e_{5,4}(x)=1-\frac{5}{2}x+\frac{65}{32}x^2-\frac{35}{64}x^3+\frac{7}{512}x^4+\frac{1}{1024}x^5.$$

Hence, by (4.5) and (4.11), $c(5,1)$ and $c(5,2)$ are determined by

$$\begin{aligned}
c(5,1)\frac{x(1-x)}{16}&+c(5,2)\frac{x^2(1-x)^2}{256}\\
&=1+\frac{1}{1023}(1-19x-494x^2-494x^3-19x^4+x^5)\\
&\quad-\frac{1024}{1023}\left(1-\frac{5}{2}x+\frac{65}{32}x^2-\frac{35}{64}x^3+\frac{7}{512}x^4+\frac{1}{1024}x^5\right)\\
&=\frac{77}{31}x-\frac{78}{31}x^2+\frac{2}{31}x^3-\frac{1}{31}x^4\\
&=\frac{77}{31}x(1-x)-\frac{1}{31}x^2(1-x)^2
\end{aligned}$$

so that

$$c(5,1)=\frac{1232}{31},\ c(5,2)=-\frac{256}{31}.$$

Hence Theorem 4.1 gives

$$\begin{aligned}
s_{20}(n)=\ &\frac{8}{31}\sigma_9(n)-\frac{8192}{31}\sigma_9\left(\frac{n}{4}\right)\\
&+\frac{1232}{31}\,\delta(-2/3,13/6,-2/3;n)-\frac{256}{31}\,\delta(1/3,1/6,1/3;n),
\end{aligned}$$

which is Theorem 4.2(v).

The formulae of Theorem 4.2(i)(ii) are classical and due to Jacobi.

By (3.22) we have

$$\delta(0,1/2,0;n)=0\ \text{ for } n\equiv 0 \pmod 2$$

so that Theorem 4.2(iii) gives

$$s_{12}(n)=8\sigma_5(n)-512\sigma_5\left(\frac{n}{4}\right),\ \ n\equiv 0 \pmod 2, \tag{4.12}$$

a result first given by Liouville [11]. The first elementary arithmetic proof of (4.12) was given by Huard and Williams [9] in 2003. A formula extending (4.12) to odd n is given in [20].

From (3.20) and (3.17) we have for $n \in \mathbb{N}$

$$\delta(-1,3,-1;n) = (-1)^{n-1}\tau(n) \tag{4.13}$$

and

$$\delta(0,1,0;n) = \tau(n/2) \tag{4.14}$$

so that Theorem 4.2(vi) becomes

$$\begin{aligned} s_{24}(n) = \quad & \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}\left(\frac{n}{2}\right) + \frac{65536}{691}\sigma_{11}\left(\frac{n}{4}\right) \\ & + \frac{33152}{691}(-1)^{n-1}\tau(n) - \frac{65536}{691}\tau(n/2), \end{aligned} \tag{4.15}$$

which is a result due to Ramanujan [17], [18, p. 162].

As in the classical case, the sums of squares formulae in Theorem 4.2 involve coefficients of cusp forms that are not easy to compute explicitly. Any given classical sums of $4k$ squares formula here can be recursively worked out, but there is no general explicit closed formula for the coefficients involved. The formulae in (iv), (vi), (viii), (ix) and (xii) of Theorem 4.2 for 16, 24, 32, 36 and 48 squares are quite different from those of Milne for $s_{16}(n)$, $s_{24}(n)$, $s_{36}(n)$ and $s_{48}(n)$, and of Chan and Chua [3, p. 83] for $s_{32}(n)$. These new formulae are all explicit and do not involve coefficients of cusp forms. Moreover, Milne's formulae for 16, 24, 36 and 48 squares are special cases of his infinite families of explicit, exact, nontrivial, closed sums of squares formulae for $4s^2$ and $4s(s+1)$ squares in [12], [13] and [14]. Milne's work has inspired many recent modular forms and combinatorial investigations into sums of squares, and related, formulae by Zagier [21], Ono [16], Getz and Mahlburg [7], Imamoglu and Kohnen [10], Chan and Chua [3], Chan and Krattenthaler [4], and Rosengren [19]. Huard and Williams [8] have given an elementary arithmetic proof of Milne's formula for $s_{16}(n)$.

The authors have investigated the possibility of using their ideas in this paper to prove the analogous results for sums of $4k+2$ squares and sums of an even number of triangular numbers. However there appear to be difficulties in proving the required analogues of Propositions 2.2 and 2.3.

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TABLE OF VALUES OF $e_{k,r}(x)$

$e_{2,1}(x)$	$1+14x+x^2$
$e_{2,2}(x)$	$1-x+x^2$
$e_{2,4}(x)$	$1-x+\frac{1}{16}x^2$
$e_{3,1}(x)$	$1-33x-33x^2+x^3$
$e_{3,2}(x)$	$1-\frac{3}{2}x-\frac{3}{2}x^2+x^3$
$e_{3,4}(x)$	$1-\frac{3}{2}x+\frac{15}{32}x^2+\frac{1}{64}x^3$
$e_{4,1}(x)$	$1+28x+198x^2+28x^3+x^4$
$e_{4,2}(x)$	$1-2x+3x^2-2x^3+x^4$
$e_{4,4}(x)$	$1-2x+\frac{9}{8}x^2-\frac{1}{8}x^3+\frac{1}{256}x^4$
$e_{5,1}(x)$	$1-19x-494x^2-494x^3-19x^4+x^5$
$e_{5,2}(x)$	$1-\frac{5}{2}x+x^2+x^3-\frac{5}{2}x^4+x^5$
$e_{5,4}(x)$	$1-\frac{5}{2}x+\frac{65}{32}x^2-\frac{35}{64}x^3+\frac{7}{512}x^4+\frac{1}{1024}x^5$
$e_{6,1}(x)$	$1+\frac{2022}{691}x+\frac{516381}{691}x^2+\frac{1792148}{691}x^3+\frac{516381}{691}x^4+\frac{2022}{691}x^5+x^6$
$e_{6,2}(x)$	$1-3x+\frac{4917}{1382}x^2-\frac{1462}{691}x^3+\frac{4917}{1382}x^4-3x^5+x^6$
$e_{6,4}(x)$	$1-3x+\frac{51}{16}x^2-\frac{11}{8}x^3+\frac{67107}{353792}x^4-\frac{771}{353792}x^5+\frac{1}{4096}x^6$
$e_{7,1}(x)$	$1-5x-759x^2-7429x^3-7429x^4-759x^5-5x^6+x^7$
$e_{7,2}(x)$	$1-\frac{7}{2}x+\frac{9}{2}x^2-\frac{5}{2}x^3-\frac{5}{2}x^4+\frac{9}{2}x^5-\frac{7}{2}x^6+x^7$
$e_{7,4}(x)$	$1-\frac{7}{2}x+\frac{147}{32}x^2-\frac{175}{64}x^3+\frac{11}{16}x^4-\frac{3}{64}x^5-\frac{1}{8192}x^6+\frac{1}{16384}x^7$

TABLE (continued)

$e_{8,1}(x)$	$1-\frac{13448}{3617}x+\frac{2108060}{3617}x^2+\frac{50891848}{3617}x^3+\frac{131063558}{3617}x^4$ $+\frac{50891848}{3617}x^5+\frac{2108060}{3617}x^6-\frac{13448}{3617}x^7+x^8$
$e_{8,2}(x)$	$1-4x+\frac{22670}{3617}x^2-\frac{17372}{3617}x^3+\frac{14723}{3617}x^4-\frac{17372}{3617}x^5+\frac{22670}{3617}x^6$ $-4x^7+x^8$
$e_{8,4}(x)$	$1-4x+\frac{25}{4}x^2-\frac{19}{4}x^3+\frac{814309}{462976}x^4-\frac{61973}{231488}x^5+\frac{16525}{1851904}x^6$ $-\frac{121}{1851904}x^7+\frac{1}{65536}x^8$
$e_{9,1}(x)$	$1-\frac{199197}{43867}x-\frac{14343876}{43867}x^2-\frac{854608020}{43867}x^3-\frac{4880628198}{43867}x^4$ $-\frac{4880628198}{43867}x^5-\frac{854608020}{43867}x^6-\frac{14343876}{43867}x^7-\frac{199197}{43867}x^8+x^9$
$e_{9,2}(x)$	$1-\frac{9}{2}x+\frac{357678}{43867}x^2-\frac{330666}{43867}x^3+\frac{104589}{43867}x^4+\frac{104589}{43867}x^5$ $-\frac{330666}{43867}x^6+\frac{357678}{43867}x^7-\frac{9}{2}x^8+x^9$
$e_{9,4}(x)$	$1-\frac{9}{2}x+\frac{261}{32}x^2-\frac{483}{64}x^3+\frac{83229183}{22459904}x^4-\frac{40293459}{44919808}x^5$ $+\frac{14951685}{179679232}x^6-\frac{448695}{359358464}x^7-\frac{97803}{5749735424}x^8+\frac{1}{262144}x^9$
$e_{10,1}(x)$	$1-\frac{872230}{174611}x+\frac{28830615}{174611}x^2+\frac{3635476920}{174611}x^3+\frac{42424739670}{174611}x^4+\frac{90916204764}{174611}x^5$ $+\frac{42424739670}{174611}x^6+\frac{3635476920}{174611}x^7+\frac{28830615}{174611}x^8-\frac{872230}{174611}x^9+x^{10}$
$e_{10,2}(x)$	$1-5x+\frac{3601455}{349222}x^2-\frac{1964580}{174611}x^3+\frac{1309995}{174611}x^4-\frac{720786}{174611}x^5+\frac{1309995}{174611}x^6$ $-\frac{1964580}{174611}x^7+\frac{3601455}{349222}x^8-5x^9+x^{10}$
$e_{10,4}(x)$	$1-5x+\frac{165}{16}x^2-\frac{45}{4}x^3+\frac{616486065}{89400832}x^4-\frac{206717907}{89400832}x^5+\frac{268139595}{715206656}x^6$ $-\frac{7530615}{357603328}x^7+\frac{3604755}{22886612992}x^8-\frac{109235}{22886612992}x^9+\frac{1}{1048576}x^{10}$

TABLE (continued)

$e_{11,1}(x)$	$1-\frac{427291}{77683}x-\frac{3533783}{77683}x^2-\frac{1390589169}{77683}x^3-\frac{31441153746}{77683}x^4-\frac{130077432510}{77683}x^5$ $-\frac{130077432510}{77683}x^6-\frac{31441153746}{77683}x^7-\frac{1390589169}{77683}x^8-\frac{3533783}{77683}x^9$ $-\frac{427291}{77683}x^{10}+x^{11}$
$e_{11,2}(x)$	$1-\frac{11}{2}x+\frac{1976057}{155366}x^2-\frac{2483409}{155366}x^3+\frac{895767}{77683}x^4-\frac{331359}{77683}x^5-\frac{331359}{77683}x^6$ $+\frac{895767}{77683}x^7-\frac{2483409}{155366}x^8+\frac{1976057}{155366}x^9-\frac{11}{2}x^{10}+x^{11}$
$e_{11,4}(x)$	$1-\frac{11}{2}x+\frac{407}{32}x^2-\frac{1023}{64}x^3+\frac{233842821}{19886848}x^4-\frac{201317907}{39773696}x^5+\frac{760649775}{636379136}x^6$ $-\frac{166870011}{1272758272}x^7+\frac{44650113}{10182066176}x^8-\frac{220883}{20364132352}x^9-\frac{213611}{162913058816}x^{10}$ $+\frac{1}{4194304}x^{11}$
$e_{12,1}(x)$	$1-\frac{1418176356}{236364091}x+\frac{7927972086}{236364091}x^2+\frac{2985206272300}{236364091}x^3+\frac{127198424569365}{236364091}x^4$ $+\frac{951313821526584}{236364091}x^5+\frac{1802523012294516}{236364091}x^6+\frac{951313821526584}{236364091}x^7$ $+\frac{127198424569365}{236364091}x^8+\frac{2985206272300}{236364091}x^9+\frac{7927972086}{236364091}x^{10}$ $-\frac{1418176356}{236364091}x^{11}+x^{12}$
$e_{12,2}(x)$	$1-6x+\frac{3634098411}{236364091}x^2-\frac{5170467050}{236364091}x^3+\frac{8914522005}{472728182}x^4-\frac{2406271716}{236364091}x^5$ $+\frac{1570763706}{236364091}x^6-\frac{2406271716}{236364091}x^7+\frac{8914522005}{472728182}x^8-\frac{5170467050}{236364091}x^9$ $+\frac{3634098411}{236364091}x^{10}-6x^{11}+x^{12}$
$e_{12,4}(x)$	$1-6x+\frac{123}{8}x^2-\frac{175}{8}x^3+\frac{1136763550665}{60509207296}x^4-\frac{149707106649}{15127301824}x^5$ $+\frac{374611238247}{121018414592}x^6-\frac{63386206137}{121018414592}x^7+\frac{1205508131955}{30980714135552}x^8$ $-\frac{11869087475}{15490357067776}x^9+\frac{247751943}{123922856542208}x^{10}-\frac{44318523}{123922856542208}x^{11}$ $+\frac{1}{16777216}x^{12}$

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