

Seven octonary quadratic forms

by

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1. Introduction. Let \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} and \mathbb{C} denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. For $k, a_1, \dots, a_k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$(1.1) \quad N(a_1, \dots, a_k; n) := \text{card}\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid n = a_1x_1^2 + \dots + a_kx_k^2\}.$$

As $N(a_1, \dots, a_k; n)$ remains invariant under a permutation of a_1, \dots, a_k , we may suppose that

$$(1.2) \quad a_1 \leq \dots \leq a_k.$$

Clearly,

$$(1.3) \quad N(a_1, \dots, a_k; 0) = 1.$$

If l of a_1, \dots, a_k are equal, say

$$a_i = a_{i+1} = \dots = a_{i+l-1} = a,$$

we indicate this in $N(a_1, \dots, a_k; n)$ by writing a^l for $a_i, a_{i+1}, \dots, a_{i+l-1}$. For $k \in \mathbb{N}$ the sum of divisors function $\sigma_k(n)$ is defined by

$$\sigma_k(n) := \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d|n}} d^k & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n \notin \mathbb{N}. \end{cases}$$

We write $\sigma(n)$ for $\sigma_1(n)$.

The authors and M. F. Lemire have recently proved formulae for $N(1^i, 4^{4-i}; n)$ for $i \in \{1, 2, 3, 4\}$ and all $n \in \mathbb{N}$ in terms of $\sigma(n), \sigma(n/2), \sigma(n/4), \sigma(n/8)$ and $\sigma(n/16)$ [1, Theorems 1.6, 1.7, 1.11 and 1.18]. The origins of these formulae are given in [1, pp. 284–286].

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PROPOSITION 1.1. *Let $n \in \mathbb{N}$. Then*

- (i) $N(1^4; n) = 8\sigma(n) - 32\sigma(n/4),$
- (ii) $N(1^3, 4; n) = \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 20\sigma(n/4)$
 $+ 24\sigma(n/8) - 32\sigma(n/16),$
- (iii) $N(1^2, 4^2; n) = \left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 2\sigma(n/2)$
 $+ 8\sigma(n/8) - 32\sigma(n/16),$
- (iv) $N(1, 4^3; n) = \left(1 + \left(\frac{-4}{n}\right)\right)\sigma(n) - 3\sigma(n/2)$
 $+ 10\sigma(n/4) - 32\sigma(n/16),$

where $\left(\frac{-4}{n}\right)$ is the Legendre–Jacobi–Kronecker symbol for discriminant -4 , that is,

$$\left(\frac{-4}{n}\right) = \begin{cases} +1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

Proof. See [1, pp. 296, 297, 298, 303]. ■

DEFINITION 1.1. For $k, n \in \mathbb{N}$ we define

$$W_k(n) := \sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n - km).$$

In recent years the convolution sums $W_k(n)$ have been evaluated explicitly for certain values of k and all $n \in \mathbb{N}$. We require the evaluations for $k = 1$ [6, eq. (3.10), p. 236], $k = 2$ [6, Theorem 2, p. 247], $k = 4$ [6, Theorem 4, p. 249], $k = 8$ [7, Theorem 1, p. 388], and $k = 16$ [2, Theorem 1.1, p. 4].

PROPOSITION 1.2. *Let $n \in \mathbb{N}$. Then*

- (i) $W_1(n) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n).$
- (ii) $W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n)$
 $+ \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/2).$
- (iii) $W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4)$
 $+ \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/4).$

$$(iv) \quad W_8(n) = \frac{1}{192} \sigma_3(n) + \frac{1}{64} \sigma_3(n/2) + \frac{1}{16} \sigma_3(n/4) + \frac{1}{3} \sigma_3(n/8) \\ + \left(\frac{1}{24} - \frac{n}{32}\right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma(n/8) - \frac{1}{64} c_8(n),$$

where the integers $c_8(n)$ ($n \in \mathbb{N}$) are given by

$$(1.4) \quad \sum_{n=1}^{\infty} c_8(n) q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4.$$

$$(v) \quad W_{16}(n) = \frac{1}{768} \sigma_3(n) + \frac{1}{256} \sigma_3(n/2) \\ + \frac{1}{64} \sigma_3(n/4) + \frac{1}{16} \sigma_3(n/8) + \frac{1}{3} \sigma_3(n/16) \\ + \left(\frac{1}{24} - \frac{n}{64}\right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma(n/16) - \frac{7}{256} c_{16}(n),$$

where the rational numbers $c_{16}(n)$ ($n \in \mathbb{N}$) are given by

$$(1.5) \quad \sum_{n=1}^{\infty} c_{16}(n) q^n = \frac{1}{32} A_1(q) + \frac{3}{112} A_2(q) + \frac{1}{224} A_3(q) \\ - \frac{1}{32} A_5(q) - \frac{3}{112} A_6(q) - \frac{1}{224} A_7(q)$$

with

$$(1.6) \quad A_k(q) := \prod_{n=1}^{\infty} (1 + q^n)^{24-4k} (1 - q^n)^8 (1 - q^{4n-2})^{16-2k}.$$

We remark that for $n \in \mathbb{N}$,

$$(1.7) \quad c_8(n) = 0 \quad \text{if } n \equiv 0 \pmod{2}$$

(see [7, p. 388]), and

$$(1.8) \quad 7c_{16}(n) \in \mathbb{Z}$$

(see [2, eq. (1.6), p. 4]).

DEFINITION 1.2. For $a \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ we define

$$S_{a,m}(n) := \sum_{\substack{l=1 \\ l \equiv a \pmod{m}}}^{n-1} \sigma(l) \sigma(n-l).$$

Clearly,

$$S_{a,m}(n) = S_{b,m}(n) \quad \text{if } a \equiv b \pmod{m},$$

and

$$\sum_{a=0}^{m-1} S_{a,m}(n) = W_1(n).$$

These sums were introduced in [6, p. 255]. We require the evaluation of $S_{a,4}(n)$ given in [3, Theorem 1.1].

PROPOSITION 1.3. *Let $n \in \mathbb{N}$.*

(i) *If $n \equiv 0 \pmod{4}$ then*

$$S_{0,4}(n) = \frac{29}{192} \sigma_3(n) + \frac{17}{64} \sigma_3(n/2) + \left(\frac{1}{12} - \frac{n}{2} \right) \sigma(n),$$

$$S_{1,4}(n) = \frac{1}{16} \sigma_3(n) - \frac{1}{16} \sigma_3(n/2),$$

$$S_{2,4}(n) = \frac{9}{64} \sigma_3(n) - \frac{9}{64} \sigma_3(n/2),$$

$$S_{3,4}(n) = \frac{1}{16} \sigma_3(n) - \frac{1}{16} \sigma_3(n/2).$$

(ii) *If $n \equiv 1 \pmod{4}$ then*

$$S_{0,4}(n) = \frac{11}{96} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n) + \frac{3}{32} c_8(n),$$

$$S_{1,4}(n) = \frac{11}{96} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n) + \frac{3}{32} c_8(n),$$

$$S_{2,4}(n) = \frac{3}{32} \sigma_3(n) - \frac{3}{32} c_8(n),$$

$$S_{3,4}(n) = \frac{3}{32} \sigma_3(n) - \frac{3}{32} c_8(n).$$

(iii) *If $n \equiv 2 \pmod{4}$ then*

$$S_{0,4}(n) = \frac{11}{72} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n),$$

$$S_{1,4}(n) = \frac{1}{18} \sigma_3(n) + \frac{1}{2} c_8(n/2),$$

$$S_{2,4}(n) = \frac{11}{72} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n),$$

$$S_{3,4}(n) = \frac{1}{18} \sigma_3(n) - \frac{1}{2} c_8(n/2).$$

(iv) *If $n \equiv 3 \pmod{4}$ then*

$$S_{0,4}(n) = \frac{11}{96} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n) + \frac{3}{32} c_8(n),$$

$$\begin{aligned}
S_{1,4}(n) &= \frac{3}{32} \sigma_3(n) - \frac{3}{32} c_8(n), \\
S_{2,4}(n) &= \frac{3}{32} \sigma_3(n) - \frac{3}{32} c_8(n), \\
S_{3,4}(n) &= \frac{11}{96} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma(n) + \frac{3}{32} c_8(n).
\end{aligned}$$

In this paper we use Propositions 1.1, 1.2 and 1.3 to determine $N(1^i, 4^{8-i}; n)$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$ and all $n \in \mathbb{N}$. We prove

THEOREM 1.1. *Let $n \in \mathbb{N}$. Then*

- (i)
$$\begin{aligned}
N(1^7, 4; n) &= \left(8 - \left(\frac{-4}{n} \right) \right) \sigma_3(n) - 16\sigma_3(n/2) + 136\sigma_3(n/4) \\
&\quad - 144\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + 7 \left(\frac{-4}{n} \right) c_8(n) + 28c_8(n/2).
\end{aligned}$$
- (ii)
$$\begin{aligned}
N(1^6, 4^2; n) &= \left(4 - \left(\frac{-4}{n} \right) \right) \sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/8) \\
&\quad + 256\sigma_3(n/16) + \left(2 + 7 \left(\frac{-4}{n} \right) \right) c_8(n) + 28c_8(n/2).
\end{aligned}$$
- (iii)
$$\begin{aligned}
N(1^5, 4^3; n) &= \left(2 - \frac{1}{2} \left(\frac{-4}{n} \right) \right) \sigma_3(n) + 2\sigma_3(n/2) - 68\sigma_3(n/4) \\
&\quad + 48\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + \left(3 + \frac{11}{2} \left(\frac{-4}{n} \right) \right) c_8(n) + 20c_8(n/2).
\end{aligned}$$
- (iv)
$$\begin{aligned}
N(1^4, 4^4; n) &= \sigma_3(n) + 3\sigma_3(n/2) - 68\sigma_3(n/4) + 48\sigma_3(n/8) \\
&\quad + 256\sigma_3(n/16) + \left(3 + 4 \left(\frac{-4}{n} \right) \right) c_8(n) + 12c_8(n/2).
\end{aligned}$$
- (v)
$$\begin{aligned}
N(1^3, 4^5; n) &= \left(\frac{1}{2} + \frac{1}{4} \left(\frac{-4}{n} \right) \right) \sigma_3(n) + \frac{3}{2} \sigma_3(n/2) - 34\sigma_3(n/4) \\
&\quad + 16\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + \left(\frac{5}{2} + \frac{11}{4} \left(\frac{-4}{n} \right) \right) c_8(n) + 6c_8(n/2).
\end{aligned}$$
- (vi)
$$\begin{aligned}
N(1^2, 4^6; n) &= \left(\frac{1}{4} + \frac{1}{4} \left(\frac{-4}{n} \right) \right) \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16\sigma_3(n/8) \\
&\quad + 256\sigma_3(n/16) + \left(\frac{7}{4} + \frac{7}{4} \left(\frac{-4}{n} \right) \right) c_8(n) + 2c_8(n/2).
\end{aligned}$$

$$(vii) \quad N(1, 4^7; n) = \left(\frac{1}{8} + \frac{1}{8} \left(\frac{-4}{n}\right)\right) \sigma_3(n) - \frac{9}{8} \sigma_3(n/2) + 17\sigma_3(n/4) \\ - 32\sigma_3(n/8) + 256\sigma_3(n/16) + \left(\frac{7}{8} + \frac{7}{8} \left(\frac{-4}{n}\right)\right) c_8(n).$$

Part (iv) of Theorem 1.1 was proved in [2, Theorem 1.2, p. 4] in terms of c_{16} rather than c_8 .

2. Some preliminary results. The following sums will be needed in the proof of Theorem 1.1.

DEFINITION 2.1. For $r, s \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we define

$$(2.1) \quad X(2^r, 2^s; n) := \sum_{m=1}^{n-1} \sigma(m/2^r) \sigma((n-m)/2^s).$$

Clearly,

$$(2.2) \quad X(2^r, 2^s; n) = X(2^s, 2^r; n).$$

PROPOSITION 2.1. Let $r, s \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then

$$X(2^r, 2^s; n) = \begin{cases} W_{2^{r-s}}(n/2^s) & \text{if } r \geq s, \\ W_{2^{s-r}}(n/2^r) & \text{if } r \leq s. \end{cases}$$

Proof. If $r \geq s$ then

$$X(2^r, 2^s; n) = \sum_{\substack{m \in \mathbb{N} \\ m < n/2^r}} \sigma(m) \sigma\left(\frac{n}{2^s} - 2^{r-s}m\right) = W_{2^{r-s}}(n/2^s).$$

If $r \leq s$ then

$$X(2^r, 2^s; n) = X(2^s, 2^r; n) = W_{2^{s-r}}(n/2^r). \blacksquare$$

PROPOSITION 2.2. For $n \in \mathbb{N}$,

$$\sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m) \sigma(n-m) \\ = \left(\frac{-4}{n}\right) \left(\frac{1}{48} \sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma(n) + \frac{3}{16} c_8(n)\right) + c_8(n/2).$$

Proof. We have

$$\begin{aligned} \sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m)\sigma(n-m) &= \sum_{\substack{m=1 \\ m \equiv 1 \pmod{4}}}^{n-1} \sigma(m)\sigma(n-m) - \sum_{\substack{m=1 \\ m \equiv 3 \pmod{4}}}^{n-1} \sigma(m)\sigma(n-m) \\ &= S_{1,4}(n) - S_{3,4}(n) \end{aligned}$$

by Definition 1.2, and the asserted result follows from Proposition 1.3. ■

PROPOSITION 2.3. For $n \in \mathbb{N}$,

$$\sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m)\sigma((n-m)/2) = \left(\frac{-4}{n}\right) (-W_2(n) + 6W_4(n) - 4W_8(n)).$$

Proof. We have

$$\begin{aligned} \sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m)\sigma((n-m)/2) &= \sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \left(\frac{-4}{n-2l}\right) \sigma(n-2l)\sigma(l) \\ &= \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \left(\frac{-4}{n-4l}\right) \sigma(2l)\sigma(n-4l) \\ &\quad + \sum_{\substack{l \in \mathbb{N} \\ l < (n+2)/4}} \left(\frac{-4}{n-2(2l-1)}\right) \sigma(2l-1)\sigma(n-2(2l-1)) \\ &= \left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \sigma(2l)\sigma(n-4l) \\ &\quad + \left(\frac{-4}{n+2}\right) \sum_{\substack{l \in \mathbb{N} \\ l < (n+2)/4}} \sigma(2l-1)\sigma(n-2(2l-1)) \\ &= \left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \sigma(2l)\sigma(n-4l) \\ &\quad - \left(\frac{-4}{n}\right) \left(\sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \sigma(l)\sigma(n-2l) - \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \sigma(2l)\sigma(n-4l) \right) \end{aligned}$$

$$\begin{aligned}
 &= -\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \sigma(l)\sigma(n-2l) + 2\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \sigma(2l)\sigma(n-4l) \\
 &= -\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \sigma(l)\sigma(n-2l) + 2\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} (3\sigma(l) - 2\sigma(l/2))\sigma(n-4l) \\
 &= \left(\frac{-4}{n}\right)(-W_2(n) + 6W_4(n) - 4W_8(n)),
 \end{aligned}$$

as asserted. ■

PROPOSITION 2.4. For $k, n \in \mathbb{N}$ with $k \geq 2$,

$$\sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m)\sigma((n-m)/2^k) = \left(\frac{-4}{n}\right) W_{2^k}(n).$$

Proof. As $k \geq 2$ we have

$$\left(\frac{-4}{n-2^k l}\right) = \left(\frac{-4}{n}\right),$$

and so

$$\begin{aligned}
 \sum_{m=1}^{n-1} \left(\frac{-4}{m}\right) \sigma(m)\sigma((n-m)/2^k) &= \sum_{\substack{l \in \mathbb{N} \\ l < n/2^k}} \left(\frac{-4}{n-2^k l}\right) \sigma(n-2^k l)\sigma(l) \\
 &= \left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2^k}} \sigma(l)\sigma(n-2^k l) = \left(\frac{-4}{n}\right) W_{2^k}(n)
 \end{aligned}$$

by Definition 1.1. ■

3. The relationship between $c_8(n)$ and $c_{16}(n)$. Let q be a complex variable with $|q| < 1$. As in [4, p. 6] we set

$$(3.1) \quad \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

The infinite product representations of $\varphi(q)$ and $\varphi(-q)$ are due to Jacobi, namely

$$(3.2) \quad \varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \quad \varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}.$$

DEFINITION 3.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty} (1-q^{kn}).$$

From (3.2) and Definition 3.1, we deduce

$$(3.3) \quad \varphi(q) = E_1^{-2} E_2^5 E_4^{-2},$$

$$(3.4) \quad \varphi(-q) = E_1^2 E_2^{-1}.$$

LEMMA 3.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$,

$$A_k(q) = \varphi^{8-k}(q)\varphi^k(-q).$$

Proof. We have

$$\prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^n} = E_1^{-1} E_2,$$

$$\prod_{n=1}^{\infty} (1 - q^{4n-2}) = \prod_{n=1}^{\infty} \frac{(1 - q^{4n-2})(1 - q^{4n})}{1 - q^{4n}} = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{4n}} = E_2 E_4^{-1}.$$

Thus, by (1.6), we obtain

$$A_k(q) = \prod_{n=1}^{\infty} (1 + q^n)^{24-4k} (1 - q^n)^8 (1 - q^{4n-2})^{16-2k} = E_1^{4k-16} E_2^{40-6k} E_4^{2k-16}$$

$$= (E_1^{-2} E_2^5 E_4^{-2})^{8-k} (E_1^2 E_2^{-1})^k = \varphi^{8-k}(q)\varphi^k(-q). \blacksquare$$

Following Berndt [4, pp. 119–120] we set

$$(3.5) \quad x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)},$$

$$(3.6) \quad z = \varphi^2(q).$$

From Berndt’s catalogue of formulae for theta fuctions [4, p. 122] we have

$$(3.7) \quad \varphi(q) = \sqrt{z},$$

$$(3.8) \quad \varphi(-q) = \sqrt{z}(1 - x)^{1/4}.$$

Following Cheng and Williams [5, p. 564] we set

$$(3.9) \quad g = (1 - x)^{1/4}.$$

LEMMA 3.2. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we have

$$A_k(q) = g^k z^4.$$

Proof. By Lemma 3.1 and (3.7)–(3.9), we have

$$A_k(q) = (\sqrt{z})^{8-k} (\sqrt{z}(1 - x)^{1/4})^k = (1 - x)^{k/4} z^4 = g^k z^4. \blacksquare$$

LEMMA 3.3.

$$\sum_{n=1}^{\infty} c_{16}(n)q^n = \left(\frac{1}{32} g + \frac{3}{112} g^2 + \frac{1}{224} g^3 - \frac{1}{224} g^5 - \frac{3}{112} g^6 - \frac{1}{224} g^7 \right) z^4.$$

Proof. This follows from (1.5) and Lemma 3.2. \blacksquare

LEMMA 3.4. For $q \in \mathbb{C}$ with $|q| < 1$, we have

$$(i) \quad \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} c_8(n)q^n = \frac{1}{64} (g + 2g^2 + g^3 - g^5 - 2g^6 - g^7)z^4.$$

$$(ii) \quad \sum_{\substack{n=1 \\ n \equiv 3 \pmod{4}}}^{\infty} c_8(n)q^n = \frac{1}{64} (-g + 2g^2 - g^3 + g^5 - 2g^6 + g^7)z^4.$$

$$(iii) \quad \sum_{\substack{n=1 \\ n \equiv 0 \pmod{2}}}^{\infty} c_8(n/2)q^n = \frac{1}{128} (g - g^3 - g^5 + g^7)z^4.$$

Proof. Part (i) is [3, Theorem 2.3(i)]. Part (ii) is [3, Theorem 2.3(ii)]. By [3, Theorem 2.4] and (1.7), we have

$$\sum_{\substack{n=1 \\ n \equiv 0 \pmod{2}}}^{\infty} c_8(n/2)q^n = \sum_{\substack{n=1 \\ n \equiv 2 \pmod{4}}}^{\infty} c_8(n/2)q^n = \frac{1}{128} (g - g^3 - g^5 + g^7)z^4. \blacksquare$$

THEOREM 3.1. For $n \in \mathbb{N}$,

$$c_{16}(n) = \begin{cases} \frac{12}{7} c_8(n/2) & \text{if } n \equiv 0 \pmod{2}, \\ c_8(n) & \text{if } n \equiv 1 \pmod{4}, \\ -\frac{1}{7} c_8(n) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof. By Lemmas 3.3 and 3.4 we have

$$\begin{aligned} \sum_{n=1}^{\infty} c_{16}(n)q^n &= \left(\frac{1}{32} g + \frac{3}{112} g^2 + \frac{1}{224} g^3 - \frac{1}{224} g^5 - \frac{3}{112} g^6 - \frac{1}{224} g^7 \right) z^4 \\ &= \left(\frac{1}{64} g + \frac{1}{32} g^2 + \frac{1}{64} g^3 - \frac{1}{64} g^5 - \frac{1}{32} g^6 - \frac{1}{64} g^7 \right) z^4 \\ &\quad - \frac{1}{7} \left(-\frac{1}{64} g + \frac{1}{32} g^2 - \frac{1}{64} g^3 + \frac{1}{64} g^5 - \frac{1}{32} g^6 + \frac{1}{64} g^7 \right) z^4 \\ &\quad + \frac{12}{7} \left(\frac{1}{128} g - \frac{1}{128} g^3 - \frac{1}{128} g^5 + \frac{1}{128} g^7 \right) z^4 \\ &= \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} c_8(n)q^n - \frac{1}{7} \sum_{\substack{n=1 \\ n \equiv 3 \pmod{4}}}^{\infty} c_8(n)q^n + \frac{12}{7} \sum_{\substack{n=1 \\ n \equiv 0 \pmod{2}}}^{\infty} c_8(n/2)q^n. \end{aligned}$$

Equating coefficients of q^n , we obtain the assertion. \blacksquare

We note that the first equality of Theorem 3.1 is Corollary 2.1 of [2]. We also observe that Theorem 3.1 can be expressed as

$$(3.10) \quad c_{16}(n) = \frac{1}{7} \left(3 + 4 \left(\frac{-4}{n} \right) \right) c_8(n) + \frac{12}{7} c_8(n/2).$$

4. Proof of Theorem 1.1. We just prove part (i) as the remaining parts can be proved similarly. Appealing to (1.3), Proposition 1.1(i)–(ii), Definition 2.1 and Propositions 2.1–2.3, we obtain

$$\begin{aligned} N(1^7, 4; n) &= \sum_{m=0}^n N(1^3, 4; m)N(1^4; n - m) \\ &= N(1^4; n) + N(1^3, 4; n) + \sum_{m=1}^{n-1} N(1^3, 4; m)N(1^4; n - m) \\ &= 8\sigma(n) - 32\sigma(n/4) \\ &\quad + \left(4 + 2 \left(\frac{-4}{n} \right) \right) \sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\ &\quad + \sum_{m=1}^{n-1} (4\sigma(m) - 20\sigma(m/4) + 24\sigma(m/8) - 32\sigma(m/16)) \\ &\quad \times (8\sigma(n - m) - 32\sigma((n - m)/4)) \\ &\quad + \sum_{m=1}^{n-1} 2 \left(\frac{-4}{m} \right) \sigma(m)(8\sigma(n - m) - 32\sigma((n - m)/4)) \\ &= \left(12 + 2 \left(\frac{-4}{m} \right) \right) \sigma(n) - 52\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\ &\quad + 32X(1, 1; n) - 160X(4, 1; n) + 192X(8, 1; n) - 256X(16, 1; n) \\ &\quad - 128X(1, 4; n) + 640X(4, 4; n) - 768X(8, 4; n) \\ &\quad + 1024X(16, 4; n) + 16 \sum_{m=1}^{n-1} \left(\frac{-4}{m} \right) \sigma(m)\sigma(n - m) \\ &\quad - 64 \sum_{m=1}^{n-1} \left(\frac{-4}{m} \right) \sigma(m)\sigma((n - m)/4) \\ &= \left(12 + 2 \left(\frac{-4}{m} \right) \right) \sigma(n) - 52\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\ &\quad + 32W_1(n) - 288W_4(n) + 192W_8(n) - 256W_{16}(n) \end{aligned}$$

$$\begin{aligned}
& + 640W_1(n/4) - 768W_2(n/4) + 1024W_4(n/4) \\
& + 16\left(\frac{-4}{n}\right)\left(\frac{1}{48}\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{16}c_8(n)\right) \\
& + 16c_8(n/2) - 64\left(\frac{-4}{n}\right)W_4(n).
\end{aligned}$$

The asserted result now follows from Proposition 1.2 and Theorem 3.1. ■

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