International Journal of Number Theory
Vol. 5, No. 1 (2009) 13–40

World Scientific
www.worldscientific.com

THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER BY CERTAIN QUATERNARY QUADRATIC FORMS

AYŞE ALACA*, ŞABAN ALACA†, MATHIEU F. LEMIRE‡
and KENNETH S. WILLIAMS§

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics, Carleton University
Ottawa, Ontario, K1S 5B6 Canada
**aalaca@connect.carleton.ca*
†salaca@connect.carleton.ca
‡mathieul@connect.carleton.ca
§kwilliam@connect.carleton.ca

Received 13 February 2007
Accepted 25 July 2007

Some theta function identities are proved and used to give formulae for the number of representations of a positive integer by certain quaternary forms $x^2 + ey^2 + fz^2 + gt^2$ with $e, f, g \in \{1, 2, 4, 8\}$.

Keywords: Quaternary quadratic forms; theta functions.

Mathematics Subject Classification 2000: 11E20, 11E25

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a, b, c, d; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}. \quad (1.1)$$

Clearly

$$N(a, b, c, d; 0) = 1. \quad (1.2)$$

Also

$$\sum_{n=0}^{\infty} N(a, b, c, d; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d), \quad (1.3)$$

where $\varphi(q)$ denotes Jacobi's theta function, namely

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \quad |q| < 1. \quad (1.4)$$

The basic properties of $\varphi(q)$ are

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{1.5}$$

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{1.6}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{1.7}$$

see, for example, [2, p. 40]. In Sec. 2, we deduce from (1.4)–(1.7) a few identities involving φ, for example

$$2\varphi(q)\varphi(q^4) = \varphi^2(q) + \varphi^2(-q^2),$$

which will be used in Sec. 3, see Lemmas 2.1–2.4. In Sec. 3, we define

$$\alpha(q) := \varphi(q)\varphi^2(-q)\varphi(q^2)$$

and

$$\beta(q) := \varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi(q)\varphi(q^2)\varphi^2(q^4),$$

and give their basic properties, see Lemmas 3.1–3.7. In the remainder of Sec. 3, we use Lemmas 3.1–3.7 to express ten products involving φ in terms of α and β, see Theorems 3.1–3.10. In Sec. 4, we define the arithmetic functions

$$R(n) := \sum_{d \mid n} d\left(\frac{8}{d}\right), \quad n \in \mathbb{N},$$

and

$$S(n) := \sum_{d \mid n} \frac{n}{d}\left(\frac{8}{d}\right), \quad n \in \mathbb{N},$$

where d runs through the positive integers dividing n and $\left(\frac{8}{d}\right)$ is the Legendre–Jacobi–Kronecker symbol for discriminant 8, that is

$$\left(\frac{8}{d}\right) = \begin{cases} +1, & \text{if } d \equiv 1, 7 \pmod 8, \\ -1, & \text{if } d \equiv 3, 5 \pmod 8, \\ 0, & \text{if } d \equiv 0 \pmod 2. \end{cases}$$

The basic properties of $R(n)$ and $S(n)$ are given in Theorem 4.1 and Corollary 4.1. By appealing to results of Petr [14] we show that

$$\sum_{n=1}^{\infty} R(n)q^n = \frac{1}{2} - \frac{1}{2}\alpha(q)$$

and

$$\sum_{n=1}^{\infty} S(n)q^n = \frac{1}{2}\beta(q),$$

see Theorem 4.2. From these we deduce immediately the series expansions of $\alpha(q)$ and $\beta(q)$ in powers of q, namely

$$\alpha(q) = 1 - 2\sum_{n=1}^{\infty} R(n)q^n$$

and

$$\beta(q) = 2\sum_{n=1}^{\infty} S(n)q^n,$$

see Theorem 4.3. Finally, in Sec. 5, we use the results obtained in Secs. 3 and 4 to determine formulae for the number of representations of $n \in \mathbb{N}$ by each of the following ten quaternary quadratic forms:

$$\begin{array}{ll}
x^2 + y^2 + z^2 + 2t^2 & \text{(Theorem 5.1)}\\
x^2 + 2y^2 + 2z^2 + 2t^2 & \text{(Theorem 5.2)}\\
x^2 + y^2 + 2z^2 + 4t^2 & \text{(Theorem 5.3)}\\
x^2 + 2y^2 + 4z^2 + 4t^2 & \text{(Theorem 5.4)}\\
x^2 + y^2 + 4z^2 + 8t^2 & \text{(Theorem 5.5)}\\
x^2 + 4y^2 + 4z^2 + 8t^2 & \text{(Theorem 5.6)}\\
x^2 + 2y^2 + 2z^2 + 8t^2 & \text{(Theorem 5.7)}\\
x^2 + 2y^2 + 8z^2 + 8t^2 & \text{(Theorem 5.8)}\\
x^2 + 8y^2 + 8z^2 + 8t^2 & \text{(Theorem 5.9)}\\
x^2 + y^2 + z^2 + 8t^2 & \text{(Theorem 5.10)}
\end{array}$$

These formulae were stated but not proved by Liouville [4–12].

2. Identities Involving $\varphi(q)$

In this section we use (1.4) and the three basic properties of $\varphi(q)$, namely (1.5), (1.6) and (1.7), to prove further identities involving $\varphi(q^{1/2}),\ \varphi(q),\ \varphi(q^2), \ldots$. The following is an immediate consequence of (1.6)

$$\varphi(q) - \varphi(-q) = 2(\varphi(q) - \varphi(q^4)) = 2(\varphi(q^4) - \varphi(-q)),$$

which we will use on a number of occasions without comment.

Lemma 2.1. $(\varphi(q^{1/2}) - \varphi(q^2))^2 = 2\varphi(q)\varphi(q^4) - 2\varphi^2(q^4)$.

Proof. We have

$$\begin{aligned}
(\varphi(q^{1/2}) - \varphi(q^2))^2 &= \varphi^2(q^{1/2}) - 2\varphi(q^{1/2})\varphi(q^2) + \varphi^2(q^2)\\
&= \varphi^2(q^{1/2}) - 2\varphi(q^{1/2})\frac{1}{2}(\varphi(q^{1/2}) + \varphi(-q^{1/2}))\\
&\quad + \frac{1}{2}(\varphi^2(q) + \varphi^2(-q)) \quad \text{(by (1.6) and (1.7))}
\end{aligned}$$

$$
\begin{aligned}
&= -\varphi(q^{1/2})\varphi(-q^{1/2}) + \frac{1}{2}\varphi^2(q) + \frac{1}{2}\varphi^2(-q) \\
&= -\varphi^2(-q) + \frac{1}{2}\varphi^2(q) + \frac{1}{2}\varphi^2(-q) \quad \text{(by (1.5))} \\
&= \frac{1}{2}\varphi^2(q) - \frac{1}{2}\varphi^2(-q) \\
&= \frac{1}{2}\varphi^2(q) - \frac{1}{2}(2\varphi(q^4) - \varphi(q))^2 \quad \text{(by (1.6))} \\
&= 2\varphi(q)\varphi(q^4) - 2\varphi^2(q^4)
\end{aligned}
$$

as asserted. □

Lemma 2.2. $2\varphi(q)\varphi(q^4) = \varphi^2(q) + \varphi^2(-q^2)$.

Proof. We have

$$
\begin{aligned}
2\varphi(q)\varphi(q^4) &= \varphi(q)(\varphi(q) + \varphi(-q)) \quad \text{(by (1.6))} \\
&= \varphi^2(q) + \varphi(q)\varphi(-q) \\
&= \varphi^2(q) + \varphi^2(-q^2) \quad \text{(by (1.5))}
\end{aligned}
$$

as asserted. □

Lemma 2.3.

$$
\begin{aligned}
\varphi(iq) &= \varphi(q^4) + i(\varphi(q) - \varphi(q^4)), \\
\varphi(-iq) &= \varphi(q^4) - i(\varphi(q) - \varphi(q^4)).
\end{aligned}
$$

Proof. We have from (1.4)

$$
\begin{aligned}
\varphi(iq) = \sum_{n=-\infty}^{\infty} (iq)^{n^2} &= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} q^{n^2} + i \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} q^{n^2} \\
&= \sum_{n=-\infty}^{\infty} q^{4n^2} + i\left(\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{4n^2}\right) \\
&= \varphi(q^4) + i(\varphi(q) - \varphi(q^4)).
\end{aligned}
$$

Then, from (1.6), we obtain

$$
\begin{aligned}
\varphi(-iq) &= 2\varphi(q^4) - \varphi(iq) \\
&= 2\varphi(q^4) - (\varphi(q^4) + i(\varphi(q) - \varphi(q^4))) \\
&= \varphi(q^4) - i(\varphi(q) - \varphi(q^4)).
\end{aligned}
$$

□

It is convenient to set $\omega = e^{2\pi i/8}$ so that

$$\begin{cases} \omega = \dfrac{1+i}{\sqrt{2}}, \quad \omega^2 = i, \quad \omega^3 = \dfrac{-1+i}{\sqrt{2}}, \quad \omega^4 = -1, \\ \omega^5 = -\omega = \dfrac{-1-i}{\sqrt{2}}, \quad \omega^6 = -i, \quad \omega^7 = -\omega^3 = \dfrac{1-i}{\sqrt{2}}, \quad \omega^8 = 1. \end{cases} \tag{2.1}$$

Lemma 2.4.

$$\begin{aligned} \varphi(\omega q) &= \varphi(-q^4) + \omega(\varphi(q) - \varphi(q^4)), \\ \varphi(\omega^3 q) &= \varphi(-q^4) + \omega^3(\varphi(q) - \varphi(q^4)), \\ \varphi(-\omega q) &= \varphi(-q^4) - \omega(\varphi(q) - \varphi(q^4)), \\ \varphi(-\omega^3 q) &= \varphi(-q^4) - \omega^3(\varphi(q) - \varphi(q^4)). \end{aligned}$$

Proof. Let $r \in \{1, 3, 5, 7\}$. Then for $n \in \mathbb{Z}$ we have

$$\omega^{rn^2} = \omega^r, \quad \text{if } n \equiv 1 \pmod 2,$$

and

$$\omega^{rn^2} = (-1)^{n/2}, \quad \text{if } n \equiv 0 \pmod 2.$$

Hence, by (1.4), we deduce

$$\begin{aligned} \varphi(\omega^r q) &= \sum_{n=-\infty}^{\infty} (\omega^r q)^{n^2} = \sum_{\substack{n=-\infty \\ n \equiv 0 \ (\text{mod } 2)}}^{\infty} (-1)^{n/2} q^{n^2} + \omega^r \sum_{\substack{n=-\infty \\ n \equiv 1 \ (\text{mod } 2)}}^{\infty} q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2} + \omega^r \left(\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{\substack{n=-\infty \\ n \equiv 0 \ (\text{mod } 2)}}^{\infty} q^{n^2} \right) \\ &= \sum_{n=-\infty}^{\infty} (-q^4)^{n^2} + \omega^r \left(\varphi(q) - \sum_{n=-\infty}^{\infty} q^{4n^2} \right) \\ &= \varphi(-q^4) + \omega^r(\varphi(q) - \varphi(q^4)). \end{aligned}$$

The asserted results now follow using (2.1). □

3. Identities Involving $\alpha(q)$, $\beta(q)$ and $\varphi(q)$

It is convenient to set

$$\alpha(q) := \varphi(q)\varphi^2(-q)\varphi(q^2) \tag{3.1}$$

and

$$\beta(q) := \varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi(q)\varphi(q^2)\varphi^2(q^4). \tag{3.2}$$

It is easy to check that α and β satisfy the basic relations

$$\alpha(q)+\alpha(-q)=2\alpha(q^2)$$

and

$$\beta(q)+\beta(-q)=4\beta(q^2).$$

In the next few lemmas we give the properties of $\alpha(q)$ and $\beta(q)$ that we shall need.

Lemma 3.1.

$$\beta(q)-\beta(-q)=\varphi(q)\varphi(q^2)\varphi^2(q^4)-\varphi(-q)\varphi(q^2)\varphi^2(q^4).$$

Proof. Appealing to (3.2) and (1.6) we obtain

$$\begin{aligned}\beta(q)-\beta(-q) &= \varphi^2(q)\varphi(q^2)\varphi(q^4)-\varphi(q)\varphi(q^2)\varphi^2(q^4)\\ &\quad -\varphi^2(-q)\varphi(q^2)\varphi(q^4)+\varphi(-q)\varphi(q^2)\varphi^2(q^4)\\ &= (\varphi^2(q)-\varphi^2(-q))\varphi(q^2)\varphi(q^4)-(\varphi(q)-\varphi(-q))\varphi(q^2)\varphi^2(q^4)\\ &= 2(\varphi(q)-\varphi(-q))\varphi(q^2)\varphi^2(q^4)-(\varphi(q)-\varphi(-q))\varphi(q^2)\varphi^2(q^4)\\ &= (\varphi(q)-\varphi(-q))\varphi(q^2)\varphi^2(q^4),\end{aligned}$$

as asserted. □

Lemma 3.2.

$$\begin{aligned}i(\beta(iq)-\beta(-iq)) &= -(\varphi(q)-\varphi(-q))\varphi(-q^2)\varphi^2(q^4)\\ &= -2(\varphi(q)-\varphi(q^4))\varphi(-q^2)\varphi^2(q^4).\end{aligned}$$

Proof. Replacing q by iq in Lemma 3.1, and appealing to Lemma 2.3, we obtain

$$\begin{aligned}i(\beta(iq)-\beta(-iq)) &= i(\varphi(iq)-\varphi(-iq))\varphi(-q^2)\varphi^2(q^4)\\ &= -(\varphi(q)-\varphi(-q))\varphi(-q^2)\varphi^2(q^4),\end{aligned}$$

as asserted. □

Lemma 3.3.

$$\beta(q)-\beta(-q)-i\beta(iq)+i\beta(-iq)=4\varphi(q)\varphi^2(q^4)\varphi(q^8)-4\varphi^3(q^4)\varphi(q^8).$$

Proof. Appealing to Lemma 3.1, Lemma 3.2 and (1.6), we have

$$\begin{aligned}&\beta(q)-\beta(-q)-i\beta(iq)+i\beta(-iq)\\ &\quad= \varphi(q)\varphi(q^2)\varphi^2(q^4)-\varphi(-q)\varphi(q^2)\varphi^2(q^4)\\ &\qquad +\varphi(q)\varphi(-q^2)\varphi^2(q^4)-\varphi(-q)\varphi(-q^2)\varphi^2(q^4)\\ &\quad= 2\varphi(q)\varphi^2(q^4)\varphi(q^8)-2\varphi(-q)\varphi^2(q^4)\varphi(q^8)\\ &\quad= 2(\varphi(q)-\varphi(-q))\varphi^2(q^4)\varphi(q^8)\end{aligned}$$

$$= 4(\varphi(q) - \varphi(q^4))\varphi^2(q^4)\varphi(q^8)$$
$$= 4\varphi(q)\varphi^2(q^4)\varphi(q^8) - 4\varphi^3(q^4)\varphi(q^8),$$

as asserted. □

Lemma 3.4.

$$\beta(q^4) = \frac{1}{2}\varphi^3(q^4)\varphi(q^8) - \frac{1}{2}\varphi(q^4)\varphi^3(q^8).$$

Proof. Appealing to (3.2), (1.5)–(1.7), we deduce

$$\begin{aligned}
\beta(q^4) &= \varphi^2(q^4)\varphi(q^8)\varphi(q^{16}) - \varphi(q^4)\varphi(q^8)\varphi^2(q^{16}) \\
&= \varphi^2(q^4)\varphi(q^8)\frac{1}{2}(\varphi(q^4) + \varphi(-q^4)) \\
&\quad - \varphi(q^4)\varphi(q^8)\frac{1}{2}(\varphi^2(q^8) + \varphi^2(-q^8)) \\
&= \frac{1}{2}\varphi^3(q^4)\varphi(q^8) + \frac{1}{2}\varphi(q^4)\varphi(q^8)\varphi^2(-q^8) \\
&\quad - \frac{1}{2}\varphi(q^4)\varphi^3(q^8) - \frac{1}{2}\varphi(q^4)\varphi(q^8)\varphi^2(-q^8) \\
&= \frac{1}{2}\varphi^3(q^4)\varphi(q^8) - \frac{1}{2}\varphi(q^4)\varphi^3(q^8),
\end{aligned}$$

as claimed. □

Lemma 3.5.

$$\omega\beta(\omega q) - \omega\beta(-\omega q) = 2i\varphi(iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4),$$
$$\omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q) = -2i\varphi(-iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4).$$

Proof. By Lemmas 3.1 and 2.4, we obtain

$$\begin{aligned}
\beta(\omega q) - \beta(-\omega q) &= \varphi(\omega q)\varphi(iq^2)\varphi^2(-q^4) - \varphi(-\omega q)\varphi(iq^2)\varphi^2(-q^4) \\
&= (\varphi(\omega q) - \varphi(-\omega q))\varphi(iq^2)\varphi^2(-q^4) \\
&= 2\omega(\varphi(q) - \varphi(q^4))\varphi(iq^2)\varphi^2(-q^4)
\end{aligned}$$

so that

$$\omega\beta(\omega q) - \omega\beta(-\omega q) = 2i\varphi(iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4).$$

Also by Lemmas 3.1 and 2.4, we have

$$\begin{aligned}
\beta(\omega^3 q) - \beta(-\omega^3 q) &= \varphi(\omega^3 q)\varphi(-iq^2)\varphi^2(-q^4) - \varphi(-\omega^3 q)\varphi(-iq^2)\varphi^2(-q^4) \\
&= (\varphi(\omega^3 q) - \varphi(-\omega^3 q))\varphi(-iq^2)\varphi^2(-q^4) \\
&= 2\omega^3(\varphi(q) - \varphi(q^4))\varphi(-iq^2)\varphi^2(-q^4)
\end{aligned}$$

so that

$$\omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q) = -2i\varphi(-iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4).$$

This completes the proof. □

Lemma 3.6.

$$\begin{aligned}&\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)\\ &\quad = -4(\varphi(q) - \varphi(q^4))(\varphi(q^2) - \varphi(q^8))\varphi^2(-q^4).\end{aligned}$$

Proof. Appealing to Lemmas 3.5 and 2.3, we obtain

$$\begin{aligned}&\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)\\ &\quad = 2i\varphi(iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\qquad - 2i\varphi(-iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\quad = 2(i\varphi(iq^2) - i\varphi(-iq^2))(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\quad = -4(\varphi(q^2) - \varphi(q^8))(\varphi(q) - \varphi(q^4))\varphi^2(-q^4),\end{aligned}$$

as required. □

Lemma 3.7.

$$\omega\beta(\omega q) - \omega\beta(-\omega q) - \omega^3\beta(\omega^3 q) + \omega^3\beta(-\omega^3 q) = 4i(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\varphi(q^8).$$

Proof. Appealing to Lemma 3.5 and (1.6), we deduce

$$\begin{aligned}&\omega\beta(\omega q) - \omega\beta(-\omega q) - \omega^3\beta(\omega^3 q) + \omega^3\beta(-\omega^3 q)\\ &\quad = 2i\varphi(iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\qquad + 2i\varphi(-iq^2)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\quad = 2i(\varphi(iq^2) + \varphi(-iq^2))(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\\ &\quad = 4i\varphi(q^8)(\varphi(q) - \varphi(q^4))\varphi^2(-q^4),\end{aligned}$$

which is the asserted result. □

We now come to the main goal of this section, which is to show that each of the ten products $\varphi^3(q)\varphi(q^2)$, $\varphi(q)\varphi^3(q^2)$, $\varphi^2(q)\varphi(q^2)\varphi(q^4)$, $\varphi(q)\varphi(q^2)\varphi^2(q^4)$, $\varphi^2(q)\varphi(q^4)\varphi(q^8)$, $\varphi(q)\varphi^2(q^4)\varphi(q^8)$, $\varphi(q)\varphi^2(q^2)\varphi(q^8)$, $\varphi(q)\varphi(q^2)\varphi^2(q^8)$, $\varphi(q)\varphi^3(q^8)$ and $\varphi^3(q)\varphi(q^8)$ can be expressed in terms of α and β.

Theorem 3.1.

$$\varphi^3(q)\varphi(q^2) = \alpha(q) + 4\beta(q).$$

Proof. By (1.6), (3.1) and (3.2), we have

$$\begin{aligned}\varphi^3(q)\varphi(q^2) &= 4(\varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi(q)\varphi(q^2)\varphi^2(q^4)) + \varphi(q)\varphi(q^2)(2\varphi(q^4) - \varphi(q))^2 \\ &= 4\beta(q) + \varphi(q)\varphi(q^2)\varphi^2(-q) \\ &= \alpha(q) + 4\beta(q).\end{aligned}$$

□

Theorem 3.2.

$$\varphi(q)\varphi^3(q^2) = \alpha(q) + 2\beta(q).$$

Proof. By (1.7), Theorem 3.1 and (3.1), we obtain

$$\begin{aligned}\varphi(q)\varphi^3(q^2) &= \frac{1}{2}\varphi(q)\varphi(q^2)(\varphi^2(q) + \varphi^2(-q)) \\ &= \frac{1}{2}\varphi^3(q)\varphi(q^2) + \frac{1}{2}\varphi(q)\varphi^2(-q)\varphi(q^2) \\ &= \frac{1}{2}(\alpha(q) + 4\beta(q)) + \frac{1}{2}\alpha(q) \\ &= \alpha(q) + 2\beta(q).\end{aligned}$$

□

Theorem 3.3.

$$\varphi^2(q)\varphi(q^2)\varphi(q^4) = \alpha(q^2) + 2\beta(q).$$

Proof. Appealing to (3.2), (1.6), (1.5) and (3.1), we obtain

$$\begin{aligned}\varphi^2(q)\varphi(q^2)\varphi(q^4) &= 2\varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi^2(q)\varphi(q^2)\varphi(q^4) \\ &= 2(\varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi(q)\varphi(q^2)\varphi^2(q^4)) \\ &\quad + \varphi(q)\varphi(q^2)\varphi(q^4)(2\varphi(q^4) - \varphi(q)) \\ &= 2\beta(q) + \varphi(q)\varphi(-q)\varphi(q^2)\varphi(q^4) \\ &= 2\beta(q) + \varphi(q^2)\varphi^2(-q^2)\varphi(q^4) \\ &= 2\beta(q) + \alpha(q^2).\end{aligned}$$

□

Theorem 3.4.

$$\varphi(q)\varphi(q^2)\varphi^2(q^4) = \alpha(q^2) + \beta(q).$$

Proof. By Theorem 3.3, we have

$$\varphi^2(q)\varphi(q^2)\varphi(q^4) = \alpha(q^2) + 2\beta(q),$$

and by (3.2) we have

$$\varphi^2(q)\varphi(q^2)\varphi(q^4) = \beta(q) + \varphi(q)\varphi(q^2)\varphi^2(q^4).$$

Hence

$$\varphi(q)\varphi(q^2)\varphi^2(q^4) = \alpha(q^2) + \beta(q)$$

as asserted. □

Theorem 3.5.

$$\varphi^2(q)\varphi(q^4)\varphi(q^8) = \alpha(q^4) + \beta(q) - \frac{1}{2}(i\beta(iq) - i\beta(-iq)).$$

Proof. By (3.1), (3.2), Lemma 3.2, (1.5)–(1.7), we obtain

$$\begin{aligned}
&\alpha(q^4) + \beta(q) - \frac{1}{2}(i\beta(iq) - i\beta(-iq))\\
&\quad = \varphi(q^4)\varphi^2(-q^4)\varphi(q^8) + \varphi^2(q)\varphi(q^2)\varphi(q^4)\\
&\qquad - \varphi(q)\varphi(q^2)\varphi^2(q^4) + \varphi(q)\varphi(-q^2)\varphi^2(q^4) - \varphi(-q^2)\varphi^3(q^4)\\
&\quad = \varphi(q^4)(\varphi^2(-q^4)\varphi(q^8) + \varphi^2(q)\varphi(q^2)\\
&\qquad - \varphi(q)\varphi(q^2)\varphi(q^4) + \varphi(q)\varphi(-q^2)\varphi(q^4) - \varphi(-q^2)\varphi^2(q^4))\\
&\quad = \varphi(q^4)\bigg(\frac{1}{2}\varphi(q^2)\varphi(-q^2)(\varphi(q^2) + \varphi(-q^2)) + \varphi^2(q)\varphi(q^2)\\
&\qquad - \varphi(q)\varphi(q^2)\varphi(q^4) + \varphi(q)\varphi(-q^2)\varphi(q^4) - \varphi(-q^2)\varphi^2(q^4)\bigg)\\
&\quad = \frac{1}{2}\varphi(q^4)(\varphi^2(q^2)\varphi(-q^2) + \varphi(q^2)\varphi^2(-q^2) + 2\varphi^2(q)\varphi(q^2)\\
&\qquad - 2\varphi(q)\varphi(q^2)\varphi(q^4) + 2\varphi(q)\varphi(-q^2)\varphi(q^4) - 2\varphi(-q^2)\varphi^2(q^4))\\
&\quad = \frac{1}{2}\varphi(q^4)(\varphi^2(q^2)\varphi(-q^2) + \varphi(q^2)\varphi^2(-q^2) + 2\varphi^2(q)\varphi(q^2)\\
&\qquad - \varphi(q)\varphi(q^2)(\varphi(q) + \varphi(-q)) + \varphi(q)\varphi(-q^2)(\varphi(q) + \varphi(-q))\\
&\qquad - \varphi(-q^2)(\varphi^2(q^2) + \varphi^2(-q^2)))\\
&\quad = \frac{1}{2}\varphi(q^4)(\varphi^2(q^2)\varphi(-q^2) + \varphi(q^2)\varphi^2(-q^2) + 2\varphi^2(q)\varphi(q^2)\\
&\qquad - \varphi^2(q)\varphi(q^2) - \varphi(q)\varphi(-q)\varphi(q^2) - \varphi^2(q^2)\varphi(-q^2) - \varphi^3(-q^2)\\
&\qquad + \varphi^2(q)\varphi(-q^2) + \varphi(q)\varphi(-q)\varphi(-q^2))\\
&\quad = \frac{1}{2}\varphi(q^4)(\varphi^2(q)\varphi(q^2) + \varphi(q)\varphi(-q)\varphi(q^2) - \varphi(q)\varphi(-q)\varphi(q^2)\\
&\qquad - \varphi(q)\varphi(-q)\varphi(-q^2) + \varphi^2(q)\varphi(-q^2) + \varphi(q)\varphi(-q)\varphi(-q^2))\\
&\quad = \frac{1}{2}\varphi(q^4)\varphi^2(q)(\varphi(q^2) + \varphi(-q^2))\\
&\quad = \frac{1}{2}\varphi(q^4)\varphi^2(q)2\varphi(q^8)\\
&\quad = \varphi^2(q)\varphi(q^4)\varphi(q^8).
\end{aligned}$$

□

Theorem 3.6.

$$\varphi(q)\varphi^2(q^4)\varphi(q^8) = \alpha(q^4) + 4\beta(q^4) + \frac{1}{4}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq)).$$

Proof. By (3.1), Lemma 3.4, Lemma 3.3 and (1.7), we have

$$\begin{aligned}
&\alpha(q^4) + 4\beta(q^4) + \frac{1}{4}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq)) \\
&\quad = \varphi(q^4)\varphi^2(-q^4)\varphi(q^8) + 2\varphi^3(q^4)\varphi(q^8) - 2\varphi(q^4)\varphi^3(q^8) \\
&\qquad + \varphi(q)\varphi^2(q^4)\varphi(q^8) - \varphi^3(q^4)\varphi(q^8) \\
&\quad = \varphi(q^4)\varphi(q^8)(\varphi^2(-q^4) + \varphi^2(q^4) - 2\varphi^2(q^8) + \varphi(q)\varphi(q^4)) \\
&\quad = \varphi(q)\varphi^2(q^4)\varphi(q^8).
\end{aligned}$$

□

Theorem 3.7.

$$\varphi(q)\varphi^2(q^2)\varphi(q^8) = \alpha(q^4) + \beta(q) - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)).$$

Proof. Appealing to (3.1), (3.2) and Lemma 3.6, we obtain

$$\begin{aligned}
&\alpha(q^4) + \beta(q) - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&\quad = \varphi^2(-q^4)(2\varphi(q^4)\varphi(q^8) + \varphi(q)\varphi(q^2) - \varphi(q)\varphi(q^8) - \varphi(q^2)\varphi(q^4)) \\
&\qquad + \varphi^2(q)\varphi(q^2)\varphi(q^4) - \varphi(q)\varphi(q^2)\varphi^2(q^4).
\end{aligned}$$

Next, by using (1.5)–(1.7), we obtain

$$\begin{aligned}
&\alpha(q^4) + \beta(q) - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&\quad = \varphi(q^2)(2\varphi(-q^2)\varphi(q^4)\varphi(q^8) + \varphi(q)\varphi(q^2)\varphi(-q^2) - \varphi(q)\varphi(-q^2)\varphi(q^8) \\
&\qquad - \varphi(q^2)\varphi(-q^2)\varphi(q^4) + \varphi^2(q)\varphi(q^4) - \varphi(q)\varphi^2(q^4)) \quad \text{(by (1.5))} \\
&\quad = \varphi(q^2)(\varphi(-q^2)\varphi(q^4)(2\varphi(q^8) - \varphi(q^2)) + \varphi(q)\varphi(q^2)\varphi(-q^2) \\
&\qquad - \varphi(q)\varphi(-q^2)\varphi(q^8) + \varphi^2(q)\varphi(q^4) - \varphi(q)\varphi^2(q^4)) \\
&\quad = \varphi(q^2)(\varphi^2(-q^2)\varphi(q^4) + \varphi(q)\varphi(q^2)\varphi(-q^2) - \varphi(q)\varphi(-q^2)\varphi(q^8) \\
&\qquad + \varphi^2(q)\varphi(q^4) - \varphi(q)\varphi^2(q^4)) \quad \text{(by (1.6))} \\
&\quad = \varphi(q)\varphi(q^2)(\varphi(-q)\varphi(q^4) + \varphi(q^2)\varphi(-q^2) \\
&\qquad - \varphi(-q^2)\varphi(q^8) + \varphi(q)\varphi(q^4) - \varphi^2(q^4)) \quad \text{(by (1.5))} \\
&\quad = \varphi(q)\varphi(q^2)(\varphi^2(q^4) + \varphi(q^2)\varphi(-q^2) - \varphi(-q^2)\varphi(q^8)) \quad \text{(by (1.6))} \\
&\quad = \varphi(q)\varphi(q^2)(\varphi^2(q^4) + \varphi^2(-q^4) - \varphi(-q^2)\varphi(q^8)) \quad \text{(by (1.5))} \\
&\quad = \varphi(q)\varphi(q^2)(2\varphi^2(q^8) - \varphi(-q^2)\varphi(q^8)) \quad \text{(by (1.7))}
\end{aligned}$$

$$= \varphi(q)\varphi(q^2)\varphi(q^8)(2\varphi(q^8) - \varphi(-q^2))$$
$$= \varphi(q)\varphi^2(q^2)\varphi(q^8) \quad \text{(by (1.6))}. \qquad \square$$

Theorem 3.8.

$$\begin{aligned}\varphi(q)\varphi(q^2)\varphi(q^8)^2 = \alpha(q^4) + \frac{1}{2}\beta(q)& \\ &- \frac{1}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\ &- \frac{i}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) - \omega^3\beta(\omega^3 q) + \omega^3\beta(-\omega^3 q)).\end{aligned}$$

Proof. By (3.1), (3.2), Lemma 3.6, Lemma 3.7, (1.5)–(1.7), we have

$$\begin{aligned}
&\alpha(q^4) + \frac{1}{2}\beta(q) - \frac{1}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&\qquad - \frac{i}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) - \omega^3\beta(\omega^3 q) + \omega^3\beta(-\omega^3 q)) \\
&\quad = \varphi(q^4)\varphi^2(-q^4)\varphi(q^8) + \frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi(q^2)\varphi^2(q^4) \\
&\qquad + \frac{1}{2}(\varphi(q) - \varphi(q^4))(\varphi(q^2) - \varphi(q^8))\varphi^2(-q^4) + \frac{1}{2}(\varphi(q) - \varphi(q^4))\varphi^2(-q^4)\varphi(q^8) \\
&\quad = \varphi(q^2)\varphi(-q^2)\varphi(q^4)\varphi(q^8) \\
&\qquad + \frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi(q^2)\varphi^2(q^4) + \frac{1}{2}(\varphi(q) - \varphi(q^4))\varphi(q^2)\varphi^2(-q^4) \\
&\quad = \varphi(q^2)\bigg(\varphi(-q^2)\varphi(q^4)\varphi(q^8) + \frac{1}{2}\varphi^2(q)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi^2(q^4) \\
&\qquad + \frac{1}{2}\varphi(q)\varphi^2(-q^4) - \frac{1}{2}\varphi(q^4)\varphi^2(-q^4)\bigg) \\
&\quad = \varphi(q^2)\bigg(\varphi(-q^2)\varphi(q^4)\frac{1}{2}(\varphi(q^2) + \varphi(-q^2)) + \frac{1}{2}\varphi^2(q)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi^2(q^4) \\
&\qquad + \frac{1}{2}\varphi(q)\varphi^2(-q^4) - \frac{1}{2}\varphi(q^4)\varphi^2(-q^4)\bigg) \\
&\quad = \frac{1}{2}\varphi(q^2)(\varphi^2(-q^4)\varphi(q^4) + \varphi(q^4)\varphi(q)\varphi(-q) + \varphi^2(q)\varphi(q^4) - \varphi(q)\varphi^2(q^4) \\
&\qquad + \varphi(q)\varphi^2(-q^4) - \varphi(q^4)\varphi^2(-q^4)) \\
&\quad = \frac{1}{2}\varphi(q)\varphi(q^2)(\varphi(-q)\varphi(q^4) + \varphi(q)\varphi(q^4) - \varphi^2(q^4) + \varphi^2(-q^4)) \\
&\quad = \frac{1}{2}\varphi(q)\varphi(q^2)(2\varphi^2(q^4) - \varphi^2(q^4) + \varphi^2(-q^4))
\end{aligned}$$

$$= \frac{1}{2}\varphi(q)\varphi(q^2)(\varphi^2(q^4) + \varphi^2(-q^4))$$
$$= \frac{1}{2}\varphi(q)\varphi(q^2)2\varphi^2(q^8)$$
$$= \varphi(q)\varphi(q^2)\varphi^2(q^8).$$ □

Theorem 3.9.

$$\varphi(q)\varphi^3(q^8) = \alpha(q^4) + \frac{1}{2}\alpha(q^2) - \frac{1}{2}\alpha(q) + 2\beta(q^4)$$
$$+ \frac{1}{8}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq))$$
$$- \frac{1}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)).$$

Proof. We have

$$4\varphi(q)\varphi^3(q^8) = 4\varphi(q)\frac{1}{2}\left(\varphi(q^2) + \varphi(-q^2))\frac{1}{2}(\varphi^2(q^4) + \varphi^2(-q^4)\right) \quad \text{(by (1.6) and (1.7))}$$
$$= \varphi(q)\varphi(q^2)\varphi^2(q^4) + \varphi(q)\varphi(-q^2)\varphi^2(q^4)$$
$$+ \varphi(q)\varphi(q^2)\varphi^2(-q^4) + \varphi(q)\varphi(-q^2)\varphi^2(-q^4)$$
$$= \varphi(q)\varphi(q^2)\varphi^2(q^4) + \varphi(q)(2\varphi(q^8) - \varphi(q^2))\varphi^2(q^4)$$
$$+ \varphi(q)\varphi(-q^2)\varphi^2(q^2) + \varphi(q)\varphi^2(-q^2)\varphi(q^2) \quad \text{(by (1.6) and (1.5))}$$
$$= 2\varphi(q)\varphi^2(q^4)\varphi(q^8) + \varphi(q)(2\varphi(q^8) - \varphi(q^2))\varphi^2(q^2)$$
$$+ \varphi(q)(2\varphi^2(q^4) - \varphi^2(q^2))\varphi(q^2) \quad \text{(by (1.6) and (1.7))}$$
$$= 2\varphi(q)\varphi^2(q^4)\varphi(q^8) + 2\varphi(q)\varphi^2(q^2)\varphi(q^8)$$
$$- \varphi(q)\varphi^3(q^2) + 2\varphi(q)\varphi(q^2)\varphi^2(q^4) - \varphi(q)\varphi^3(q^2)$$
$$= 2\varphi(q)\varphi^2(q^4)\varphi(q^8) + 2\varphi(q)\varphi^2(q^2)\varphi(q^8)$$
$$+ 2\varphi(q)\varphi(q^2)\varphi^2(q^4) - 2\varphi(q)\varphi^3(q^2).$$

Thus

$$\varphi(q)\varphi^3(q^8) = \frac{1}{2}\varphi(q)\varphi^2(q^4)\varphi(q^8) + \frac{1}{2}\varphi(q)\varphi^2(q^2)\varphi(q^8)$$
$$+ \frac{1}{2}\varphi(q)\varphi(q^2)\varphi^2(q^4) - \frac{1}{2}\varphi(q)\varphi^3(q^2)$$
$$= \frac{1}{2}\left(\alpha(q^4) + 4\beta(q^4) + \frac{1}{4}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq))\right)$$
$$+ \frac{1}{2}\left(\alpha(q^4) + \beta(q) - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q))\right)$$

$$
\begin{aligned}
&+\frac{1}{2}(\alpha(q^2)+\beta(q))\\
&-\frac{1}{2}(\alpha(q)+2\beta(q)) \quad \text{(by Theorems 3.6, 3.7, 3.4 and 3.2)}\\
&=\alpha(q^4)+\frac{1}{2}\alpha(q^2)-\frac{1}{2}\alpha(q)+2\beta(q^4)\\
&+\frac{1}{8}(\beta(q)-\beta(-q)-i\beta(iq)+i\beta(-iq))\\
&-\frac{1}{8}(\omega\beta(\omega q)-\omega\beta(-\omega q)+\omega^3\beta(\omega^3 q)-\omega^3\beta(-\omega^3 q))
\end{aligned}
$$

as asserted. □

Theorem 3.10.

$$
\begin{aligned}
\varphi^3(q)\varphi(q^8) &= \alpha(q^4)+3\beta(q)-8\beta(q^4)\\
&\quad -\frac{1}{2}(\beta(q)-\beta(-q)+i\beta(iq)-i\beta(-iq))\\
&\quad -\frac{1}{4}(\omega\beta(\omega q)-\omega\beta(-\omega q)+\omega^3\beta(\omega^3 q)-\omega^3\beta(-\omega^3 q)).
\end{aligned}
$$

Proof. Appealing to Lemma 2.2, we have

$$
\begin{aligned}
\varphi^3(q)\varphi(q^8)+\varphi(q)\varphi^2(-q^2)\varphi(q^8) &= \varphi(q)\varphi(q^8)(\varphi^2(q)+\varphi^2(-q^2))\\
&= 2\varphi^2(q)\varphi(q^4)\varphi(q^8).
\end{aligned}
$$

Next, by (1.7), we deduce

$$
\begin{aligned}
\varphi(q)\varphi^2(-q^2)\varphi(q^8) &= \varphi(q)(2\varphi^2(q^4)-\varphi^2(q^2))\varphi(q^8)\\
&= 2\varphi(q)\varphi^2(q^4)\varphi(q^8)-\varphi(q)\varphi^2(q^2)\varphi(q^8).
\end{aligned}
$$

Hence

$$
\varphi^3(q)\varphi(q^8) = 2\varphi^2(q)\varphi(q^4)\varphi(q^8)-2\varphi(q)\varphi^2(q^4)\varphi(q^8)+\varphi(q)\varphi^2(q^2)\varphi(q^8).
$$

Appealing to Theorems 3.5–3.7, we obtain

$$
\begin{aligned}
\varphi^3(q)\varphi(q^8) = 2\bigg(&\alpha(q^4)+\beta(q)-\frac{1}{2}(i\beta(iq)-i\beta(-iq))\bigg)\\
&-2\bigg(\alpha(q^4)+4\beta(q^4)+\frac{1}{4}(\beta(q)-\beta(-q)-i\beta(iq)+i\beta(-iq))\bigg)\\
&+\alpha(q^4)+\beta(q)-\frac{1}{4}(\omega\beta(\omega q)-\omega\beta(-\omega q)+\omega^3\beta(\omega^3 q)-\omega^3\beta(-\omega^3 q))
\end{aligned}
$$

$$\begin{aligned}&= \alpha(q^4) + 3\beta(q) - 8\beta(q^4)\\ &\quad - \frac{1}{2}(\beta(q) - \beta(-q)) - \frac{1}{2}(i\beta(iq) - i\beta(-iq))\\ &\quad - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q))\end{aligned}$$

as claimed. □

4. The Functions $R(n)$ and $S(n)$

For $n \in \mathbb{N}$ we define

$$R(n) := \sum_{d|n} d\left(\frac{8}{d}\right) \tag{4.1}$$

and

$$S(n) := \sum_{d|n} \frac{n}{d}\left(\frac{8}{d}\right), \tag{4.2}$$

where d runs through the positive integers dividing n and $\left(\frac{8}{d}\right)$ is the Legendre–Jacobi–Kronecker symbol for discriminant 8, that is

$$\left(\frac{8}{d}\right) = \begin{cases} +1, & \text{if } d \equiv 1, 7 \pmod 8, \\ -1, & \text{if } d \equiv 3, 5 \pmod 8, \\ 0, & \text{if } d \equiv 0 \pmod 2. \end{cases}$$

Using the properties of the Legendre–Jacobi–Kronecker symbol it is easy to prove the following theorem.

Theorem 4.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and N is odd. Then*

(a) $R(n) = R(N) = \left(\frac{8}{N}\right)S(N)$

and

(b) $S(n) = 2^\alpha \left(\frac{8}{N}\right)R(N) = 2^\alpha S(N)$.

An immediate consequence of Theorem 4.1 is the following result.

Corollary 4.1. *Let $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ be such that $n_1 = 2^\beta n_2$ for some $\beta \in \mathbb{N}_0$. Then*

(a) $R(n_1) = R(n_2)$

and

(b) $S(n_1) = 2^\beta S(n_2)$.

Our next theorem of this section gives the power series $\sum_{n=1}^{\infty} R(n)q^n$ and $\sum_{n=1}^{\infty} S(n)q^n$ in terms of Jacobi's theta function φ.

Theorem 4.2. *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then*

(a) $$\sum_{n=1}^{\infty} R(n)q^n = \frac{1}{2} - \frac{1}{2}\varphi(q)\varphi^2(-q)\varphi(q^2)$$

and

(b) $$\sum_{n=1}^{\infty} S(n)q^n = \frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^4) - \frac{1}{2}\varphi(q)\varphi(q^2)\varphi^2(q^4).$$

Proof. We appeal to classical results of Petr [14]. Petr uses the following notation:

$$\begin{cases} \Theta_1 = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/4}, & \Theta_1(0,2\tau) = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/2}, \\ \Theta_2 = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, & \Theta_2(0,2\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}, \\ \Theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, & \Theta_3(0,2\tau) = 1 + 2\sum_{n=1}^{\infty} q^{2n^2}, \\ & \Theta_3(0,\tau/2) = 1 + 2\sum_{n=1}^{\infty} q^{n^2/2}. \end{cases} \tag{4.3}$$

He proves [14, (19′), third equation]

$$\Theta_1^2\Theta_3\Theta_3(0,\tau/2) = 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)\right)q^{n/2} \tag{4.4}$$

and [14, (19′), second equation]

$$\Theta_2^2\Theta_3\Theta_3(0,2\tau) = 1 - 2\sum_{n=1}^{\infty}\left(\sum_{d|n} d\left(\frac{8}{d}\right)\right)q^n. \tag{4.5}$$

From (1.4) and (4.3) we obtain

$$\Theta_1 = \varphi(q^{1/4}) - \varphi(q), \quad \Theta_1(0,2\tau) = \varphi(q^{1/2}) - \varphi(q^2), \tag{4.6}$$

$$\Theta_2 = \varphi(-q), \quad \Theta_2(0,2\tau) = \varphi(-q^2), \tag{4.7}$$

$$\Theta_3 = \varphi(q), \quad \Theta_3(0,2\tau) = \varphi(q^2), \quad \Theta_3(0,\tau/2) = \varphi(q^{1/2}). \tag{4.8}$$

Replacing q by q^2 in (4.4), we obtain

$$\Theta_1^2(0,2\tau)\Theta_3\Theta_3(0,2\tau) = 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)\right)q^n. \tag{4.9}$$

Appealing to (4.2), (4.6) and (4.8), we deduce

$$(\varphi(q^{1/2}) - \varphi(q^2))^2\varphi(q)\varphi(q^2) = 4\sum_{n=1}^{\infty} S(n)q^n. \tag{4.10}$$

Next, appealing to Lemma 2.1, we obtain

$$4\sum_{n=1}^{\infty} S(n)q^n = 2\varphi^2(q)\varphi(q^2)\varphi(q^4) - 2\varphi(q)\varphi(q^2)\varphi^2(q^4)$$

from which (b) follows on dividing both sides by 4.

Appealing to (4.1), (4.7) and (4.8), (4.5) becomes

$$\varphi^2(-q)\varphi(q)\varphi(q^2) = 1 - 2\sum_{n=1}^{\infty} R(n)q^n$$

from which (a) follows. □

From (3.1), (3.2) and Theorem 4.2 we obtain immediately

Theorem 4.3. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(a) $\alpha(q) = 1 - 2\sum_{n=1}^{\infty} R(n)q^n$

and

(b) $\beta(q) = 2\sum_{n=1}^{\infty} S(n)q^n.$

5. Representations by Quaternary Quadratic Forms

In this section, we use the theta function identities proved in Theorems 3.1–3.10 in conjunction with Theorems 4.1 and 4.3 to determine the number of representations of $n \in \mathbb{N}$ by the ten quaternary quadratic forms listed at the end of Sec. 1.

Theorem 5.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N, 2) = 1$. Then*

$$N(1,1,1,2;n) = 2\left(2^{\alpha+2} - \left(\frac{8}{N}\right)\right) S(N).$$

Proof. By (1.3), Theorems 3.1 and 4.3, we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,2;n)q^n &= \varphi^3(q)\varphi(q^2)\\ &= \alpha(q) + 4\beta(q)\\ &= 1 - 2\sum_{n=1}^{\infty} R(n)q^n + 8\sum_{n=1}^{\infty} S(n)q^n.\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we deduce

$$N(1,1,1,2;n) = 8S(n) - 2R(n), \quad n \in \mathbb{N}.$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.1 was stated but not proved by Liouville in [4]. Pepin [13] used Liouville's elementary methods and recurrence relations to prove Theorem 5.1. Petr [14] evaluated $N(1,1,1,2;n)$ in terms of the class number of binary quadratic forms. Benz [1, pp. 168–175] gave a proof of Theorem 5.1 using recurrence relations and theta function identities. Demuth [3, pp. 241–243] deduced Theorem 5.1 from Siegel's mass formula. Wild [15] used modular forms to prove Theorem 5.1. Recently, Williams [16] has given a completely arithmetic proof of Theorem 5.1 without recourse to recurrence relations.

Theorem 5.2. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,2,2,2;n) = 2\left(2^{\alpha+1} - \left(\frac{8}{N}\right)\right) S(N).$$

Proof. By (1.3), Theorems 3.2 and 4.3, we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,2,2,2;n)q^n &= \varphi(q)\varphi^3(q^2) \\ &= \alpha(q) + 2\beta(q) \\ &= 1 - 2\sum_{n=1}^{\infty} R(n)q^n + 4\sum_{n=1}^{\infty} S(n)q^n.\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we deduce

$$N(1,2,2,2;n) = 4S(n) - 2R(n), \quad n \in \mathbb{N}.$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.2 was stated without proof by Liouville [4] in 1861. In 1884, Pepin [13] proved Theorem 5.2 using Liouville's elementary methods and recurrence relations between $N(1,1,1,2;n)$ and $N(1,2,2,2;n)$ as well as between $N(1,2,2,2;2^\alpha N)$ and $N(1,2,2,2;N)$. In 1964, Benz [1] gave a proof of Liouville's formula for $N(1,2,2,2;n)$ using theta functions and recurrence relations such as $N(1,2,2,2;2n) = N(1,1,1,2;n)$. Recently, Williams [16] has given a completely arithmetic proof of Theorem 5.2.

Theorem 5.3. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,1,2,4;n) = \left(2^{\alpha+2} - (1 + (-1)^n)\right)\left(\frac{8}{N}\right)\Big) S(N).$$

Proof. By (1.3), Theorem 3.3, Theorem 4.3 and Corollary 4.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,2,4;n)q^n &= \varphi^2(q)\varphi(q^2)\varphi(q^4) \\
&= \alpha(q^2) + 2\beta(q) \\
&= 1 - 2\sum_{n=1}^{\infty} R(n)q^{2n} + 4\sum_{n=1}^{\infty} S(n)q^n \\
&= 1 - 2\sum_{n=1}^{\infty} R(2n)q^{2n} + 4\sum_{n=1}^{\infty} S(n)q^n \\
&= 1 - 2\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} R(n)q^n + 4\sum_{n=1}^{\infty} S(n)q^n \\
&= 1 - \sum_{n=1}^{\infty} R(n)(1+(-1)^n)q^n + 4\sum_{n=1}^{\infty} S(n)q^n.
\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,1,2,4;n) = 4S(n) - (1+(-1)^n)R(n).$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.3 was stated without proof by Liouville in [10]. The authors have not located a proof of this theorem in the literature.

Theorem 5.4. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,2,4,4;n) = \left(2^{\alpha+1} - (1+(-1)^n)\left(\frac{8}{N}\right)\right)S(N).$$

Proof. By (1.3), Theorem 3.4, Theorem 4.3 and Corollary 4.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,2,4,4;n)q^n &= \varphi(q)\varphi(q^2)\varphi^2(q^4) \\
&= \alpha(q^2) + \beta(q) \\
&= 1 - 2\sum_{n=1}^{\infty} R(n)q^{2n} + 2\sum_{n=1}^{\infty} S(n)q^n \\
&= 1 - 2\sum_{n=1}^{\infty} R(2n)q^{2n} + 2\sum_{n=1}^{\infty} S(n)q^n
\end{aligned}$$

$$= 1 - 2 \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} R(n)q^n + 2 \sum_{n=1}^{\infty} S(n)q^n$$

$$= 1 - \sum_{n=1}^{\infty} R(n)(1 + (-1)^n)q^n + 2 \sum_{n=1}^{\infty} S(n)q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we deduce

$$N(1, 2, 4, 4; n) = 2S(n) - (1 + (-1)^n)R(n).$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.4 was stated without proof by Liouville [8]. The authors have not found a proof of this theorem in the literature.

Theorem 5.5. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N, 2) = 1$. *Then*

$$N(1, 1, 4, 8; n) = \begin{cases} 4S(N), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 3 \pmod 4, \\ 4S(N), & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^\alpha - \left(\dfrac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.5 and 4.3, we obtain

$$\sum_{n=0}^{\infty} N(1, 1, 4, 8; n)q^n$$

$$= \varphi^2(q)\varphi(q^4)\varphi(q^8)$$

$$= \alpha(q^4) + \beta(q) - \frac{1}{2}(i\beta(iq) - i\beta(-iq))$$

$$= 1 - 2\sum_{n=1}^{\infty} R(n)q^{4n} + 2\sum_{n=1}^{\infty} S(n)q^n - \sum_{n=1}^{\infty} S(n)q^n(i^{n+1} - (-1)^n i^{n+1})$$

$$= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + 2\sum_{n=1}^{\infty} S(n)q^n$$

$$+ 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n - 2 \sum_{\substack{n=1 \\ n \equiv 3 \pmod 4}}^{\infty} S(n)q^n$$

$$= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + 4 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n + 2 \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} S(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,1,4,8;n) = \begin{cases} 4S(n), & \text{if } n \equiv 1 \pmod 4, \\ 2S(n), & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 3 \pmod 4, \\ -2R(n/4) + 2S(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Appealing to Theorem 4.1 and Corollory 4.1, we deduce

$$N(1,1,4,8;n) = \begin{cases} 4S(N), & \text{if } n \equiv 1 \pmod 4, \\ 4S(N), & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 3 \pmod 4, \\ 2\left(2^\alpha - \left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

as asserted. □

This result was stated without proof by Liouville [11]. The authors have not located a proof in the literature.

Theorem 5.6. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and N is odd. Then*

$$N(1,4,4,8;n) = \begin{cases} 2S(N), & \textit{if } n \equiv 1 \pmod 4, \\ 0, & \textit{if } n \equiv 3 \pmod 4, \\ 0, & \textit{if } n \equiv 2 \pmod 4, \\ 2\left(2^\alpha - \left(\frac{8}{N}\right)\right) S(N), & \textit{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.6 and 4.3, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(1,4,4,8;n)q^n &= \varphi(q)\varphi^2(q^4)\varphi(q^8) \\ &= \alpha(q^4) + 4\beta(q^4) + \frac{1}{4}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq)) \\ &= 1 - 2\sum_{n=1}^{\infty} R(n)q^{4n} + 8\sum_{n=1}^{\infty} S(n)q^{4n} \\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty} S(n)q^n(1 - (-1)^n - i^{n+1} + (-1)^n i^{n+1}) \\ &= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + 8\sum_{n=1}^{\infty} S(n/4)q^n + 2\sum_{\substack{n=1 \\ n \equiv 1 \pmod 4}}^{\infty} S(n)q^n. \end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,4,4,8;n) = \begin{cases} 2S(n), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2,3 \pmod 4, \\ -2R(n/4) + 8S(n/4), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

This result was stated by Liouville [7] without proof. The authors have not found a proof in the literature.

Theorem 5.7. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and N is odd. Then*

$$N(1,2,2,8;n) = \begin{cases} 0, & \text{if } n \equiv 7 \pmod 8, \\ 2S(N), & \text{if } n \equiv 1 \pmod 4, \\ 4S(N), & \text{if } n \equiv 3 \pmod 8, \\ 4S(N), & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^\alpha - \left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.7 and 4.3, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(1,2,2,8;n)q^n &= \varphi(q)\varphi^2(q^2)\varphi(q^8) \\ &= \alpha(q^4) + \beta(q) - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) \\ &\quad + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\ &= 1 - 2\sum_{n=1}^{\infty} R(n)q^{4n} + 2\sum_{n=1}^{\infty} S(n)q^n \\ &\quad - \frac{1}{2}\sum_{n=1}^{\infty} S(n)(\omega^{n+1} - (-1)^n\omega^{n+1} + \omega^{3n+3} - (-1)^n\omega^{3n+3})q^n \\ &= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + 2\sum_{n=1}^{\infty} S(n)q^n \\ &\quad + 2\sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty} S(n)q^n - 2\sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty} S(n)q^n. \end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,2,2,8;n) = \begin{cases} 2S(n), & \text{if } n \equiv 2 \pmod 4, \\ 2S(n) - 2R(n/4), & \text{if } n \equiv 0 \pmod 4, \\ 2S(n), & \text{if } n \equiv 1 \pmod 4, \\ 4S(n), & \text{if } n \equiv 3 \pmod 8, \\ 0, & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.7 was stated without proof by Liouville [12]. No proof appears to exist in the literature.

Theorem 5.8. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and N is odd. Then*

$$N(1,2,8,8;n) = \begin{cases} 0, & \text{if } n \equiv 5,7 \pmod 8, \\ 2S(N), & \text{if } n \equiv 1,3 \pmod 8, \\ 2S(N), & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.8 and 4.3, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,2,8,8;n) q^n \\
&\quad = \varphi(q)\varphi(q^2)\varphi^2(q^8) \\
&\quad = \alpha(q^4) + \frac{1}{2}\beta(q) - \frac{1}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&\qquad - \frac{i}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) - \omega^3\beta(\omega^3 q) + \omega^3\beta(-\omega^3 q)) \\
&\quad = 1 - 2\sum_{n=1}^{\infty} R(n) q^{4n} + \sum_{n=1}^{\infty} S(n) q^n \\
&\qquad - \frac{1}{4}\sum_{n=1}^{\infty} S(n)(\omega^{n+1} - (-1)^n\omega^{n+1} + \omega^{3n+3} - (-1)^n\omega^{3n+3}) q^n \\
&\qquad - \frac{i}{4}\sum_{n=1}^{\infty} S(n)(\omega^{n+1} - (-1)^n\omega^{n+1} - \omega^{3n+3} + (-1)^n\omega^{3n+3}) q^n
\end{aligned}$$

$$= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + \sum_{n=1}^{\infty} S(n)q^n + \sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty} S(n)q^n$$

$$- \sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty} S(n)q^n + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 8}}^{\infty} S(n)q^n - \sum_{\substack{n=1 \\ n\equiv 5 \pmod 8}}^{\infty} S(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,2,8,8;n) = \begin{cases} 2S(n), & \text{if } n \equiv 1,3 \pmod 8, \\ 0, & \text{if } n \equiv 5,7 \pmod 8, \\ S(n), & \text{if } n \equiv 2 \pmod 4, \\ S(n) - 2R(n/4), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Appealing to Theorem 4.1, we obtain the assertion of the theorem. □

Theorem 5.8 was stated by Liouville [9] without proof. We have not found a proof in the literature.

Theorem 5.9. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and N is odd. Then*

$$N(1,8,8,8;n) = \begin{cases} 0, & \text{if } n \equiv 3,5,7 \pmod 8, \\ 2S(N), & \text{if } n \equiv 1 \pmod 8, \\ 0, & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.9 and 4.3, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,8,8,8;n)q^n \\
&\quad = \varphi(q)\varphi^3(q^8) \\
&\quad = \alpha(q^4) + \frac{1}{2}\alpha(q^2) - \frac{1}{2}\alpha(q) + 2\beta(q^4) \\
&\qquad + \frac{1}{8}(\beta(q) - \beta(-q) - i\beta(iq) + i\beta(-iq)) \\
&\qquad - \frac{1}{8}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&\quad = 1 - 2\sum_{n=1}^{\infty} R(n)q^{4n} + \frac{1}{2} - \sum_{n=1}^{\infty} R(n)q^{2n} - \frac{1}{2} + \sum_{n=1}^{\infty} R(n)q^n
\end{aligned}$$

$$+4\sum_{n=1}^{\infty} S(n)q^{4n} + \frac{1}{4}\sum_{n=1}^{\infty} S(n)(1-(-1)^n - i^{n+1} + (-1)^n i^{n+1})q^n$$

$$-\frac{1}{4}\sum_{n=1}^{\infty} S(n)(\omega^{n+1} - (-1)^n\omega^{n+1} + \omega^{3n+3} - (-1)^n\omega^{3n+3})q^n$$

$$= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n - \sum_{n=1}^{\infty} R(n/2)q^n + \sum_{n=1}^{\infty} R(n)q^n + 4\sum_{n=1}^{\infty} S(n/4)q^n$$

$$+ \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} S(n)q^n + \sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty} S(n)q^n - \sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty} S(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we deduce

$$N(1,8,8,8;n) = \begin{cases} -2R(n/4) - R(n/2) + R(n) + 4S(n/4), & \text{if } n \equiv 0 \pmod 4, \\ -R(n/2) + R(n), & \text{if } n \equiv 2 \pmod 4, \\ R(n) + S(n), & \text{if } n \equiv 1 \pmod 4, \\ R(n) + S(n), & \text{if } n \equiv 3 \pmod 8, \\ R(n) - S(n), & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Appealing to Theorem 4.1, we obtain

$$N(1,8,8,8;n) = \begin{cases} \left(2^\alpha - 2\left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4, \\ 0, & \text{if } n \equiv 2 \pmod 4, \\ \left(1 + \left(\frac{8}{N}\right)\right) S(n), & \text{if } n \equiv 1 \pmod 4, \\ \left(1 + \left(\frac{8}{N}\right)\right) S(n) = 0, & \text{if } n \equiv 3 \pmod 8, \\ \left(-1 + \left(\frac{8}{N}\right)\right) S(n) = 0, & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

which gives the assertion of the theorem. □

Theorem 5.9 was stated by Liouville [6] without proof. We have not located a proof of Theorem 5.9 in the literature.

Theorem 5.10. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$.

Then

$$N(1,1,1,8;n) = \begin{cases} 6S(N), & \text{if } n \equiv 1 \pmod 4, \\ 4S(N), & \text{if } n \equiv 3 \pmod 8, \\ 0, & \text{if } n \equiv 7 \pmod 8, \\ 12S(N), & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^\alpha - \left(\frac{8}{N}\right)\right) S(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. By (1.3), Theorems 3.10 and 4.3, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,1,8;n)q^n &= \varphi^3(q)\varphi(q^8) \\
&= \alpha(q^4) + 3\beta(q) - 8\beta(q^4) \\
&\quad - \frac{1}{2}(\beta(q) - \beta(-q) + i\beta(iq) - i\beta(-iq)) \\
&\quad - \frac{1}{4}(\omega\beta(\omega q) - \omega\beta(-\omega q) + \omega^3\beta(\omega^3 q) - \omega^3\beta(-\omega^3 q)) \\
&= 1 - 2\sum_{n=1}^{\infty} R(n)q^{4n} + 6\sum_{n=1}^{\infty} S(n)q^n - 16\sum_{n=1}^{\infty} S(n)q^{4n} \\
&\quad - \sum_{n=1}^{\infty} S(n)(1 - (-1)^n + i^{n+1} - (-1)^n i^{n+1})q^n \\
&\quad - \frac{1}{2}\sum_{n=1}^{\infty} S(n)(\omega^{n+1} - (-1)^n\omega^{n+1} + \omega^{3n+3} - (-1)^n\omega^{3n+3})q^n \\
&= 1 - 2\sum_{n=1}^{\infty} R(n/4)q^n + 6\sum_{n=1}^{\infty} S(n)q^n - 16\sum_{n=1}^{\infty} S(n/4)q^n \\
&\quad - 4\sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} S(n)q^n + 2\sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty} S(n)q^n \\
&\quad - 2\sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty} S(n)q^n.
\end{aligned}$$

If $n \equiv 1 \pmod 4$, we have

$$N(1,1,1,8;n) = 6S(n) = 6S(N).$$

If $n \equiv 3 \pmod 8$, we have

$$N(1,1,1,8;n) = 6S(n) - 4S(n) + 2S(n) = 4S(n) = 4S(N).$$

If $n \equiv 7 \pmod 8$, we have

$$N(1,1,1,8;n) = 6S(n) - 4S(n) - 2S(n) = 0.$$

If $n \equiv 2 \pmod 4$, we have

$$N(1,1,1,8;n) = 6S(n) = 12S(N).$$

If $n \equiv 0 \pmod 4$, we have

$$\begin{aligned} N(1,1,1,8;n) &= -2R(n/4) + 6S(n) - 16S(n/4) \\ &= -2\left(\frac{8}{N}\right)S(N) + 6\cdot 2^{\alpha}S(N) - 16\cdot 2^{\alpha-2}S(N) \\ &= 2\left(2^{\alpha} - \left(\frac{8}{N}\right)\right)S(N). \end{aligned}$$ □

Theorem 5.10 was stated by Liouville [5] without proof. The authors have not located a proof of Theorem 5.10 in the literature.

6. Conclusion

There are twenty quaternary forms $x^2 + ey^2 + fz^2 + gt^2$ with

$$e, f, g \in \{1, 2, 4, 8\}, \quad e \le f \le g.$$

In this paper $N(1, e, f, g; n)$ $(n \in \mathbb{N})$ was evaluated for ten of these forms in terms of the sum $S(n) = \sum_{d|n} \frac{n}{d}\left(\frac{8}{d}\right)$. Of the remaining ten forms, $N(1, e, f, g; n)$ can be evaluated in terms of $\sigma(n) = \sum_{d|n} d$ for six of them and in terms of $\sigma(n)$ and the sum $\sum_{\substack{(i,\,s)\,\in\,\mathbb{N}_0\,\times\,\mathbb{Z} \\ i^2\,+\,4s^2\,=\,n}} (-1)^{\frac{i-1}{2}} i$ for the remaining four forms, see *Acta Arith.* **130** (2007) 277–310 and *Int. J. Modern Math.* **2** (2007) 143–176.

Acknowledgments

The fourth author was supported by research grant A-7233 from the Natural Sciences and Engineering Research Council of Canada.

References

[1] E. Benz, Über die Anzahl Darstellungen einer Zahlen durch gewisse quaternäre quadratische Formen: Beweise, welche auf Identiäten aus dem Gebiete der Thetafunktionen basierren. Dissertation, Zürich 1964, in *Studien zur Theorie der quadratischen Formen*, eds. B. L. van der Waerden and H. Gross (Birkhäuser Verlag, 1968), pp. 165–198.

[2] B. C. Berndt, *Ramanujan's Notebooks, Part III* (Springer-Verlag, New York, 1991).

[3] P. Demuth, Die Zahl der Darstellungen einer natürlichen Zahl durch spezielle quaternäre quadratishe Formen aufgrund der Siegelschen Massformel, in *Studien zur Theorie der quadratischen Formen*, eds. B. L. van der Waerden and H. Gross, (Birkhäuser Verlag, 1968), pp. 224–254.

[4] J. Liouville, Sur les deux formes $x^2 + y^2 + z^2 + 2t^2$, $x^2 + 2(y^2 + z^2 + t^2)$, *J. Math. Pures Appl.* **6** (1861) 225–230.

[5] J. Liouville, Sur la forme $x^2 + y^2 + z^2 + 8t^2$, *J. Math. Pures Appl.* **6** (1861) 324–328.

[6] J. Liouville, Sur la forme $x^2 + 8(y^2 + z^2 + t^2)$, *J. Math. Pures Appl.* **7** (1862) 5–8.

[7] J. Liouville, Sur la forme $x^2 + 4y^2 + 4z^2 + 8t^2$, *J. Math. Pures Appl.* **7** (1862) 9–12.

[8] J. Liouville, Sur la forme $x^2 + 2y^2 + 4z^2 + 4t^2$, *J. Math. Pures Appl.* **7** (1862) 62–64.

[9] J. Liouville, Sur la forme $x^2 + 2y^2 + 8z^2 + 8t^2$, *J. Math. Pures Appl.* **7** (1862) 65–68.

[10] J. Liouville, Sur la forme $x^2 + y^2 + 2z^2 + 4t^2$, *J. Math. Pures Appl.* **7** (1862) 99–100.

[11] J. Liouville, Sur la forme $x^2 + y^2 + 4z^2 + 8t^2$, *J. Math. Pures Appl.* **7** (1862) 103–104.

[12] J. Liouville, Sur la forme $x^2 + 2y^2 + 2z^2 + 8t^2$, *J. Math. Pures Appl.* **7** (1862) 148–149.

[13] T. Pepin, Étude sur quelques formules d'analyse utiles dans la thèorie des nombres, *Atti Accad. Pont. Nuovi Lincei* **38** (1884-5) 139–196.

[14] K. Petr, O počtu třid forem kvadratických záporného diskriminantu, *Rozpravy Ceské Akademie Cisare Frantiska Josefa I* **10** (1901) 1–22.

[15] H. Wild, Die Anzahl der Darstellungen einer natürlichen Zahl durch die Form $x^2 + y^2 + z^2 + 2t^2$, *Abh. Math. Sem. Univ. Hamburg* **40** (1974) 132–135.

[16] K. S. Williams, On the representations of a positive integer by the forms $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2$, *Int. J. Modern Math.* **3** (2008) 225–230.