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Some identities involving theta functions

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A B S T R A C T

Let $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ $(|q| < 1)$. For $k \in \mathbb{N}$ it is shown that there exist k rational numbers $A(k,0), \dots, A(k,k-1)$ such that

$$1 + \frac{4}{E_{2k}} \sum_{n=1}^{\infty} \Biggl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^{2k} \Biggr) q^n = \sum_{j=0}^{k-1} A(k,j) \varphi^{4j+2}(q) \varphi^{4k-4j}(-q),$$

where E_{2k} is an Euler number. Similarly it is shown that there exist $k+1$ rational numbers $B(k,0), \dots, B(k,k)$ such that

$$\sum_{n=1}^{\infty} \Biggl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^{2k} \Biggr) q^n = \sum_{j=0}^{k} B(k,j) \varphi^{4j+2}(q) \varphi^{4k-4j}(-q).$$

Recurrence relations are given for the $A(k,j)$ and $B(k,j)$.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this paper q denotes a complex variable with $|q| < 1$. The Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi, \tag{1.1}$$

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and the Euler numbers E_{2n} $(n \in \mathbb{N}_0)$ by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \quad |x| < \pi/2. \tag{1.2}$$

The first few values are

$$\begin{cases} B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \\ B_6 = \dfrac{1}{42}, \quad B_7 = 0, \quad B_8 = -\dfrac{1}{30}, \quad B_9 = 0, \quad B_{10} = \dfrac{5}{66}, \quad B_{11} = 0, \\ B_{12} = -\dfrac{691}{2730}, \quad B_{13} = 0, \quad B_{14} = \dfrac{7}{6}, \quad B_{15} = 0, \quad B_{16} = -\dfrac{3617}{510}, \quad \dots \end{cases} \tag{1.3}$$

and

$$\begin{cases} E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50521, \\ E_{12} = 2702765, \quad E_{14} = -199360981, \quad E_{16} = 19391512145, \quad \dots. \end{cases} \tag{1.4}$$

The theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1. \tag{1.5}$$

The Legendre–Jacobi–Kronecker symbol for discriminant -4 is defined for $k \in \mathbb{N}$ by

$$\left(\frac{-4}{k}\right) = \begin{cases} 1, & \text{if } k \equiv 1 \pmod 4, \\ -1, & \text{if } k \equiv 3 \pmod 4, \\ 0, & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{1.6}$$

It is a classical result of Jacobi (see for example [3, Eq. (3.2.8), p. 58]) that

$$1 + 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right)\Bigg) q^n = 1 + 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right)\Bigg) q^n = \varphi^2(q). \tag{1.7}$$

It is also known (see for example [1, Lemmas 1 and 2]) that

$$1 - 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^2\Bigg) q^n = \varphi^2(q)\varphi^4(-q) \tag{1.8}$$

and

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^2\Bigg) q^n = \frac{1}{16}\varphi^6(q) - \frac{1}{16}\varphi^2(q)\varphi^4(-q). \tag{1.9}$$

In this paper we generalize the identities in (1.8) and (1.9) by showing that each of

$$F_k(q) := 1 + \frac{4}{E_{2k}} \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^{2k} \Bigg) q^n \quad (k \in \mathbb{N}) \tag{1.10}$$

and

$$G_k(q) := \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^{2k} \Bigg) q^n \quad (k \in \mathbb{N}) \tag{1.11}$$

can be expressed as a polynomial in $\varphi(q)$ and $\varphi(-q)$ with rational coefficients, and we give recurrence relations for these coefficients.

For $k \in \mathbb{N}$ with $k \geqslant 2$ and $(t,u) \in \mathbb{N}_0^2$ with $2t + 3u = k$ we define the rational numbers $a_k(t,u)$ recursively by

$$a_2(1,0) = 1, \tag{1.12}$$

$$a_3(0,1) = 1, \tag{1.13}$$

$$a_k(t,u) = \frac{-3k(k-1)(2k-3)}{(k-3)(2k+1)B_{2k}} \sum_{l=2}^{k-2} \frac{B_{2l}B_{2k-2l}}{l(k-l)} \binom{2k-4}{2l-2} \times \sum_{\substack{r,s=0 \\ 2r+3s=l}}^{t,u} a_l(r,s) a_{k-l}(t-r, u-s), \quad k \geqslant 4. \tag{1.14}$$

The first few values are

$$a_4(2,0) = 1, \tag{1.15}$$

$$a_5(1,1) = 1, \tag{1.16}$$

$$a_6(0,2) = \frac{250}{691}, \qquad a_6(3,0) = \frac{441}{691}, \tag{1.17}$$

$$a_7(2,1) = 1, \tag{1.18}$$

$$a_8(1,2) = \frac{2000}{3617}, \qquad a_8(4,0) = \frac{1617}{3617}. \tag{1.19}$$

For $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we set

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k. \tag{1.20}$$

If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We also set $\sigma(n) := \sigma_1(n)$. The Eisenstein series $E_k(q)$ $(k \in \mathbb{N})$ is defined by

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{1.21}$$

so that

$$E_k(0) = 1, \qquad E_k'(0) = -\frac{4k}{B_{2k}}. \tag{1.22}$$

It is known that $E_1(q)$, $E_2(q)$ and $E_3(q)$ are algebraically independent [9, p. 69]. In Section 2 we prove the following result, which makes explicit a result of Ramanujan. This result, in which we identify a recursion for the coefficients $a_k(t,u)$, is one of the main tools of this paper.

Theorem 1.1. *For $k \in \mathbb{N}$ with $k \geqslant 2$*

$$E_k(q) = \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) E_2(q)^t E_3(q)^u, \tag{1.23}$$

where $a_k(t,u)$ is given recursively by (1.12)–(1.14).

Taking $q = 0$ in Theorem 1.1, and appealing to (1.22), we deduce

$$\sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) = 1. \tag{1.24}$$

Differentiating (1.23), and setting $q = 0$, we obtain

$$\sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} (10t - 21u) a_k(t,u) = -\frac{k}{6B_{2k}}. \tag{1.25}$$

It is easy to obtain more identities of the type (1.24) and (1.25) by further differentiations of (1.23).

Next for $k \in \mathbb{N}$ and $j \in \{0, 1, 2, \ldots, k\}$ we define $e_1(k,j)$, $e_2(k,j)$ and $e_4(k,j)$ as follows:

$$e_1(k,j) := \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7 \\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k \\ t_2+2t_3+u_2+2u_3+3u_4=j}} (-1)^{t_2+u_1+u_3+u_4} 2^{4t_1+4t_2+6u_1+5u_2+u_3} 3^{u_2+u_3} 5^{u_3}$$
$$\times \binom{t_1+t_2+t_3}{t_1,t_2,t_3} \binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4} a_k(t_1+t_2+t_3, u_1+u_2+u_3+u_4), \tag{1.26}$$

$$e_2(k,j) := \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7 \\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k \\ t_2+2t_3+u_2+2u_3+3u_4=j}} (-1)^{t_2+u_1+u_4} 2^{-u_2-u_3} 3^{u_2+u_3}$$
$$\times \binom{t_1+t_2+t_3}{t_1,t_2,t_3} \binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4} a_k(t_1+t_2+t_3, u_1+u_2+u_3+u_4), \tag{1.27}$$

$$e_4(k,j) := \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7 \\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k \\ t_2+2t_3+u_2+2u_3+3u_4=j}} (-1)^{u_1+u_4} 2^{t_2-2k} 3^{u_2+u_3} 7^{t_2} 11^{u_2+u_3}$$
$$\times \binom{t_1+t_2+t_3}{t_1,t_2,t_3} \binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4} a_k(t_1+t_2+t_3, u_1+u_2+u_3+u_4). \tag{1.28}$$

In Section 3 we prove the following result.

Table 1
Values of $e_1(j,k)$.

$e_1(2,0) = 16$	$e_1(5,2) = -2080$	$e_1(7,3) = 44800$
$e_1(2,1) = -16$	$e_1(5,3) = 560$	$e_1(7,4) = -11264$
$e_1(2,2) = 1$	$e_1(5,4) = -14$	$e_1(7,5) = 768$
$e_1(3,0) = -64$	$e_1(5,5) = -1$	$e_1(7,6) = 2$
$e_1(3,1) = 96$	$e_1(6,0) = 4096$	$e_1(7,7) = -1$
$e_1(3,2) = -30$	$e_1(6,1) = -12288$	$e_1(8,0) = 65536$
$e_1(3,3) = -1$	$e_1(6,2) = 13056$	$e_1(8,1) = -262144$
$e_1(4,0) = 256$	$e_1(6,3) = -5632$	$e_1(8,2) = 409600$
$e_1(4,1) = -512$	$e_1(6,4) = \frac{536856}{691}$	$e_1(8,3) = -311296$
$e_1(4,2) = 288$	$e_1(6,5) = \frac{-6168}{691}$	$e_1(8,4) = \frac{416926208}{3617}$
$e_1(4,3) = -32$	$e_1(6,6) = 1$	$e_1(8,5) = \frac{-63460352}{3617}$
$e_1(4,4) = 1$	$e_1(7,0) = -16384$	$e_1(8,6) = \frac{2115200}{3617}$
$e_1(5,0) = -1024$	$e_1(7,1) = 57344$	$e_1(8,7) = \frac{-15488}{3617}$
$e_1(5,1) = 2560$	$e_1(7,2) = -75264$	$e_1(8,8) = 1$

Theorem 1.2. *For $k \in \mathbb{N}$ with $k \geqslant 2$ and $r \in \{1,2,4\}$*

$$E_k(q^r) = \sum_{j=0}^{k} e_r(k,j)\varphi^{4k-4j}(q)\varphi^{4j}(-q). \tag{1.29}$$

In Sections 4–6, using (1.23) and (1.24), we prove Theorems 1.3–1.5 respectively. These theorems give some of the values of the $e_r(k,j)$.

Theorem 1.3. *For $k \in \mathbb{N}$ with $k \geqslant 2$*

$$e_1(k,0) = (-1)^k 2^{2k}, \tag{1.30}$$

$$e_1(k,1) = (-1)^{k+1} k 2^{2k-1}, \tag{1.31}$$

$$e_1(k,2) = (-1)^k k(4k-7)2^{2k-5}, \tag{1.32}$$

$$e_1(k,3) = \frac{1}{3}(-1)^{k+1} k(k-2)(4k-13)2^{2k-6}, \tag{1.33}$$

$$e_1(k,k-1) = \frac{(-1)^k k}{4B_{2k}}(1-2B_{2k}), \tag{1.34}$$

$$e_1(k,k) = (-1)^k, \tag{1.35}$$

$$\sum_{j=0}^{k} e_1(k,j) = 1. \tag{1.36}$$

The first few values of the $e_1(k,j)$ are given in Table 1.

Theorem 1.4. *For $k \in \mathbb{N}$ with $k \geqslant 2$*

$$e_2(k,0) = (-1)^k, \tag{1.37}$$

$$e_2(k,1) = \frac{1}{2}(-1)^{k+1} k, \tag{1.38}$$

Table 2
Values of $e_2(j,k)$.

$e_2(2,0)=1$	$e_2(5,2)=-1$	$e_2(7,3)=\frac{5}{2}$
$e_2(2,1)=-1$	$e_2(5,3)=-1$	$e_2(7,4)=\frac{5}{2}$
$e_2(2,2)=1$	$e_2(5,4)=\frac{5}{2}$	$e_2(7,5)=\frac{-9}{2}$
$e_2(3,0)=-1$	$e_2(5,5)=-1$	$e_2(7,6)=\frac{7}{2}$
$e_2(3,1)=\frac{3}{2}$	$e_2(6,0)=1$	$e_2(7,7)=-1$
$e_2(3,2)=\frac{3}{2}$	$e_2(6,1)=-3$	$e_2(8,0)=1$
$e_2(3,3)=-1$	$e_2(6,2)=\frac{4917}{1382}$	$e_2(8,1)=-4$
$e_2(4,0)=1$	$e_2(6,3)=\frac{-1462}{691}$	$e_2(8,2)=\frac{22670}{3617}$
$e_2(4,1)=-2$	$e_2(6,4)=\frac{4917}{1382}$	$e_2(8,3)=\frac{-17372}{3617}$
$e_2(4,2)=3$	$e_2(6,5)=-3$	$e_2(8,4)=\frac{14723}{3617}$
$e_2(4,3)=-2$	$e_2(6,6)=1$	$e_2(8,5)=\frac{-17372}{3617}$
$e_2(4,4)=1$	$e_2(7,0)=-1$	$e_2(8,6)=\frac{22670}{3617}$
$e_2(5,0)=-1$	$e_2(7,1)=\frac{7}{2}$	$e_2(8,7)=-4$
$e_2(5,1)=\frac{5}{2}$	$e_2(7,2)=\frac{-9}{2}$	$e_2(8,8)=1$

$$e_2(k,2)=\frac{(-1)^{k+1}k}{64B_{2k}}\big(1-(8k-14)B_{2k}\big), \tag{1.39}$$

$$e_2(k,3)=\frac{(-1)^{k}k(k-2)}{384B_{2k}}\big(3-2(4k-13)B_{2k}\big), \tag{1.40}$$

$$e_2(k,k-j)=e_2(k,j), \quad j=0,1,\dots,k, \tag{1.41}$$

$$\sum_{j=0}^{k} e_2(k,j)=1. \tag{1.42}$$

The first few values of the $e_2(k,j)$ are given in Table 2.

Theorem 1.5. *For $k\in\mathbb{N}$ with $k\geqslant 2$*

$$e_4(k,0)=\frac{(-1)^k}{2^{2k}}, \tag{1.43}$$

$$e_4(k,1)=\frac{(-1)^{k+1}k(1+2B_{2k})}{2^{2k+2}B_{2k}}, \tag{1.44}$$

$$e_4(k,k-j)=e_4(k,j), \quad j=0,1,\dots,k, \tag{1.45}$$

$$\sum_{j=0}^{k} e_4(k,j)=1. \tag{1.46}$$

Table 3
Values of $e_4(j,k)$.

$e_4(2,0)=\frac{1}{16}$	$e_4(5,2)=\frac{247}{512}$	$e_4(7,3)=\frac{7429}{16384}$
$e_4(2,1)=\frac{7}{8}$	$e_4(5,3)=\frac{247}{512}$	$e_4(7,4)=\frac{7429}{16384}$
$e_4(2,2)=\frac{1}{16}$	$e_4(5,4)=\frac{19}{1024}$	$e_4(7,5)=\frac{759}{16384}$
$e_4(3,0)=\frac{-1}{64}$	$e_4(5,5)=\frac{-1}{1024}$	$e_4(7,6)=\frac{5}{16384}$
$e_4(3,1)=\frac{33}{64}$	$e_4(6,0)=\frac{1}{4096}$	$e_4(7,7)=\frac{-1}{16384}$
$e_4(3,2)=\frac{33}{64}$	$e_4(6,1)=\frac{1011}{1415168}$	$e_4(8,0)=\frac{1}{65536}$
$e_4(3,3)=\frac{-1}{64}$	$e_4(6,2)=\frac{516381}{2830336}$	$e_4(8,1)=\frac{-1681}{29630464}$
$e_4(4,0)=\frac{1}{256}$	$e_4(6,3)=\frac{448037}{707584}$	$e_4(8,2)=\frac{527015}{59260928}$
$e_4(4,1)=\frac{7}{64}$	$e_4(6,4)=\frac{516381}{2830336}$	$e_4(8,3)=\frac{6361481}{29630464}$
$e_4(4,2)=\frac{99}{128}$	$e_4(6,5)=\frac{1011}{1415168}$	$e_4(8,4)=\frac{65531779}{118521856}$
$e_4(4,3)=\frac{7}{64}$	$e_4(6,6)=\frac{1}{4096}$	$e_4(8,5)=\frac{6361481}{29630464}$
$e_4(4,4)=\frac{1}{256}$	$e_4(7,0)=\frac{-1}{16384}$	$e_4(8,6)=\frac{527015}{59260928}$
$e_4(5,0)=\frac{-1}{1024}$	$e_4(7,1)=\frac{5}{16384}$	$e_4(8,7)=\frac{-1681}{29630464}$
$e_4(5,1)=\frac{19}{1024}$	$e_4(7,2)=\frac{759}{16384}$	$e_4(8,8)=\frac{1}{65536}$

The first few values of the $e_4(k,j)$ are given in Table 3.
For $k\in\mathbb{N}$ and $j\in\{1,2,\dots,k\}$, we set

$$f(k,j):=e_1(k,j)-\left(2^{2k}+2\right)e_2(k,j)+2^{2k+1}e_4(k,j),\quad k\geqslant 2, \tag{1.47}$$

and

$$f(1,1)=3. \tag{1.48}$$

For $k\in\mathbb{N}$ and $j\in\{0,1,\dots,k\}$, we set

$$g(k,j):=e_1(k,j)-\left(2^{2k-1}+1\right)e_2(k,j)+2^{2k-1}e_4(k,j) \tag{1.49}$$

and

$$g(1,0)=-\frac{3}{2},\qquad g(1,1)=\frac{3}{2}. \tag{1.50}$$

From (1.47)–(1.50) and Theorem 1.2, we obtain the following two results, see Sections 7 and 8.

Theorem 1.6. *For $k\in\mathbb{N}$*

$$E_k(q)-\left(2^{2k}+2\right)E_k\left(q^2\right)+2^{2k+1}E_k\left(q^4\right)=\sum_{j=1}^{k}f(k,j)\varphi^{4k-4j}(q)\varphi^{4j}(-q). \tag{1.51}$$

Table 4
Values of $f(j,k)$.

$f(2,1)=30$	$f(5,4)=-2541$	$f(7,5)=76023$
$f(2,2)=-15$	$f(5,5)=1023$	$f(7,6)=-57339$
$f(3,1)=63$	$f(6,1)=\frac{8190}{691}$	$f(7,7)=16383$
$f(3,2)=-63$	$f(6,2)=\frac{-20475}{691}$	$f(8,1)=\frac{2040}{3617}$
$f(3,3)=63$	$f(6,3)=\frac{5683860}{691}$	$f(8,2)=\frac{-7140}{3617}$
$f(4,1)=60$	$f(6,4)=\frac{-8505315}{691}$	$f(8,3)=\frac{114352200}{3617}$
$f(4,2)=-90$	$f(6,5)=\frac{8493030}{691}$	$f(8,4)=\frac{-285862650}{3617}$
$f(4,3)=540$	$f(6,6)=-4095$	$f(8,5)=\frac{1176849480}{3617}$
$f(4,4)=-255$	$f(7,1)=3$	$f(8,6)=\frac{-1479415140}{3617}$
$f(5,1)=33$	$f(7,2)=-9$	$f(8,7)=\frac{948161400}{3617}$
$f(5,2)=-66$	$f(7,3)=18693$	$f(8,8)=-65535$
$f(5,3)=2574$	$f(7,4)=-37371$	

Theorem 1.7. *For* $k \in \mathbb{N}$

$$E_k(q) - \left(2^{2k-1}+1\right)E_k\left(q^2\right) + 2^{2k-1}E_k\left(q^4\right) = \sum_{j=0}^{k} g(k,j)\varphi^{4k-4j}(q)\varphi^{4j}(-q). \tag{1.52}$$

From Theorems 1.3–1.5 and (1.47)–(1.50) we obtain the values of the $f(k,j)$ and $g(k,j)$ given in Theorems 1.8 and 1.9, see Sections 9 and 10.

Theorem 1.8. *For* $k \in \mathbb{N}$

$$f(k,1) = \frac{(-1)^{k+1}k}{2B_{2k}}, \tag{1.53}$$

$$f(k,k-1) = \frac{(-1)^{k+1}k}{4B_{2k}}\left(1-2\left(2^{2k}-1\right)B_{2k}\right), \quad k \geqslant 2, \tag{1.54}$$

$$f(k,k) = (-1)^{k+1}\left(2^{2k}-1\right), \tag{1.55}$$

$$\sum_{j=1}^{k} f(k,j) = 2^{2k}-1. \tag{1.56}$$

The first few values of the $f(k,j)$ are given in Table 4.

Theorem 1.9. *For* $k \in \mathbb{N}$

$$g(k,0) = \frac{(-1)^k}{2}\left(2^{2k}-1\right), \tag{1.57}$$

$$g(k,1) = \frac{(-1)^{k+1}k}{8B_{2k}}\left(1+2\left(2^{2k}-1\right)B_{2k}\right), \tag{1.58}$$

Table 5
Values of $g(j,k)$.

$g(2,0)=\frac{15}{2}$	$g(5,2)=-1320$	$g(7,3)=28032$
$g(2,1)=0$	$g(5,3)=1320$	$g(7,4)=-28032$
$g(2,2)=\frac{-15}{2}$	$g(5,4)=-1287$	$g(7,5)=38016$
$g(3,0)=\frac{-63}{2}$	$g(5,5)=\frac{1023}{2}$	$g(7,6)=-28671$
$g(3,1)=63$	$g(6,0)=\frac{4095}{2}$	$g(7,7)=\frac{16383}{2}$
$g(3,2)=-63$	$g(6,1)=\frac{-4242420}{691}$	$g(8,0)=\frac{65535}{2}$
$g(3,3)=\frac{63}{2}$	$g(6,2)=\frac{4242420}{691}$	$g(8,1)=\frac{-474079680}{3617}$
$g(4,0)=\frac{255}{2}$	$g(6,3)=0$	$g(8,2)=\frac{739704000}{3617}$
$g(4,1)=-240$	$g(6,4)=\frac{-4242420}{691}$	$g(8,3)=\frac{-531248640}{3617}$
$g(4,2)=0$	$g(6,5)=\frac{4242420}{691}$	$g(8,4)=0$
$g(4,3)=240$	$g(6,6)=\frac{-4095}{2}$	$g(8,5)=\frac{531248640}{3617}$
$g(4,4)=\frac{-255}{2}$	$g(7,0)=\frac{-16383}{2}$	$g(8,6)=\frac{-739704000}{3617}$
$g(5,0)=\frac{-1023}{2}$	$g(7,1)=28671$	$g(8,7)=\frac{474079680}{3617}$
$g(5,1)=1287$	$g(7,2)=-38016$	$g(8,8)=\frac{-65535}{2}$

$$g(k,k-j)=-g(k,j), \quad j=0,1,\dots,k, \tag{1.59}$$

$$\sum_{j=0}^{k} g(k,j)=0. \tag{1.60}$$

The first few values of the $g(k,j)$ are given in Table 5.

We are now ready to define the $A(k,j)$ $(k\in\mathbb{N},\ j\in\{0,1,\dots,k-1\})$ recursively by

$$\begin{aligned}A(k,j)={}&-\frac{B_{2k}}{E_{2k}}\frac{2^{2k-1}}{k}f(k,k-j)\\&-\frac{1}{E_{2k}}\sum_{l=1}^{k-1}\sum_{\substack{m=0\\ l+m-n=k-j}}^{k-l}\sum_{n=0}^{l-1}B_{2k-2l}E_{2l}\frac{2^{2k-2l-1}}{k-l}\binom{2k-1}{2l}f(k-l,m)A(l,n)\end{aligned} \tag{1.61}$$

and

$$A(1,0)=1, \tag{1.62}$$

and the $B(k,j)$ $(k\in\mathbb{N},\ j\in\{0,1,\dots,k\})$ by

Table 6
Values of $A(j,k)$.

$A(1,0)=1$	$A(5,2)=\frac{30768}{50521}$	$A(7,3)=\frac{106923008}{199360981}$
$A(2,0)=\frac{1}{5}$	$A(5,3)=\frac{832}{2659}$	$A(7,4)=\frac{1383168}{4241723}$
$A(2,1)=\frac{4}{5}$	$A(5,4)=\frac{256}{50521}$	$A(7,5)=\frac{4180992}{199360981}$
$A(3,0)=\frac{1}{61}$	$A(6,0)=\frac{1}{2702765}$	$A(7,6)=\frac{4096}{199360981}$
$A(3,1)=\frac{44}{61}$	$A(6,1)=\frac{33212}{2702765}$	$A(8,0)=\frac{1}{19391512145}$
$A(3,2)=\frac{16}{61}$	$A(6,2)=\frac{174128}{540553}$	$A(8,1)=\frac{2690416}{19391512145}$
$A(4,0)=\frac{1}{1385}$	$A(6,3)=\frac{307712}{540553}$	$A(8,2)=\frac{586629984}{19391512145}$
$A(4,1)=\frac{408}{1385}$	$A(6,4)=\frac{259328}{2702765}$	$A(8,3)=\frac{6337665152}{19391512145}$
$A(4,2)=\frac{912}{1385}$	$A(6,5)=\frac{1024}{2702765}$	$A(8,4)=\frac{9860488448}{19391512145}$
$A(4,3)=\frac{64}{1385}$	$A(7,0)=\frac{1}{199360981}$	$A(8,5)=\frac{2536974336}{19391512145}$
$A(5,0)=\frac{1}{50521}$	$A(7,1)=\frac{298932}{199360981}$	$A(8,6)=\frac{67047424}{19391512145}$
$A(5,1)=\frac{3688}{50521}$	$A(7,2)=\frac{22945056}{199360981}$	$A(8,7)=\frac{16384}{19391512145}$

$$B(k,j)=\frac{B_{2k}}{4k}g(k,j)-\sum_{l=1}^{k-1}\sum_{\substack{m=0\\ l+m-n=k-j}}^{k-l}\sum_{n=0}^{l}\frac{B_{2k-2l}}{k-l}\binom{2k-1}{2l}g(k-l,m)B(l,n) \tag{1.63}$$

and

$$B(1,0)=-\frac{1}{16},\qquad B(1,1)=\frac{1}{16}. \tag{1.64}$$

The first few values of the $A(k,j)$ are given in Table 6 and those of $B(k,j)$ in Table 7.

In Sections 11 and 12 we prove the two main results of this paper.

Theorem 1.10. *For $k\in\mathbb{N}$*

$$1+\frac{4}{E_{2k}}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)d^{2k}\Bigg)q^n=\sum_{j=0}^{k-1}A(k,j)\varphi^{4j+2}(q)\varphi^{4k-4j}(-q),$$

where the $A(k,j)$ are given recursively by (1.61) *and* (1.62).

Theorem 1.11. *For $k\in\mathbb{N}$*

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{n/d}\right)d^{2k}\Bigg)q^n=\sum_{j=0}^{k}B(k,j)\varphi^{4j+2}(q)\varphi^{4k-4j}(-q),$$

where the $B(k,j)$ are given recursively by (1.63) *and* (1.64).

Table 7
Values of $B(j,k)$.

$B(1,0)=\frac{-1}{16}$	$B(5,1)=\frac{3693}{4096}$	$B(7,3)=\frac{226132303}{65536}$
$B(1,1)=\frac{1}{16}$	$B(5,2)=\frac{-22765}{2048}$	$B(7,4)=\frac{-728130163}{65536}$
$B(2,0)=\frac{1}{64}$	$B(5,3)=\frac{65125}{2048}$	$B(7,5)=\frac{1074680289}{65536}$
$B(2,1)=\frac{-3}{32}$	$B(5,4)=\frac{-138933}{4096}$	$B(7,6)=\frac{-747603679}{65536}$
$B(2,2)=\frac{5}{64}$	$B(5,5)=\frac{50521}{4096}$	$B(7,7)=\frac{199360981}{65536}$
$B(3,0)=\frac{-1}{256}$	$B(6,0)=\frac{1}{16384}$	$B(8,0)=\frac{1}{262144}$
$B(3,1)=\frac{47}{256}$	$B(6,1)=\frac{-16609}{8192}$	$B(8,1)=\frac{-336303}{32768}$
$B(3,2)=\frac{-107}{256}$	$B(6,2)=\frac{1036715}{16384}$	$B(8,2)=\frac{151365731}{65536}$
$B(3,3)=\frac{61}{256}$	$B(6,3)=\frac{-1338315}{4096}$	$B(8,3)=\frac{-1239242981}{32768}$
$B(4,0)=\frac{1}{1024}$	$B(6,4)=\frac{10430983}{16384}$	$B(8,4)=\frac{25221214299}{131072}$
$B(4,1)=\frac{-103}{256}$	$B(6,5)=\frac{-4391993}{8192}$	$B(8,5)=\frac{-14647792993}{32768}$
$B(4,2)=\frac{1071}{512}$	$B(6,6)=\frac{2702765}{16384}$	$B(8,6)=\frac{34768375291}{65536}$
$B(4,3)=\frac{-779}{256}$	$B(7,0)=\frac{-1}{65536}$	$B(8,7)=\frac{-10301740827}{32768}$
$B(4,4)=\frac{1385}{1024}$	$B(7,1)=\frac{298939}{65536}$	$B(8,8)=\frac{19391512145}{262144}$
$B(5,0)=\frac{-1}{4096}$	$B(7,2)=\frac{-24738669}{65536}$	

Theorem 1.10 is reasonably deep, at the level of Watson's quintuple product identity via Eq. (11.9), and Theorem 1.11 is even deeper, as it relies on Ramanujan's striking equation (12.9) via the Ramanujan ${}_1\psi_1$ summation. Our proofs of Theorems 1.10 and 1.11 extend Jacobi's formulae for six squares, and the case $k=2$ of these two theorems allow us to recover the classical formula for ten squares presented in Section 13. The value of this paper is that it aids in the effective computation of various aspects of Ramanujan's identities for Eisenstein series and theta functions.

2. Proof of Theorem 1.1

Let $k \in \mathbb{N}$. Ramanujan [6], [7, p. 141] has asserted, and Berndt [3, p. 96] has proved, that there exist rational numbers $a_k(t,u)$ $((t,u) \in \mathbb{N}_0^2, 2t+3u=k)$, which do not depend upon q, such that

$$E_k(q) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) E_2(q)^t E_3(q)^u, \quad k \geqslant 2. \tag{2.1}$$

Clearly

$$a_2(1,0) = 1, \qquad a_3(0,1) = 1. \tag{2.2}$$

Following Skoruppa [9, p. 68] we set

$$G_{2k} := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \quad k \geqslant 1, \tag{2.3}$$

so that by (1.21) we have

$$G_{2k} = -\frac{B_{2k}}{4k} E_k(q), \quad k \geqslant 1. \tag{2.4}$$

From [9, Eq. (2), p. 69] (see also [5, p. 266] and [8, p. 108]) we have

$$\frac{(2k+1)(2k-6)}{12(2k-2)(2k-3)} G_{2k} = \sum_{l=2}^{k-2} \binom{2k-4}{2l-2} G_{2l} G_{2k-2l}, \quad k \geqslant 4. \tag{2.5}$$

Using (2.4) in (2.5) we obtain

$$E_k(q) = -\frac{3k(k-1)(2k-3)}{(k-3)(2k+1)B_{2k}} \sum_{l=2}^{k-2} \frac{B_{2l} B_{2k-2l}}{l(k-l)} \binom{2k-4}{2l-2} E_l(q) E_{k-l}(q), \quad k \geqslant 4. \tag{2.6}$$

Appealing to (2.1) we deduce for $k \geqslant 4$

$$\begin{aligned}
\sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} a_k(t,u) E_2(q)^t E_3(q)^u &= -\frac{3k(k-1)(2k-3)}{(k-3)(2k+1)B_{2k}} \sum_{l=2}^{k-2} \frac{B_{2l} B_{2k-2l}}{l(k-l)} \binom{2k-4}{2l-2} \\
&\quad \times \sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2\\ 2t_1+3u_1=l}} a_l(t_1,u_1) E_2(q)^{t_1} E_3(q)^{u_1} \\
&\quad \times \sum_{\substack{(t_2,u_2)\in\mathbb{N}_0^2\\ 2t_2+3u_2=k-l}} a_{k-l}(t_2,u_2) E_2(q)^{t_2} E_3(q)^{u_2} \\
&= -\frac{3k(k-1)(2k-3)}{(k-3)(2k+1)B_{2k}} \sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} \sum_{l=2}^{k-2} \frac{B_{2l} B_{2k-2l}}{l(k-l)} \binom{2k-4}{2l-2} \\
&\quad \times \sum_{\substack{r,s=0\\ 2r+3s=l}}^{t,u} a_l(r,s) a_{k-l}(t-r,u-s) E_2(q)^t E_3(q)^u.
\end{aligned}$$

As $E_2(q)$ and $E_3(q)$ are algebraically independent over $\mathbb{Q}$, we deduce for $(t,u) \in \mathbb{N}_0^2$, $2t+3u=k$, $k \geqslant 4$

$$a_k(t,u) = -\frac{3k(k-1)(2k-3)}{(k-3)(2k+1)B_{2k}} \sum_{l=2}^{k-2} \frac{B_{2l} B_{2k-2l}}{l(k-l)} \binom{2k-4}{2l-2} \sum_{\substack{r,s=0\\ 2r+3s=l}}^{t,u} a_l(r,s) a_{k-l}(t-r,u-s).$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let $k \in \mathbb{N}$ be such that $k \geqslant 2$. We just prove the theorem for $r = 1$ as the cases $r = 2$ and $r = 4$ can be treated similarly. We give a verification proof in which we start with the right-hand side. We set (as in [3, Eqs. (5.2.27), (5.2.29), p. 120])

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \qquad z = \varphi^2(q). \tag{3.1}$$

From [3, Theorem 5.4.11, p. 126; Theorem 5.4.12, p. 127], we have

$$E_2(q) = \left(1 + 14x + x^2\right)z^4 \tag{3.2}$$

and

$$E_3(q) = \left(1 - 33x - 33x^2 + x^3\right)z^6. \tag{3.3}$$

From (3.1)–(3.3) we deduce

$$E_2(q) = 16\varphi^8(q) - 16\varphi^4(q)\varphi^4(-q) + \varphi^8(-q) \tag{3.4}$$

and

$$E_3(q) = -64\varphi^{12}(q) + 96\varphi^8(q)\varphi^4(-q) - 30\varphi^4(q)\varphi^8(-q) - \varphi^{12}(-q). \tag{3.5}$$

By the multinomial theorem we have

$$\begin{aligned} E_2(q)^t &= \left(16\varphi^8(q) - 16\varphi^4(q)\varphi^4(-q) + \varphi^8(-q)\right)^t \\ &= \sum_{\substack{(t_1,t_2,t_3) \in \mathbb{N}_0^3 \\ t_1+t_2+t_3=t}} \binom{t}{t_1, t_2, t_3} (-1)^{t_2} 2^{4t_1+4t_2} \varphi^{8t_1+4t_2}(q)\varphi^{4t_2+8t_3}(-q) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} E_3(q)^u &= \left(-64\varphi^{12}(q) + 96\varphi^8(q)\varphi^4(-q) - 30\varphi^4(q)\varphi^8(-q) - \varphi^{12}(-q)\right)^u \\ &= \sum_{\substack{(u_1,u_2,u_3,u_4) \in \mathbb{N}_0^4 \\ u_1+u_2+u_3+u_4=u}} \binom{u}{u_1, u_2, u_3, u_4} (-1)^{u_1+u_3+u_4} 2^{6u_1+5u_2+u_3} 3^{u_2+u_3} 5^{u_3} \\ &\quad \times \varphi^{12u_1+8u_2+4u_3}(q)\varphi^{4u_2+8u_3+12u_4}(-q). \end{aligned} \tag{3.7}$$

Appealing to (1.26), (3.6), (3.7) and Theorem 1.1, we obtain

$$\begin{aligned} &\sum_{j=0}^{k} e_1(k, j)\varphi^{4k-4j}(q)\varphi^{4j}(-q) \\ &= \sum_{j=0}^{k} \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4) \in \mathbb{N}_0^7 \\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k \\ t_2+2t_3+u_2+2u_3+3u_4=j}} (-1)^{t_2+u_1+u_3+u_4} 2^{4t_1+4t_2+6u_1+5u_2+u_3} 3^{u_2+u_3} 5^{u_3} \end{aligned}$$

$$
\begin{aligned}
&\times \binom{t_1+t_2+t_3}{t_1,t_2,t_3}\binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4} a_k(t_1+t_2+t_3, u_1+u_2+u_3+u_4) \\
&\times \varphi^{4k-4j}(q)\varphi^{4j}(-q) \\
= &\sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7 \\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k}} (-1)^{t_2+u_1+u_3+u_4} 2^{4t_1+4t_2+6u_1+5u_2+u_3} 3^{u_2+u_3} 5^{u_3} \\
&\times \binom{t_1+t_2+t_3}{t_1,t_2,t_3}\binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4} a_k(t_1+t_2+t_3, u_1+u_2+u_3+u_4) \\
&\times \varphi^{8t_1+4t_2+12u_1+8u_2+4u_3}(q)\varphi^{4t_2+8t_3+4u_2+8u_3+12u_4}(-q) \\
= &\sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \sum_{\substack{(t_1,t_2,t_3)\in\mathbb{N}_0^3 \\ t_1+t_2+t_3=t}} (-1)^{t_2} 2^{4t_1+4t_2}\binom{t}{t_1,t_2,t_3}\varphi^{8t_1+4t_2}(q)\varphi^{4t_2+8t_3}(-q) \\
&\times \sum_{\substack{(u_1,u_2,u_3,u_4)\in\mathbb{N}_0^4 \\ u_1+u_2+u_3+u_4=u}} (-1)^{u_1+u_3+u_4} 2^{6u_1+5u_2+u_3} 3^{u_2+u_3} 5^{u_3}\binom{u}{u_1,u_2,u_3,u_4} \\
&\times \varphi^{12u_1+8u_2+4u_3}(q)\varphi^{4u_2+8u_3+12u_4}(-q) \\
= &\sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) E_2(q)^t E_3(q)^u \\
= &E_k(q),
\end{aligned}
$$

which completes the proof in the case $r=1$.

4. Proof of Theorem 1.3

We prove (1.30) and (1.31). Eqs. (1.32)–(1.36) can be proved by similar techniques. By (1.26) we have

$$
\begin{aligned}
e_1(k,0) &= \sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2 \\ 2t_1+3u_1=k}} (-1)^{u_1} 2^{4t_1+6u_1}\binom{t_1}{t_1,0,0}\binom{u_1}{u_1,0,0,0} a_k(t_1,u_1) \\
&= \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} (-1)^k 2^{2k} a_k(t,u) \\
&= (-1)^k 2^{2k} \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \\
&= (-1)^k 2^{2k}
\end{aligned}
$$

by (1.24). This proves (1.30).

By (1.26) we have

$$\begin{aligned}
e_1(k,1) &= \sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2\\ 2t_1+2+3u_1=k}} (-1)^{1+u_1}2^{4t_1+4+6u_1}\binom{t_1+1}{t_1,1,0}\binom{u_1}{u_1,0,0,0}a_k(t_1+1,u_1)\\
&\quad+\sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2\\ 2t_1+3u_1+3=k}} (-1)^{u_1}2^{4t_1+6u_1+5}3\binom{t_1}{t_1,0,0}\binom{u_1+1}{u_1,1,0,0}a_k(t_1,u_1+1)\\
&= \sum_{\substack{(t,u)\in\mathbb{N}\times\mathbb{N}_0\\ 2t+3u=k}} (-1)^{k+1}2^{2k}a_k(t,u) + \sum_{\substack{(t,u)\in\mathbb{N}_0\times\mathbb{N}\\ 2t+3u=k}} (-1)^{k+1}2^{2k-1}3ua_k(t,u)\\
&= (-1)^{k+1}2^{2k}\sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} ta_k(t,u) + (-1)^{k+1}2^{2k-1}3\sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} ua_k(t,u)\\
&= (-1)^{k+1}2^{2k-1}\sum_{\substack{(t,u)\in\mathbb{N}_0^2\\ 2t+3u=k}} (2t+3u)a_k(t,u)\\
&= (-1)^{k+1}2^{2k-1}k,
\end{aligned}$$

by (1.24). This proves (1.31).

5. Proof of Theorem 1.4

We prove (1.41) and (1.42). Formulae (1.37)–(1.40) can be proved using similar techniques. For $k\in\mathbb{N}$ with $k\geqslant 2$ and $j\in\{0,1,\dots,k\}$ we have

$$\begin{aligned}
e_2(k,k-j) &= \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7\\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k\\ t_2+2t_3+u_2+2u_3+3u_4=k-j}} (-1)^{t_2+u_1+u_4}2^{-u_2-u_3}3^{u_2+u_3}\\
&\quad\times\binom{t_1+t_2+t_3}{t_1,t_2,t_3}\binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4}a_k(t_1+t_2+t_3,u_1+u_2+u_3+u_4)\\
&= \sum_{\substack{(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\in\mathbb{N}_0^7\\ 2t_1+2t_2+2t_3+3u_1+3u_2+3u_3+3u_4=k\\ 2t_1+t_2+3u_1+2u_2+u_3=j}} (-1)^{t_2+u_1+u_4}2^{-u_2-u_3}3^{u_2+u_3}\\
&\quad\times\binom{t_1+t_2+t_3}{t_1,t_2,t_3}\binom{u_1+u_2+u_3+u_4}{u_1,u_2,u_3,u_4}a_k(t_1+t_2+t_3,u_1+u_2+u_3+u_4).
\end{aligned}$$

In this sum we map $(t_1,t_2,t_3,u_1,u_2,u_3,u_4)\to(t_3,t_2,t_1,u_4,u_3,u_2,u_1)$. The quantities $t_2+u_1+u_4$, u_2+u_3, $t_1+t_2+t_3$ and $u_1+u_2+u_3+u_4$ are preserved under this transformation, and the condition $2t_1+t_2+3u_1+2u_2+u_3=j$ becomes $t_2+2t_3+u_2+2u_3+3u_4=j$. Thus $e_2(k,k-j)=e_2(k,j)$ as asserted.

Also from Theorem 1.2 with $r = 2$ we have

$$\sum_{j=0}^{k} e_2(k, j)\varphi^{4k-4j}(q)\varphi^{4j}(-q) = E_k\bigl(q^2\bigr).$$

Taking $q = 0$, we obtain (as $\varphi(0) = 1$ and $E_k(0) = 1$)

$$\sum_{j=0}^{k} e_2(k, j) = 1,$$

which is (1.42).

6. Proof of Theorem 1.5

We prove (1.44). Formulae (1.43), (1.45) and (1.46) can be proved by similar techniques. By (1.28) we have

$$\begin{aligned} e_4(k, 1) = {} & \sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2 \\ 2t_1+2+3u_1=k}} (-1)^{u_1} 2^{-4t_1-3-6u_1} 7 \binom{t_1+1}{t_1, 1, 0}\binom{u_1}{u_1, 0, 0, 0} a_k(t_1+1, u_1) \\ & + \sum_{\substack{(t_1,u_1)\in\mathbb{N}_0^2 \\ 2t_1+3u_1+3=k}} (-1)^{u_1} 2^{-4t_1-6u_1-6} 3\cdot 11 \binom{t_1}{t_1, 0, 0}\binom{u_1+1}{u_1, 1, 0, 0} a_k(t_1, u_1+1) \\ = {} & (-1)^k 2^{-2k+1} 7 \sum_{\substack{(t,u)\in\mathbb{N}\times\mathbb{N}_0 \\ 2t+3u=k}} t a_k(t, u) \\ & + (-1)^{k+1} 2^{-2k} 33 \sum_{\substack{(t,u)\in\mathbb{N}_0\times\mathbb{N} \\ 2t+3u=k}} u a_k(t, u) \\ = {} & (-1)^k 2^{-2k} \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} (14t - 33u) a_k(t, u). \end{aligned}$$

Thus

$$\sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} (14t - 33u) a_k(t, u) = (-1)^k 2^{2k} e_4(k, 1).$$

Then, appealing to (1.24) and (1.25), we obtain

$$\begin{aligned} k &= k \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) \\ &= \sum_{\substack{(t,u)\in\mathbb{N}_0^2 \\ 2t+3u=k}} (2t + 3u) a_k(t, u) \end{aligned}$$

$$= 3 \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} (10t - 21u) a_k(t,u)$$

$$- 2 \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} (14t - 33u) a_k(t,u)$$

$$= -\frac{k}{2B_{2k}} + (-1)^{k+1} 2^{2k+1} e_4(k,1)$$

so

$$e_4(k,1) = \frac{(-1)^{k+1} k(1 + 2B_{2k})}{2^{2k+2} B_{2k}},$$

which is (1.44).

7. Proof of Theorem 1.6

For $k \in \mathbb{N}$ with $k \geqslant 2$ we have by (1.47) and Theorems 1.2–1.5

$$\sum_{j=1}^{k} f(k,j) \varphi^{4k-4j}(q) \varphi^{4j}(-q) = \sum_{j=1}^{k} \left(e_1(k,j) - \left(2^{2k}+2\right) e_2(k,j) + 2^{2k+1} e_4(k,j)\right) \varphi^{4k-4j}(q) \varphi^{4j}(-q)$$

$$= E_k(q) - \left(2^{2k}+2\right) E_k\left(q^2\right) + 2^{2k+1} E_k\left(q^4\right),$$

as

$$e_1(k,0) - \left(2^{2k}+2\right) e_2(k,0) + 2^{2k+1} e_4(k,0)$$

$$= (-1)^k 2^{2k} - \left(2^{2k}+2\right)(-1)^k + 2(-1)^k$$

$$= 0.$$

It remains to treat the case $k = 1$. By [3, pp. 125, 128] we have

$$E_1(q) = (1 - 5x) z^2 + 12x(1-x) z \frac{dz}{dx}, \tag{7.1}$$

$$E_1\left(q^2\right) = (1 - 2x) z^2 + 6x(1-x) z \frac{dz}{dx}, \tag{7.2}$$

$$E_1\left(q^4\right) = \left(1 - \frac{5}{4}x\right) z^2 + 3x(1-x) z \frac{dz}{dx}. \tag{7.3}$$

Thus, by (7.1)–(7.3), (3.1) and (1.48), we have

$$E_1(q) - 6E_1\left(q^2\right) + 8E_1\left(q^4\right) = 3(1-x) z^2 = 3\varphi^4(-q) = f(1,1) \varphi^4(-q),$$

as required.

8. Proof of Theorem 1.7

For $k \in \mathbb{N}$ with $k \geqslant 2$ we have by Theorem 1.2 and (1.49)

$$\begin{aligned}
&E_k(q) - \left(2^{2k-1}+1\right)E_k\left(q^2\right) + 2^{2k-1}E_k\left(q^4\right) \\
&\quad = \sum_{j=0}^{k}\left(e_1(k,j) - \left(2^{2k-1}+1\right)e_2(k,j) + 2^{2k-1}e_4(k,j)\right)\varphi^{4k-4j}(q)\varphi^{4j}(-q) \\
&\quad = \sum_{j=0}^{k} g(k,j)\varphi^{4k-4j}(q)\varphi^{4j}(-q).
\end{aligned}$$

It remains to consider the case $k=1$. We have by (7.1)–(7.3), (3.1) and (1.50)

$$\begin{aligned}
E_1(q) - 3E_1\left(q^2\right) + 2E_1\left(q^4\right) &= -\frac{3}{2}xz^2 \\
&= -\frac{3}{2}\varphi^4(q) + \frac{3}{2}\varphi^4(-q) \\
&= g(1,0)\varphi^4(q) + g(1,1)\varphi^4(-q),
\end{aligned}$$

as required.

9. Proof of Theorem 1.8

This theorem follows from (1.47), (1.48) and Theorems 1.3–1.5.

10. Proof of Theorem 1.9

Eqs. (1.57), (1.58) and (1.60) follow from (1.49), (1.50) and Theorems 1.3–1.5.
Appealing to (1.26)–(1.28), we obtain after some calculation

$$e_1(k,j) + e_1(k,k-j) - \left(2^{2k}+2\right)e_2(k,j) + 2^{2k}e_4(k,j) = 0, \quad k \in \mathbb{N},\ k \geqslant 2,\ j \in \{0,1,\ldots,k\}.$$

Then, by (1.41), (1.45) and (1.49), we have

$$\begin{aligned}
g_1(k,k-j) &= e_1(k,k-j) - \left(2^{2k-1}+1\right)e_2(k,k-j) + 2^{2k-1}e_4(k,k-j) \\
&= e_1(k,k-j) - \left(2^{2k-1}+1\right)e_2(k,j) + 2^{2k-1}e_4(k,j) \\
&= -e_1(k,j) + \left(2^{2k-1}+1\right)e_2(k,j) - 2^{2k-1}e_4(k,j) \\
&= -g_1(k,j),
\end{aligned}$$

which proves (1.59).

11. Proof of Theorem 1.10

For $n \in \mathbb{N}$ we define

$$P_n(q) := \frac{(1-2^n)B_n}{2n} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^{n-1} q^m}{1+q^m}. \tag{11.1}$$

For $k \in \mathbb{N}$ we have

$$\begin{aligned} P_{2k}(q) + \frac{(2^{2k}-1)B_{2k}}{4k} &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^{2k-1} q^m}{1+q^m} \\ &= \sum_{m,r=1}^{\infty} (-1)^{m+r} m^{2k-1} q^{mr} \\ &= \sum_{N=1}^{\infty} \Bigg(\sum_{\substack{m \in \mathbb{N} \\ m \mid N}} (-1)^{m+(N/m)} m^{2k-1} \Bigg) q^N \\ &= \sum_{N=1}^{\infty} \left(\sigma_{2k-1}(N) - \left(2^{2k}+2\right)\sigma_{2k-1}\left(\frac{N}{2}\right) + 2^{2k+1}\sigma_{2k-1}\left(\frac{N}{4}\right)\right) q^N \\ &= \left(2^{2k}-1\right)\frac{B_{2k}}{4k} - \frac{B_{2k}}{4k}\left(E_k(q) - \left(2^{2k}+2\right)E_k\left(q^2\right) + 2^{2k+1}E_k\left(q^4\right)\right), \end{aligned}$$

so that

$$P_{2k}(q) = -\frac{B_{2k}}{4k}\left(E_k(q) - \left(2^{2k}+2\right)E_k\left(q^2\right) + 2^{2k+1}E_k\left(q^4\right)\right), \quad k \in \mathbb{N}. \tag{11.2}$$

From (11.2) and Theorem 1.6 we deduce

$$P_{2k}(q) = -\frac{B_{2k}}{4k} \sum_{j=1}^{k} f(k,j) \varphi^{4k-4j}(q) \varphi^{4j}(-q), \quad k \in \mathbb{N}, \tag{11.3}$$

where the rational numbers $f(k,j)$ are defined in (1.47). Appealing to the values of $f(k,j)$ given in Table 4 we obtain $P_2(q), P_4(q), \ldots, P_{16}(q)$, see Table 8.

Recall from (1.10) that for $k \in \mathbb{N}_0$

$$F_k(q) = 1 + \frac{4}{E_{2k}} \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^{2k} \Bigg) q^n. \tag{11.4}$$

From (1.7) we have (as $E_0 = 1$)

$$F_0(q) = \varphi^2(q).$$

Table 8
Values of $P_{2k}(q)$, $k = 1, 2, \dots, 8$.

$$P_2(q) = \frac{-1}{8}\varphi^4(-q),$$

$$P_4(q) = \frac{1}{8}\varphi^4(q)\varphi^4(-q) - \frac{1}{16}\varphi^8(-q),$$

$$P_6(q) = \frac{-1}{8}\varphi^8(q)\varphi^4(-q) + \frac{1}{8}\varphi^4(q)\varphi^8(-q) - \frac{1}{8}\varphi^{12}(-q),$$

$$P_8(q) = \frac{1}{8}\varphi^{12}(q)\varphi^4(-q) - \frac{3}{16}\varphi^8(q)\varphi^8(-q) + \frac{9}{8}\varphi^4(q)\varphi^{12}(-q) - \frac{17}{32}\varphi^{16}(-q),$$

$$P_{10}(q) = \frac{-1}{8}\varphi^{16}(q)\varphi^4(-q) + \frac{1}{4}\varphi^{12}(q)\varphi^8(-q) - \frac{39}{4}\varphi^8(q)\varphi^{12}(-q) + \frac{77}{8}\varphi^4(q)\varphi^{16}(-q) - \frac{31}{8}\varphi^{20}(-q),$$

$$P_{12}(q) = \frac{1}{8}\varphi^{20}(q)\varphi^4(-q) - \frac{5}{16}\varphi^{16}(q)\varphi^8(-q) + \frac{347}{4}\varphi^{12}(q)\varphi^{12}(-q) - \frac{2077}{16}\varphi^8(q)\varphi^{16}(-q) + \frac{1037}{8}\varphi^4(q)\varphi^{20}(-q) - \frac{691}{16}\varphi^{24}(-q),$$

$$P_{14}(q) = \frac{-1}{8}\varphi^{24}(q)\varphi^4(-q) + \frac{3}{8}\varphi^{20}(q)\varphi^8(-q) - \frac{6231}{8}\varphi^{16}(q)\varphi^{12}(-q) + \frac{12457}{8}\varphi^{12}(q)\varphi^{16}(-q) - \frac{25341}{8}\varphi^8(q)\varphi^{20}(-q)$$
$$+ \frac{19113}{8}\varphi^4(q)\varphi^{24}(-q) - \frac{5461}{8}\varphi^{28}(-q),$$

$$P_{16}(q) = \frac{1}{8}\varphi^{28}(q)\varphi^4(-q) - \frac{7}{16}\varphi^{24}(q)\varphi^8(-q) + \frac{56055}{8}\varphi^{20}(q)\varphi^{12}(-q) - \frac{560515}{32}\varphi^{16}(q)\varphi^{16}(-q) + \frac{576887}{8}\varphi^{12}(q)\varphi^{20}(-q)$$
$$- \frac{1450407}{16}\varphi^8(q)\varphi^{24}(-q) + \frac{464785}{8}\varphi^4(q)\varphi^{28}(-q) - \frac{929569}{64}\varphi^{32}(-q).$$

Now

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^{2k}\Bigg) q^n = \sum_{d,e=1}^{\infty} \left(\frac{-4}{d}\right) d^{2k} q^{de}$$
$$= \sum_{d=1}^{\infty} \left(\frac{-4}{d}\right) d^{2k} \frac{q^d}{1-q^d}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)^{2k} q^{2n-1}}{1-q^{2n-1}}$$

so that by (11.4) we have

$$F_k(q) = 1 + \frac{4}{E_{2k}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)^{2k} q^{2n-1}}{1-q^{2n-1}}, \quad k \in \mathbb{N}_0. \tag{11.5}$$

Next we set

$$Q_{2k}(q) := \frac{E_{2k} F_k(q)}{E_0 F_0(q)}, \quad k \in \mathbb{N}_0, \tag{11.6}$$

so that

$$Q_0(q) = 1. \tag{11.7}$$

By (11.5) and (11.6) we have

$$Q_{2k}(q) = \frac{\frac{E_{2k}}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)^{2k} q^{2n-1}}{1-q^{2n-1}}}{\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n-1}}{1-q^{2n-1}}}, \quad k \in \mathbb{N}_0. \tag{11.8}$$

From [2, Part III, Eq. (35.6), p. 63] we have

$$Q_{2k}(q) = \sum_{l=1}^{\infty} \binom{2k-1}{2l-1} 2^{2l+1} P_{2l}(q) Q_{2k-2l}(q), \quad k \in \mathbb{N}. \tag{11.9}$$

Eq. (11.9) forms the basis for the inductive step. Moreover, (11.9) is what led us to the recursion in (1.61). Appealing to (11.6), we deduce

$$F_k(q) = \frac{1}{E_{2k}} \sum_{l=1}^{k} \binom{2k-1}{2l-1} 2^{2l+1} P_{2l}(q) E_{2k-2l} F_{k-l}(q), \quad k \in \mathbb{N},$$

so that

$$F_k(q) = \frac{1}{E_{2k}} \sum_{l=0}^{k-1} E_{2l} \binom{2k-1}{2l} 2^{2k-2l+1} P_{2k-2l}(q) F_l(q), \quad k \in \mathbb{N}. \tag{11.10}$$

From (11.3) and (11.10) we deduce by induction that

$$F_k(q) = \sum_{j=0}^{k-1} A(k,j) \varphi^{4j+2}(q) \varphi^{4k-4j}(-q),$$

where the $A(k,j)$ satisfy (1.61).

12. Proof of Theorem 1.11

For $n \in \mathbb{N}$ we define

$$P_n^*(q) := \sum_{m=1}^{\infty} \frac{(2m-1)^{n-1} q^{2m-1}}{1-q^{4m-2}}. \tag{12.1}$$

Our first task is to prove that

$$P_{2k}^*(q) = -\frac{B_{2k}}{4k} \left(E_k(q) - \left(2^{2k-1}+1\right) E_k\left(q^2\right) + 2^{2k-1} E_k\left(q^4\right) \right), \quad k \in \mathbb{N}. \tag{12.2}$$

We have

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{(2m-1)^{2k-1} q^{2m-1}}{1-q^{4m-2}} &= \sum_{m,r=1}^{\infty} (2m-1)^{2k-1} q^{(2m-1)(2r-1)} \\
&= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} \left(\sum_{\substack{m,r \in \mathbb{N} \\ (2m-1)(2r-1)=N}} (2m-1)^{2k-1} \right) q^N \\
&= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} \left(\sum_{\substack{d \in \mathbb{N} \\ d \mid N}} d^{2k-1} \right) q^N \\
&= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} \sigma_{2k-1}(N) q^N
\end{aligned}$$

$$= \sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^N - \sum_{\substack{N=1 \\ N \text{ even}}}^{\infty} \sigma_{2k-1}(N)q^N$$

$$= \sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^N - \sum_{N=1}^{\infty} \sigma_{2k-1}(2N)q^{2N}.$$

Using the elementary identity

$$\sigma_{2k-1}(2N) - \left(2^{2k-1}+1\right)\sigma_{2k-1}(N) + 2^{2k-1}\sigma_{2k-1}(N/2) = 0, \quad N \in \mathbb{N},$$

we obtain

$$\sum_{N=1}^{\infty} \sigma_{2k-1}(2N)q^{2N} = \sum_{N=1}^{\infty} \left(\left(2^{2k-1}+1\right)\sigma_{2k-1}(N)q^{2N} - 2^{2k-1}\sigma_{2k-1}(N/2)q^{2N}\right)$$

$$= \left(2^{2k-1}+1\right)\sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^{2N} - 2^{2k-1}\sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^{4N}$$

$$= \left(2^{2k-1}+1\right)\frac{B_{2k}}{4k}\left(1 - E_k\left(q^2\right)\right) - 2^{2k-1}\frac{B_{2k}}{4k}\left(1 - E_k\left(q^4\right)\right),$$

by (1.21). Thus

$$P^*_{2k}(q) = \frac{B_{2k}}{4k}\left(\left(1 - E_k(q)\right) - \left(2^{2k-1}+1\right)\left(1 - E_k\left(q^2\right)\right) + 2^{2k-1}\left(1 - E_k\left(q^4\right)\right)\right)$$

$$= -\frac{B_{2k}}{4k}\left(E_k(q) - \left(2^{2k-1}+1\right)E_k\left(q^2\right) + 2^{2k-1}E_k\left(q^4\right)\right),$$

which is (12.2).

Then, appealing to Theorem 1.7, we obtain

$$P^*_{2k}(q) = -\frac{B_{2k}}{4k}\sum_{j=0}^{k} g(k,j)\varphi^{4k-4j}(q)\varphi^{4j}(-q), \quad k \in \mathbb{N}. \tag{12.3}$$

Next, for $n \in \mathbb{N}_0$, we define

$$S^*_{2n}(q) := \sum_{m=1}^{\infty}\left(\sum_{\substack{d \in \mathbb{N} \\ d \mid m}} \left(\frac{-4}{m/d}\right) d^{2n}\right) q^m. \tag{12.4}$$

In particular by (1.7) we have

$$S^*_0(q) = \frac{1}{4}\varphi^2(q) - \frac{1}{4}. \tag{12.5}$$

Our next task is to show that for $n \in \mathbb{N}$ we have

$$S^*_{2n}(q) = \sum_{m=1}^{\infty} m^{2n}\frac{q^m}{1+q^{2m}}. \tag{12.6}$$

We have

$$\begin{aligned}
\sum_{m=1}^{\infty}\left(\sum_{\substack{d\in\mathbb{N}\\ d\mid m}}\left(\frac{-4}{m/d}\right)d^{2n}\right)q^m &= \sum_{d,e=1}^{\infty}\left(\frac{-4}{e}\right)d^{2n}q^{de}\\
&= \sum_{d=1}^{\infty}d^{2n}\sum_{e=1}^{\infty}\left(\frac{-4}{e}\right)q^{de}\\
&= \sum_{d=1}^{\infty}d^{2n}\left(\sum_{\substack{e=1\\ e\equiv 1 \pmod 4}}^{\infty}q^{de}-\sum_{\substack{e=1\\ e\equiv 3 \pmod 4}}^{\infty}q^{de}\right)\\
&= \sum_{d=1}^{\infty}d^{2n}\left(\sum_{k=0}^{\infty}q^{d(4k+1)}-\sum_{k=0}^{\infty}q^{d(4k+3)}\right)\\
&= \sum_{d=1}^{\infty}d^{2n}\left(\frac{q^d}{1-q^{4d}}-\frac{q^{3d}}{1-q^{4d}}\right)\\
&= \sum_{d=1}^{\infty}d^{2n}\frac{q^d}{1+q^{2d}},
\end{aligned}$$

proving (12.6).

Next, by (12.6), we obtain for $\theta\in\mathbb{R}$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}\theta^{2n}}{(2n)!}S_{2n}^*(q) &= \sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}\theta^{2n}}{(2n)!}\sum_{k=1}^{\infty}\frac{k^{2n}q^k}{1+q^{2k}}\\
&= \sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{n=0}^{\infty}\frac{(-1)^n(2k\theta)^{2n}}{(2n)!},
\end{aligned}$$

that is

$$\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}\theta^{2n}}{(2n)!}S_{2n}^*(q)=\sum_{k=1}^{\infty}\frac{q^k\cos 2k\theta}{1+q^{2k}}. \tag{12.7}$$

Also

$$\begin{aligned}
2\sum_{k=1}^{\infty}\frac{q^{2k-1}\sin^2(2k-1)\theta}{(2k-1)(1-q^{4k-2})} &= \sum_{k=1}^{\infty}\frac{q^{2k-1}}{(2k-1)(1-q^{4k-2})}\big(1-\cos(4k-2)\theta\big)\\
&= \sum_{k=1}^{\infty}\frac{q^{2k-1}}{(2k-1)(1-q^{4k-2})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4k-2)^{2n}\theta^{2n}}{(2n)!}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\sum_{k=1}^{\infty}\frac{q^{2k-1}(4k-2)^{2n}}{(2k-1)(1-q^{4k-2})}\right)\theta^{2n}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n)!}\left(\sum_{k=1}^{\infty}(2k-1)^{2n-1}\frac{q^{2k-1}}{1-q^{4k-2}}\right)\theta^{2n},
\end{aligned}$$

that is

$$2\sum_{k=1}^{\infty}\frac{q^{2k-1}\sin^2(2k-1)\theta}{(2k-1)(1-q^{4k-2})}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n)!}P_{2n}^*(q)\theta^{2n}, \tag{12.8}$$

by (12.1).

From [2, Part III, p. 56] we have one of the key steps in this proof, namely,

$$\frac{1+4\sum_{k=1}^{\infty}\frac{q^k\cos 2k\theta}{1+q^{2k}}}{\varphi^2(q)}=e^{-8\sum_{k=1}^{\infty}\frac{q^{2k-1}\sin^2(2k-1)\theta}{(2k-1)(1-q^{4k-2})}}. \tag{12.9}$$

Appealing to (12.7)–(12.9) we obtain

$$1+4\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}S_{2n}^*(q)}{(2n)!}\theta^{2n}=\varphi^2(q)\ e^{-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n)!}P_{2n}^*(q)\theta^{2n}}. \tag{12.10}$$

Differentiating (12.10) with respect to θ, we deduce

$$\begin{aligned}4\sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}S_{2n}^*(q)}{(2n-1)!}\theta^{2n-1}&=\varphi^2(q)\left(-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n-1)!}P_{2n}^*(q)\theta^{2n-1}\right)e^{-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n)!}P_{2n}^*(q)\theta^{2n}}\\&=\left(-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n-1)!}P_{2n}^*(q)\theta^{2n-1}\right)\left(1+4\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}S_{2n}^*(q)\theta^{2n}\right).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}S_{2n}^*(q)}{(2n-1)!}\theta^{2n-1}&=-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}P_{2n}^*(q)}{(2n-1)!}\theta^{2n-1}\\&\quad-4\sum_{n=1}^{\infty}\sum_{\substack{m\geqslant 1,\,l\geqslant 0\\ l+m=n}}\frac{(-1)^{n-1}2^{2n}P_{2m}^*(q)S_{2l}^*(q)}{(2m-1)!(2l)!}\theta^{2n-1}.\end{aligned}$$

Equating coefficients of θ^{2n-1} $(n\in\mathbb{N})$, we obtain

$$\frac{(-1)^n2^{2n}S_{2n}^*(q)}{(2n-1)!}=\frac{(-1)^n2^{2n}P_{2n}^*(q)}{(2n-1)!}+4(-1)^n2^{2n}\sum_{l=0}^{n-1}\frac{P_{2n-2l}^*(q)S_{2l}^*(q)}{(2l)!(2n-2l-1)!}$$

so that

$$S_{2n}^*(q)=P_{2n}^*(q)+4\sum_{l=0}^{n-1}\binom{2n-1}{2l}P_{2n-2l}^*(q)S_{2l}^*(q).$$

By (12.5) we have $1+4S_0^*(q)=\varphi^2(q)$ so (as $S_{2n}^*(q)=G_n(q)$)

$$G_n(q)=\varphi^2(q)P_{2n}^*(q)+4\sum_{l=1}^{n-1}\binom{2n-1}{2l}P_{2n-2l}^*(q)G_l(q). \tag{12.11}$$

From (12.3) and (12.11) we deduce by induction that

$$G_k(q) = \sum_{j=0}^{k} B(k, j)\varphi^{4j+2}(q)\varphi^{4k-4j}(-q),$$

where the $B(k, j)$ are given recursively by (1.63) and (1.64).

13. Sums of ten squares

As an application of Theorems 1.10 and 1.11 we recover the classical formula [4] for the number of representations of n ($n \in \mathbb{N}$) as the sum of ten squares, that is for

$$r_{10}(n) = \operatorname{card}\{(x_1, \ldots, x_{10}) \in \mathbb{Z}^{10} \mid n = x_1^2 + \cdots + x_{10}^2\}. \tag{13.1}$$

From Theorems 1.10 and 1.11 with $k = 2$ we have

$$1 + \frac{4}{5}\sum_{n=1}^{\infty}\Bigl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \Bigl(\frac{-4}{d}\Bigr) d^4\Bigr) q^n = \frac{1}{5}\varphi^2(q)\varphi^8(-q) + \frac{4}{5}\varphi^6(q)\varphi^4(-q) \tag{13.2}$$

and

$$\sum_{n=1}^{\infty}\Bigl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \Bigl(\frac{-4}{n/d}\Bigr) d^4\Bigr) q^n = \frac{1}{64}\varphi^2(q)\varphi^8(-q) - \frac{3}{32}\varphi^6(q)\varphi^4(-q) + \frac{5}{64}\varphi^{10}(q). \tag{13.3}$$

From (13.2) and (13.3) we obtain

$$\begin{aligned}\varphi^{10}(q) = 1 &+ \frac{2}{5}\bigl(\varphi^6(q)\varphi^4(-q) - \varphi^2(q)\varphi^8(-q)\bigr) \\ &+ \frac{4}{5}\sum_{n=1}^{\infty}\Bigl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \Bigl(\frac{-4}{d}\Bigr) d^4\Bigr) q^n + \frac{64}{5}\sum_{n=1}^{\infty}\Bigl(\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \Bigl(\frac{-4}{n/d}\Bigr) d^4\Bigr) q^n.\end{aligned} \tag{13.4}$$

Now, motivated by Eq. (9.15) on p. 119 of [4], we have

$$\begin{aligned}\varphi^6(q)\varphi^4(-q) - \varphi^2(q)\varphi^8(-q) &= \varphi^2(q)\varphi^4(-q)\bigl(\varphi^4(q) - \varphi^4(-q)\bigr) \\ &= \varphi^2(q)\varphi^4(-q)16q\psi^4(q^2),\end{aligned}$$

by [3, Eq. (3.6.8), p. 72], where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)}.$$

As

$$\varphi(q) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}$$

and

$$\varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{2n})},$$

we obtain

$$\varphi^6(q)\varphi^4(-q) - \varphi^2(q)\varphi^8(-q) = 16q \prod_{n=1}^{\infty} (1-q^n)^4 (1-q^{2n})^2 (1-q^{4n})^4.$$

Define $w(n) \in \mathbb{Z}$ $(n \in \mathbb{N})$ by

$$q \prod_{n=1}^{\infty} (1-q^n)^4 (1-q^{2n})^2 (1-q^{4n})^4 = \sum_{n=1}^{\infty} w(n) q^n.$$

Then

$$\sum_{n=0}^{\infty} r_{10}(n) q^n = \varphi^{10}(q)$$
$$= 1 + \sum_{n=1}^{\infty} \left(\frac{4}{5} \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^4 + \frac{64}{5} \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^4 + \frac{32}{5} w(n) \right) q^n$$

so that

$$r_{10}(n) = \frac{4}{5} \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^4 + \frac{64}{5} \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^4 + \frac{32}{5} w(n), \quad n \in \mathbb{N}.$$

This is the fifth formula in Eq. (9.19) on p. 121 of [4].

n	$\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^4$	$\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^4$	$w(n)$	$r_{10}(n)$
1	1	1	1	20
2	1	16	−4	180
3	−80	80	0	960
4	1	256	16	3380
5	626	626	−14	8424

14. Final remarks

Taking $q = 0$ in Theorems 1.10 and 1.11 we obtain for $k \in \mathbb{N}$

$$\sum_{j=0}^{k-1} A(k, j) = 1 \tag{14.1}$$

Table 9
Values of $P_{2k}^*(q)$, $k = 1, 2, \dots, 8$.

$$P_2^*(q) = \frac{1}{16}\varphi^4(q) - \frac{1}{16}\varphi^4(-q),$$

$$P_4^*(q) = \frac{1}{32}\varphi^8(q) - \frac{1}{32}\varphi^8(-q),$$

$$P_6^*(q) = \frac{1}{16}\varphi^{12}(q) - \frac{1}{8}\varphi^8(q)\varphi^4(-q) + \frac{1}{8}\varphi^4(q)\varphi^8(-q) - \frac{1}{16}\varphi^{12}(-q),$$

$$P_8^*(q) = \frac{17}{64}\varphi^{16}(q) - \frac{1}{2}\varphi^{12}(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^{12}(-q) - \frac{17}{64}\varphi^{16}(-q),$$

$$P_{10}^*(q) = \frac{31}{16}\varphi^{20}(q) - \frac{39}{8}\varphi^{16}(q)\varphi^4(-q) + 5\varphi^{12}(q)\varphi^8(-q) - 5\varphi^8(q)\varphi^{12}(-q) + \frac{39}{8}\varphi^4(q)\varphi^{16}(-q) - \frac{31}{16}\varphi^{20}(-q),$$

$$P_{12}^*(q) = \frac{691}{32}\varphi^{24}(q) - \frac{259}{4}\varphi^{20}(q)\varphi^4(-q) + \frac{259}{4}\varphi^{16}(q)\varphi^8(-q) - \frac{259}{4}\varphi^8(q)\varphi^{16}(-q) + \frac{259}{4}\varphi^4(q)\varphi^{20}(-q) - \frac{691}{32}\varphi^{24}(q),$$

$$P_{14}^*(q) = \frac{5461}{16}\varphi^{28}(q) - \frac{9557}{8}\varphi^{24}(q)\varphi^4(-q) + 1584\varphi^{20}(q)\varphi^8(-q) - 1168\varphi^{16}(q)\varphi^{12}(-q) + 1168\varphi^{12}(q)\varphi^{16}(-q)$$
$$- 1584\varphi^8(q)\varphi^{20}(-q) + \frac{9557}{8}\varphi^4(q)\varphi^{24}(-q) - \frac{5461}{16}\varphi^{28}(-q),$$

$$P_{16}^*(q) = \frac{929569}{128}\varphi^{32}(q) - 29049\varphi^{28}(q)\varphi^4(-q) + 45325\varphi^{24}(q)\varphi^8(-q) - 32552\varphi^{20}(q)\varphi^{12}(-q) + 32552\varphi^{12}(q)\varphi^{20}(-q)$$
$$- 45325\varphi^8(q)\varphi^{24}(-q) + 29049\varphi^4(q)\varphi^{28}(-q) - \frac{929569}{128}\varphi^{32}(-q).$$

and

$$\sum_{j=0}^{k} B(k, j) = 0. \tag{14.2}$$

We leave it to the reader to show that

$$A(k, 0) = \frac{(-1)^k}{E_{2k}}, \qquad A(k, k-1) = \frac{(-1)^k 2^{2(k-1)}}{E_{2k}}, \tag{14.3}$$

and

$$B(k, 0) = \frac{(-1)^k}{2^{2k+2}}, \qquad B(k, k) = \frac{(-1)^k E_{2k}}{2^{2k+2}}. \tag{14.4}$$

The referee has pointed out that Tables 4–9 suggest some conjectures concerning the signs of the quantities appearing in them. We state these conjectures and comment briefly upon each of them.

For a real number x, we define as usual

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

We recall that for $k \in \mathbb{N}$ we have

$$\operatorname{sgn} B_{2k} = (-1)^{k-1}, \qquad \operatorname{sgn} E_{2k} = (-1)^k, \qquad (-1)^{k-1} B_{2k} \geqslant \frac{1}{2(2^{2k}-1)}. \tag{14.5}$$

Conjecture 14.1. *For $j, k \in \mathbb{N}$ with $1 \leqslant j \leqslant k$ we have*

$$\operatorname{sgn} f(k, j) = (-1)^{j-1}.$$

From (1.53)–(1.55) and (14.5), we see that Conjecture 14.1 is true for $j = 1$, $k - 1$ and k. We have verified Conjecture 14.1 numerically for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$ satisfying $k \leqslant 20$ and $1 \leqslant j \leqslant k$.

Conjecture 14.2. *For $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $0 \leqslant j \leqslant k$ we have for $k \equiv 1 \pmod 2$*

$$\operatorname{sgn} g(k, j) = (-1)^{j-1},$$

and for $k \equiv 0 \pmod 2$

$$\operatorname{sgn} g(k, j) = \begin{cases} (-1)^j, & \text{if } 0 \leqslant j \leqslant (k/2) - 1, \\ 0, & \text{if } j = k/2, \\ (-1)^{j+1}, & \text{if } (k/2) + 1 \leqslant j \leqslant k. \end{cases}$$

By (1.57)–(1.59) and (14.5), Conjecture 14.2 is true for $j = 0$, 1, $k - 1$ and k, as well as for $j = k/2$ when k is even. Further, when k is even, the conjecture for $(k/2) + 1 \leqslant j \leqslant k$ follows from that for $0 \leqslant j \leqslant (k/2) - 1$ by (1.59). We have verified Conjecture 14.2 numerically for all $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ satisfying $k \leqslant 20$ and $0 \leqslant j \leqslant k$.

Conjecture 14.3. *For $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $0 \leqslant j \leqslant k - 1$ we have*

$$A(k, j) > 0.$$

We have verified this conjecture for $k \leqslant 30$. It is not clear whether it follows from Conjecture 14.1 and (1.61).

Conjecture 14.4. *For $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $0 \leqslant j \leqslant k$ we have*

$$\operatorname{sgn} B(k, j) = (-1)^{j+k}.$$

We have verified this conjecture for all $k \leqslant 30$. It is not apparent whether it follows from Conjecture 14.2 and (1.62).

Assuming the truth of Conjecture 14.1, we see from (14.5) that the sign of the coefficient of $\varphi^{4k-4j}(q)\varphi^{4j}(-q)$ in (11.3) is $(-1)^{j+k-1}$. Similarly, assuming the truth of Conjecture 14.2, we see from (14.5) that the sign of the coefficient of $\varphi^{4k-4j}(q)\varphi^{4j}(-q)$ in (12.3) is also $(-1)^{j+k-1}$.

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