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Theta Function Identities and Representations by Certain Quaternary Quadratic Forms II

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Abstract

Some new theta function identities are proved and used to determine the number of representations of a positive integer n by certain quaternary quadratic forms.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers respectively. For $q \in \mathbb{C}$ with $|q| < 1$ Ramanujan defined the one-dimensional theta function $\varphi(q)$ by

$$\varphi(q) := \sum_{n \in \mathbb{Z}} q^{n^2}. \tag{1.1}$$

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define

$$N(a, b, c, d; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}. \tag{1.2}$$

In [1] the authors determined $N(a, b, c, d; n)$ $(n \in \mathbb{N})$ for the following quaternary quadratic forms:

$$x^2 + y^2 + z^2 + 3t^2$$

$$x^2 + y^2 + 2z^2 + 6t^2$$
$$x^2 + 2y^2 + 2z^2 + 3t^2$$
$$x^2 + 2y^2 + 4z^2 + 6t^2$$
$$x^2 + 3y^2 + 3z^2 + 3t^2$$
$$x^2 + 3y^2 + 6z^2 + 6t^2$$
$$2x^2 + 3y^2 + 3z^2 + 6t^2.$$

For all of these seven forms, $N(a,b,c,d;n)$ was given in terms of the arithmetic functions $A(n)$, $B(n)$, $C(n)$ and $D(n)$ given in Definition 2.1. In this paper we consider the sixteen forms

$$x^2 + y^2 + z^2 + 12t^2$$
$$x^2 + y^2 + 3z^2 + 4t^2$$
$$x^2 + y^2 + 4z^2 + 12t^2$$
$$x^2 + 2y^2 + 2z^2 + 12t^2$$
$$x^2 + 3y^2 + 3z^2 + 12t^2$$
$$x^2 + 3y^2 + 4z^2 + 4t^2$$
$$x^2 + 3y^2 + 12z^2 + 12t^2$$
$$x^2 + 4y^2 + 4z^2 + 12t^2$$
$$x^2 + 6y^2 + 6z^2 + 12t^2$$
$$x^2 + 12y^2 + 12z^2 + 12t^2$$
$$2x^2 + 2y^2 + 3z^2 + 4t^2$$
$$3x^2 + 3y^2 + 3z^2 + 4t^2$$
$$3x^2 + 3y^2 + 4z^2 + 12t^2$$
$$3x^2 + 4y^2 + 4z^2 + 4t^2$$
$$3x^2 + 4y^2 + 6z^2 + 6t^2$$
$$3x^2 + 4y^2 + 12z^2 + 12t^2$$

and show that $N(a,b,c,d;n)$ for these forms can be given in terms of $A(n)$, $B(n)$, $C(n)$, $D(n)$ and the two additional arithmetic functions $E(n)$ and $F(n)$ defined in Definition 2.2.

2 Notation and Preliminary Results

For $q \in \mathbb{C}$ with $|q| < 1$ we set

$$\varphi(q) := \sum_{n \in \mathbb{N}} q^{n^2}. \tag{2.1}$$

As in [5, pp. 32, 33] we define

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{2.2}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.3}$$

The representations of $\varphi(q^j)$ ($j \in \{1, 2, 3, 4, 6, 12\}$) in terms of p and k are given in Proposition 2.1 (see [1, Theorem 2.4].

Proposition 2.1. *Define p and k by* (2.2) *and* (2.3) *respectively. Then*

(a) $\varphi(q) = (1+2p)^{3/4}k^{1/2}$,

(b) $\varphi(q^2) = \frac{1}{\sqrt{2}}\left((1+2p)^{3/2} + (1-p)^{3/2}(1+p)^{1/2}\right)^{1/2} k^{1/2}$,

(c) $\varphi(q^3) = (1+2p)^{1/4}k^{1/2}$,

(d) $\varphi(q^4) = \frac{1}{2}\left((1+2p)^{3/4} + (1-p)^{3/4}(1+p)^{1/4}\right) k^{1/2}$,

(e) $\varphi(q^6) = \frac{1}{\sqrt{2}}\left((1+2p)^{1/2} + (1-p)^{1/2}(1+p)^{3/2}\right)^{1/2} k^{1/2}$,

(f) $\varphi(q^{12}) = \frac{1}{2}\left((1+2p)^{1/4} + (1-p)^{1/4}(1+p)^{3/4}\right) k^{1/2}$.

Using Proposition 2.1 in conjunction with the following well-known basic properties of $\varphi(q)$

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.4}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{2.5}$$

and

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.6}$$

we obtain $\varphi(-q^j)$ ($j \in \{1, 2, 3, 4, 6, 12\}$) in terms of p and k.

Proposition 2.2.

(a) $\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2}$,

(b) $\varphi(-q^2) = (1+2p)^{3/8}(1-p)^{3/8}(1+p)^{1/8}k^{1/2}$,

(c) $\varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2}$,

(d) $$\varphi(-q^4) = 2^{-1/4}(1+2p)^{3/16}(1+p)^{1/16}(1-p)^{3/16} \times \big((1+2p)^{3/2} + (1+p)^{1/2}(1-p)^{3/2}\big)^{1/4}k^{1/2},$$

(e) $\varphi(-q^6) = (1+2p)^{1/8}(1+p)^{3/8}(1-p)^{1/8}k^{1/2}$,

(f) $$\varphi(-q^{12}) = 2^{-1/4}(1+2p)^{1/16}(1+p)^{3/16}(1-p)^{1/16} \times \big((1+2p)^{1/2} + (1+p)^{3/2}(1-p)^{1/2}\big)^{1/4}k^{1/2}.$$

In [1, Definition 3.1] we introduced the multiplicative arithmetic functions $A(n)$, $B(n)$, $C(n)$ and $D(n)$.

Definition 2.1. *For $n \in \mathbb{N}$ we set*

(a) $$A(n) := \sum_{d|n} d\Big(\frac{12}{n/d}\Big) = \sum_{d|n} \frac{n}{d}\Big(\frac{12}{d}\Big),$$

(b) $$B(n) := \sum_{d|n} d\Big(\frac{-3}{d}\Big)\Big(\frac{-4}{n/d}\Big) = \sum_{d|n} \frac{n}{d}\Big(\frac{-3}{n/d}\Big)\Big(\frac{-4}{d}\Big),$$

(c) $$C(n) := \sum_{d|n} d\Big(\frac{-3}{n/d}\Big)\Big(\frac{-4}{d}\Big) = \sum_{d|n} \frac{n}{d}\Big(\frac{-3}{d}\Big)\Big(\frac{-4}{n/d}\Big),$$

(d) $$D(n) := \sum_{d|n} d\Big(\frac{12}{d}\Big) = \sum_{d|n} \frac{n}{d}\Big(\frac{12}{n/d}\Big),$$

where $\left(\frac{D}{k}\right)$ $(k \in \mathbb{N})$ is the Legendre-Jacobi-Kronecker symbol for discriminant D.

The following result was given in [1, Theorem 3.1].

Proposition 2.3. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

(a) $A(n) = 2^\alpha 3^\beta A(N)$,

(b) $B(n) = (-1)^{\alpha+\beta} 2^\alpha \Big(\frac{N}{3}\Big) A(N)$,

(c) $C(n) = (-1)^{\alpha+\beta+(N-1)/2} 3^\beta A(N)$,

(d) $D(n) = (-1)^{(N-1)/2}\Big(\frac{N}{3}\Big) A(N) = \Big(\frac{3}{N}\Big) A(N)$.

Simple consequences of Proposition 2.3 are

$$\begin{aligned}
A(n) &= B(n), & C(n) &= D(n), & &\text{if } n \equiv 1 \pmod 3,\\
A(n) &= -B(n), & C(n) &= -D(n), & &\text{if } n \equiv 2 \pmod 3,\\
A(n) &= C(n), & B(n) &= D(n), & &\text{if } n \equiv 1 \pmod 4,\\
A(n) &= -C(n), & B(n) &= -D(n), & &\text{if } n \equiv 3 \pmod 4.
\end{aligned}$$

The next result was deduced from the work of Petr [20], see [1, Theorem 3.2].

Proposition 2.4. *For* $|q| < 1$

$$\begin{aligned}
\sum_{n=1}^{\infty} A(n)q^n &= \frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi^3(q)\varphi(q^3)\\
&\quad -\frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{8}\varphi(-q)\varphi^2(q^3)\varphi(-q^3),\\
\sum_{n=1}^{\infty} B(n)q^n &= \frac{3}{8}\varphi(q)\varphi^3(q^3) - \frac{1}{8}\varphi^3(q)\varphi(q^3)\\
&\quad +\frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{8}\varphi(-q)\varphi^2(q^3)\varphi(-q^3),\\
\sum_{n=1}^{\infty} C(n)q^n &= \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{4}\varphi(-q)\varphi^2(q^3)\varphi(-q^3),\\
\sum_{n=1}^{\infty} D(n)q^n &= 1 - \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{4}\varphi(-q)\varphi^2(q^3)\varphi(-q^3).
\end{aligned}$$

Solving for the quantities $\varphi(q)\varphi^3(q^3)$, $\varphi^3(q)\varphi(q^3)$, $\varphi^2(q)\varphi(-q)\varphi(-q^3)$ and $\varphi(-q)\varphi^2(q^3)\varphi(-q^3)$ in Proposition 2.4, we obtain the following result, see [1, Theorem 3.3].

Proposition 2.5. *For* $|q| < 1$

(a) $$\varphi(q)\varphi^3(q^3) = 1 + \sum_{n=1}^{\infty}(2A(n) + 2B(n) - C(n) - D(n))q^n,$$

(b) $$\varphi^3(q)\varphi(q^3) = 1 + \sum_{n=1}^{\infty}(6A(n) - 2B(n) + 3C(n) - D(n))q^n,$$

(c) $$\varphi^2(q)\varphi(-q)\varphi(-q^3) = 1 + \sum_{n=1}^{\infty}(3C(n) - D(n))q^n,$$

(d) $\varphi(-q)\varphi^2(q^3)\varphi(-q^3) = 1 - \sum_{n=1}^{\infty}(C(n)+D(n))q^n.$

Proposition 2.6.

$$\sum_{\substack{m=-\infty \\ m \text{ odd}}}^{\infty} (-1)^{(m-1)/2} m q^{m^2} = \frac{1}{2}\big(\varphi(q)-\varphi(-q)\big)\varphi^2(-q^8).$$

Proof. See [2, Theorem 2.1]. ∎

Proposition 2.7.

$$\sum_{\substack{m=-\infty \\ m \text{ odd}}}^{\infty} (-1)^{(m-1)/2} m q^{3m^2} = \frac{1}{2}\big(\varphi(q^3)-\varphi(-q^3)\big)\varphi(q^{12})\varphi(-q^{12}).$$

Proof. Replacing q by q^3 in Proposition 2.6, and using the identity (2.6) with q replaced by q^{12}, we obtain Proposition 2.7. ∎

Definition 2.2. *For $n \in \mathbb{N}$ we set*

$$E(n) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(i-1)/2} i \quad \text{and} \quad F(n) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(j-1)/2} j.$$

Proposition 2.8. *Let $n \in \mathbb{N}$. Then*

$$E(n) = F(n) = 0, \text{ if } n \text{ is even.}$$

Proof. If i and j are both odd and $4n = i^2+3j^2$ then $4n \equiv 1+3 \equiv 4 \pmod 8$, so n is odd. ∎

Clearly, as

$$4n = (\pm i)^2 + 3(\pm j)^2, \quad (-1)^{(-i-1)/2}(-i) = (-1)^{(i-1)/2} i$$

and

$$(-1)^{(-j-1)/2}(-j) = (-1)^{(j-1)/2} j,$$

we have

$$E(n) := \frac{1}{4}\sum_{\substack{(i,j)\in\mathbb{Z}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(i-1)/2} i \quad \text{and} \quad F(n) := \frac{1}{4}\sum_{\substack{(i,j)\in\mathbb{Z}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(j-1)/2} j. \tag{2.7}$$

Theorem 2.1.

$$\text{(a)}\quad \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} E(n)q^{4n} = \frac{1}{16}\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big)\varphi(q^4)\varphi(-q^4),$$

$$\text{(b)}\quad \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} F(n)q^{4n} = \frac{1}{16}\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big)\varphi(q^{12})\varphi(-q^{12}).$$

Proof. (a) Appealing to Proposition 2.8 and (2.7), we obtain

$$(2.8)\quad \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} E(n)q^{4n} = \sum_{n=1}^{\infty} E(n)q^{4n} = \frac{1}{4}\Big(\sum_{\substack{i=-\infty\\ i\text{ odd}}}^{\infty} (-1)^{(i-1)/2} i q^{i^2}\Big)\Big(\sum_{\substack{j=-\infty\\ j\text{ odd}}}^{\infty} q^{3j^2}\Big).$$

Now, by (2.4), we have

$$(2.9)\quad \sum_{\substack{n=-\infty\\ n\text{ odd}}}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{4n^2} = \varphi(q)-\varphi(q^4) = \frac{1}{2}\big(\varphi(q)-\varphi(-q)\big).$$

Appealing to (2.8), Proposition 2.6, (2.6) (with q replaced by q^4) and (2.9) (with q replaced by q^3) we obtain part (a).

(b) Appealing to Proposition 2.8 and (2.7), we obtain

$$(2.10)\quad \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} F(n)q^{4n} = \sum_{n=1}^{\infty} F(n)q^{4n} = \frac{1}{4}\Big(\sum_{\substack{i=-\infty\\ i\text{ odd}}}^{\infty} q^{i^2}\Big)\Big(\sum_{\substack{j=-\infty\\ j\text{ odd}}}^{\infty} (-1)^{(j-1)/2} j q^{3j^2}\Big).$$

Appealing to (2.10), (2.9) and Proposition 2.7, we obtain part (b). ∎

Theorem 2.2.

$$\text{(a)}\quad \varphi^3(q)\varphi(-q^3)-\varphi^3(-q)\varphi(q^3) = 3\big(\varphi(q)\varphi^2(q^3)\varphi(-q^3)-\varphi(-q)\varphi^2(-q^3)\varphi(q^3)\big).$$

$$\text{(b)}\quad \big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big) = 4\big(\varphi(q^4)\varphi(q^{12})-\varphi(-q^4)\varphi(-q^{12})\big).$$

$$\text{(c)}\quad 4\big(\varphi^3(q^4)\varphi(-q^{12})-\varphi^3(-q^4)\varphi(q^{12})\big) = 3\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big)\varphi(q^{12})\varphi(-q^{12}).$$

$$\text{(d)}\quad \varphi(q)\varphi^3(-q^3)-\varphi(-q)\varphi^3(q^3) = \varphi^2(q)\varphi(-q)\varphi(q^3)-\varphi^2(-q)\varphi(q)\varphi(-q^3).$$

Proof. (a) We start with the identity

$$(1+2p)^2-(1-p)^2=3(1+2p)-3(1-p)(1+p).$$

Multiplying both sides by

$$(1+2p)^{1/4}(1-p)^{1/4}(1+p)^{3/4}k^2,$$

we obtain

$$\begin{aligned}(1+2p)^{9/4}&(1-p)^{1/4}(1+p)^{3/4}k^2-(1+2p)^{1/4}(1-p)^{9/4}(1+p)^{3/4}k^2\\&=3(1+2p)^{5/4}(1-p)^{1/4}(1+p)^{3/4}k^2-3(1+2p)^{1/4}(1-p)^{5/4}(1+p)^{7/4}k^2.\end{aligned}$$

By Propositions 2.1 and 2.2, we have

$$\begin{aligned}\varphi^3(q)\varphi(-q^3)&=(1+2p)^{9/4}(1-p)^{1/4}(1+p)^{3/4}k^2,\\ \varphi^3(-q)\varphi(q^3)&=(1+2p)^{1/4}(1-p)^{9/4}(1+p)^{3/4}k^2,\\ \varphi(q)\varphi^2(q^3)\varphi(-q^3)&=(1+2p)^{5/4}(1-p)^{1/4}(1+p)^{3/4}k^2,\\ \varphi(-q)\varphi^2(-q^3)\varphi(q^3)&=(1+2p)^{1/4}(1-p)^{5/4}(1+p)^{7/4}k^2,\end{aligned}$$

and part (a) follows.

(b) We have

$$\begin{aligned}&2\varphi(-q^4)\varphi(-q^{12})\\&\quad=2\cdot2^{-1/4}(1-p)^{3/16}(1+p)^{1/16}(1+2p)^{3/16}\\&\qquad\times\big((1+2p)^{3/2}+(1-p)^{3/2}(1+p)^{1/2}\big)^{1/4}k^{1/2}\\&\qquad\times2^{-1/4}(1-p)^{1/16}(1+p)^{3/16}(1+2p)^{1/16}\\&\qquad\times\big((1+2p)^{1/2}+(1-p)^{1/2}(1+p)^{3/2}\big)^{1/4}k^{1/2}\\&\quad=2^{1/2}(1-p)^{1/4}(1+p)^{1/4}(1+2p)^{1/4}\\&\qquad\times\big(2+4p+2p^2+p^4+2(1+p+p^2)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\big)^{1/4}k\\&\quad=2^{1/2}(1-p)^{1/4}(1+p)^{1/4}(1+2p)^{1/4}\\&\qquad\qquad\times\big(1+p+p^2+(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\big)^{1/2}k\\&\quad=(1-p)^{1/4}(1+p)^{1/4}(1+2p)^{1/4}\\&\qquad\qquad\times\big(2+2p+2p^2+2(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\big)^{1/2}k\end{aligned}$$

$$= (1-p)^{1/4}(1+p)^{1/4}(1+2p)^{1/4}\big((1+p)^{1/2}(1+2p)^{1/2}+(1-p)^{1/2}\big)k$$
$$= (1-p)^{1/4}(1+p)^{3/4}(1+2p)^{3/4}k+(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{1/4}k$$
$$= \varphi(q)\varphi(-q^3)+\varphi(-q)\varphi(q^3).$$

Also

$$\begin{aligned}4\varphi(q^4)\varphi(q^{12}) &= \big(\varphi(q)+\varphi(-q)\big)\big(\varphi(q^3)+\varphi(-q^3)\big)\\ &= \varphi(q)\varphi(q^3)+\varphi(-q)\varphi(q^3)+\varphi(q)\varphi(-q^3)+\varphi(-q)\varphi(-q^3).\end{aligned}$$

Hence

$$\begin{aligned}&4\big(\varphi(q^4)\varphi(q^{12})-\varphi(-q^4)\varphi(-q^{12})\big)\\ &= \varphi(q)\varphi(q^3)+\varphi(-q)\varphi(q^3)+\varphi(q)\varphi(-q^3)+\varphi(-q)\varphi(-q^3)\\ &\quad -2\varphi(q)\varphi(-q^3)-2\varphi(-q)\varphi(q^3)\\ &= \varphi(q)\varphi(q^3)-\varphi(-q)\varphi(q^3)-\varphi(q)\varphi(-q^3)+\varphi(-q)\varphi(-q^3)\\ &= \big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big).\end{aligned}$$

(c) We have

$$\begin{aligned}&3\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^3)-\varphi(-q^3)\big)\varphi(q^{12})\varphi(-q^{12})\\ &= 12\big(\varphi(q^4)\varphi(q^{12})-\varphi(-q^4)\varphi(-q^{12})\big)\varphi(q^{12})\varphi(-q^{12}) \quad \text{(by part (b))}\\ &= 12\big(\varphi(q^4)\varphi^2(q^{12})\varphi(-q^{12})-\varphi(-q^4)\varphi(q^{12})\varphi^2(-q^{12})\big)\\ &= 4\big(\varphi^3(q^4)\varphi(-q^{12})-\varphi^3(-q^4)\varphi(q^{12})\big) \quad \text{(by part (a) with } q \text{ replaced by } q^4)\end{aligned}$$

as asserted.

(d) We start with the identity

$$(1+p)^2-1=(1+2p)-(1-p)(1+p).$$

Multiplying both sides by

$$(1+2p)^{3/4}(1-p)^{3/4}(1+p)^{1/4}k^2,$$

we obtain

$$\begin{aligned}&(1+2p)^{3/4}(1-p)^{3/4}(1+p)^{9/4}k^2-(1+2p)^{3/4}(1-p)^{3/4}(1+p)^{1/4}k^2\\ &\quad = (1+2p)^{7/4}(1-p)^{3/4}(1+p)^{1/4}k^2-(1+2p)^{3/4}(1-p)^{7/4}(1+p)^{5/4}k^2.\end{aligned}$$

By Propositions 2.1 and 2.2 we obtain

$$\varphi(q)\varphi^3(-q^3)-\varphi(-q)\varphi^3(q^3)=\varphi^2(q)\varphi(-q)\varphi(q^3)-\varphi^2(-q)\varphi(q)\varphi(-q^3)$$

as asserted. ∎

Theorem 2.3.

$$\text{(a)} \quad \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n)q^n = \frac{1}{4}\big(\varphi^2(q)\varphi(-q)\varphi(q^3) - \varphi(q)\varphi^2(-q)\varphi(-q^3)\big)$$
$$= \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3) - \varphi(-q)\varphi^3(q^3)\big).$$

$$\text{(b)} \quad \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} F(n)q^n = \frac{1}{4}\big(\varphi(q)\varphi^2(q^3)\varphi(-q^3) - \varphi(-q)\varphi^2(-q^3)\varphi(q^3)\big)$$
$$= \frac{1}{12}\big(\varphi^3(q)\varphi(-q^3) - \varphi^3(-q)\varphi(q^3)\big).$$

Proof. (a) By Theorem 2.1(a) and Theorem 2.2(b), we deduce

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n)q^{4n} = \frac{1}{16}\big(\varphi(q) - \varphi(-q)\big)\big(\varphi(q^3) - \varphi(-q^3)\big)\varphi(q^4)\varphi(-q^4)$$
$$= \frac{1}{4}\big(\varphi(q^4)\varphi(q^{12}) - \varphi(-q^4)\varphi(-q^{12})\big)\varphi(q^4)\varphi(-q^4).$$

Replacing q^4 by q, we obtain appealing to Theorem 2.2(d)

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n)q^n = \frac{1}{4}\big(\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3)\big)\varphi(q)\varphi(-q)$$
$$= \frac{1}{4}\big(\varphi^2(q)\varphi(-q)\varphi(q^3) - \varphi(q)\varphi^2(-q)\varphi(-q^3)\big)$$
$$= \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3) - \varphi(-q)\varphi^3(q^3)\big),$$

which is part (a).

(b) By Theorem 2.1(b) and Theorem 2.2(b), we have

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} F(n)q^{4n} = \frac{1}{4}\big(\varphi(q^4)\varphi(q^{12}) - \varphi(-q^4)\varphi(-q^{12})\big)\varphi(q^{12})\varphi(-q^{12}).$$

Replacing q^4 by q, we obtain appealing to Theorem 2.2(a)

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} F(n)q^n = \frac{1}{4}\big(\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3)\big)\varphi(q^3)\varphi(-q^3)$$
$$= \frac{1}{4}\big(\varphi(q)\varphi^2(q^3)\varphi(-q^3) - \varphi(-q)\varphi(q^3)\varphi^2(-q^3)\big)$$
$$= \frac{1}{12}\big(\varphi^3(q)\varphi(-q^3) - \varphi^3(-q)\varphi(q^3)\big),$$

which is part (b). ∎

We conclude this section by giving some arithmetic properties of $E(n)$ and $F(n)$. These properties can be used to slightly simplify Theorem 7.2 and Corollary 7.1 when n is odd by splitting into subcases modulo 3 and/or modulo 4. However we do not do this.

Theorem 2.4. *Let* $n \in \mathbb{N}$. *Then*

(a) $$F(3n) = E(n)$$

and

(b) $$E(3n) = -3F(n).$$

Proof. (a) We have

$$\begin{aligned} F(3n) &= \sum_{\substack{(i,j) \in \mathbb{N}^2 \\ i,j \text{ odd} \\ 12n = i^2 + 3j^2}} (-1)^{(j-1)/2} j = \sum_{\substack{(k,j) \in \mathbb{N}^2 \\ k,j \text{ odd} \\ 12n = (3k)^2 + 3j^2}} (-1)^{(j-1)/2} j \\ &= \sum_{\substack{(j,k) \in \mathbb{N}^2 \\ j,k \text{ odd} \\ 4n = j^2 + 3k^2}} (-1)^{(j-1)/2} j = E(n). \end{aligned}$$

(b) We have

$$\begin{aligned} E(3n) &= \sum_{\substack{(i,j) \in \mathbb{N}^2 \\ i,j \text{ odd} \\ 12n = i^2 + 3j^2}} (-1)^{(i-1)/2} i = \sum_{\substack{(k,j) \in \mathbb{N}^2 \\ k,j \text{ odd} \\ 12n = (3k)^2 + 3j^2}} (-1)^{(3k-1)/2} 3k \\ &= -3 \sum_{\substack{(j,k) \in \mathbb{N}^2 \\ j,k \text{ odd} \\ 4n = j^2 + 3k^2}} (-1)^{(k-1)/2} k = -3F(n), \end{aligned}$$

as asserted. ∎

Theorem 2.5. *Let* $n \in \mathbb{N}$. *Then*

$$E(n) = F(n) = 0, \quad \text{if} \quad n \equiv 2 \pmod 3.$$

Proof. If $(i,j) \in \mathbb{N}^2$ is such that $4n = i^2 + 3j^2$, then $n \equiv 4n \equiv i^2 + 3j^2 \equiv i^2 \equiv 0,\ 1 \pmod 3$, so $n \equiv 2 \pmod 3$ implies $E(n) = F(n) = 0$. ■

Theorem 2.6. *Let $n \in \mathbb{N}$. Then*

$$E(n) = \begin{cases} F(n), & \text{if } n \equiv 1 \pmod 4, \\ -3F(n), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $n \in \mathbb{N}$ be odd. First we observe that by (2.7) we have (replacing i by $j + 2k$)

$$(2.11) \qquad 4F(n) = \sum_{\substack{i,j=-\infty \\ i,j \text{ odd} \\ 4n = i^2+3j^2}}^{\infty} (-1)^{(j-1)/2} j = \sum_{\substack{j,k=-\infty \\ j \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j.$$

Secondly, we see that (as $n = j^2 + jk + k^2 \iff n = j^2 + j(-j-k) + (-j-k)^2$)

$$\begin{aligned} \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ even} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j &= \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j \\ &= \frac{1}{2} \sum_{\substack{j,k=-\infty \\ j \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j \qquad (2.12) \\ &= 2F(n) \end{aligned}$$

by (2.11).

Thirdly, we see that (as $n = j^2 + jk + k^2 \equiv jk + 2 \equiv j + k + 1 \pmod 4$ for $j \equiv k \equiv 1 \pmod 2$)

$$\begin{aligned} \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k &= \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(n+1)/2+(k-1)/2} k \\ &= (-1)^{(n+1)/2} \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j, \end{aligned}$$

that is

$$(2.13) \qquad \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n = j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k = -2(-1)^{(n-1)/2} F(n),$$

by (2.12).

Fourthly, we have (as $n = j^2 + jk + k^2 \iff n = j^2 + j(-j-k) + (-j-k)^2$)

$$\begin{aligned}
\sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ even} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k &= \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} (-j-k) \\
&= - \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j - \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k,
\end{aligned}$$

that is

$$\sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ even} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k = -2F(n) + 2(-1)^{(n-1)/2} F(n), \tag{2.14}$$

by (2.12) and (2.13).

Finally

$$\begin{aligned}
4E(n) &= \sum_{\substack{i,j=-\infty \\ i,j \text{ odd} \\ 4n=i^2+3j^2}}^{\infty} (-1)^{(i-1)/2} i = \sum_{\substack{j,k=-\infty \\ j \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2+k} (j+2k) \\
&= \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ even} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j - \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} j \\
&\quad +2 \sum_{\substack{j,k=-\infty \\ j \text{ odd}, k \text{ even} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k - 2 \sum_{\substack{j,k=-\infty \\ j,k \text{ odd} \\ n=j^2+jk+k^2}}^{\infty} (-1)^{(j-1)/2} k \\
&= 2F(n) - 2F(n) - 4F(n) + 4(-1)^{(n-1)/2} F(n) + 4(-1)^{(n-1)/2} F(n) \\
&= 4(2(-1)^{(n-1)/2} - 1) F(n),
\end{aligned}$$

by (2.7), (2.12), (2.13) and (2.14), so that

$$E(n) = (2(-1)^{\frac{n-1}{2}} - 1) F(n)$$

as asserted. ∎

3 The power series of $\varphi^3(q)\varphi(-q^3)$

We see from Proposition 2.5 that the coefficients of q in the power series expansions of $\varphi(q)\varphi^3(q^3)$, $\varphi^3(q)\varphi(q^3)$, $\varphi^2(q)\varphi(-q)\varphi(-q^3)$ and $\varphi^2(q^3)\varphi(-q)\varphi(-q^3)$ involve $A(n)$, $B(n)$, $C(n)$ and $D(n)$. In Sections 3-6 we determine the power series expansions of $\varphi^3(q)\varphi(-q^3)$, $\varphi^2(q)\varphi(-q)\varphi(q^3)$, $\varphi(q)\varphi^2(q^3)\varphi(-q^3)$ and $\varphi(q)\varphi^3(-q^3)$ in powers of q and show that they involve $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$ and $F(n)$, see Theorems 3.1, 4.1, 5.1 and 6.1. These four products of theta functions occur in Theorem 2.3 together with those obtained from them by replacing q by $-q$.

In this section we determine the power series expansion of $\varphi^3(q)\varphi(-q^3)$ in powers of q.

Theorem 3.1.

$$\varphi^3(q)\varphi(-q^3) = 1 - \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(3A(n) - B(n) - 3C(n) + D(n)\big)q^n + 6\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} F(n)q^n.$$

We require a lemma before proving Theorem 3.1.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then*

$$N(1,1,1,12;n) = \begin{cases} \frac{9}{2}A(n) - \frac{3}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. If $x^2 + y^2 + z^2 + 12t^2 = n \equiv 2 \pmod 4$ then exactly one of x, y and z is even. Thus, by [1, Theorem 5.1] and Proposition 2.3(a)(b), we have

$$\begin{aligned} N(1,1,1,12;n) &= 3N(1,1,4,12;n) \\ &= 3N(1,1,2,6;n/2) \\ &= 9A(n/2) + 3B(n/2) \\ &= \frac{9}{2}A(n) - \frac{3}{2}B(n). \end{aligned}$$

If $x^2 + y^2 + z^2 + 12t^2 = n \equiv 0 \pmod 4$ then x, y and z are all even and, by [1, Theorem 4.1] and Proposition 2.3, we have

$$\begin{aligned} N(1,1,1,12;n) &= N(1,1,1,3;n/4) \\ &= 6A(n/4) - 2B(n/4) + 3C(n/4) - D(n/4) \end{aligned}$$

$$= \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n).$$

■

Proof of Theorem 3.1. By Lemma 3.1 we have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} & N(1,1,1,1,12;n)q^n \\
&= \sum_{\substack{n=1 \\ n \equiv 0 \pmod 4}}^{\infty} \left(\frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n)\right) q^n \\
&\quad + \sum_{\substack{n=1 \\ n \equiv 2 \pmod 4}}^{\infty} \left(\frac{9}{2}A(n) - \frac{3}{2}B(n)\right) q^n \\
&= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \left(\left(3 - \frac{3}{2}(-1)^{n/2}\right)A(n) - \left(1 - \frac{1}{2}(-1)^{n/2}\right)B(n)\right. \\
&\quad \left. + \left(\frac{3}{2} + \frac{3}{2}(-1)^{n/2}\right)C(n) - \left(\frac{1}{2} + \frac{1}{2}(-1)^{n/2}\right)D(n)\right)q^n \\
&= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \left(3A(n) - B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n)\right)q^n \\
&\quad - \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\left(3A(n) - B(n) - 3C(n) + D(n)\right)q^n.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} & N(1,1,1,1,12;n)q^n \\
&= \frac{1}{2}\sum_{n=1}^{\infty} N(1,1,1,1,12;n)q^n + \frac{1}{2}\sum_{n=1}^{\infty} N(1,1,1,1,12;n)(-q)^n \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} N(1,1,1,1,12;n)q^n - 1\right) + \frac{1}{2}\left(\sum_{n=0}^{\infty} N(1,1,1,1,12;n)(-q)^n - 1\right) \\
&= \frac{1}{2}\varphi^3(q)\varphi(q^{12}) + \frac{1}{2}\varphi^3(-q)\varphi(q^{12}) - 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big(\varphi^3(q)+\varphi^3(-q)\big)\varphi(q^{12})-1 \\
&= \frac{1}{4}\big(\varphi^3(q)+\varphi^3(-q)\big)\big(\varphi(q^3)+\varphi(-q^3)\big)-1 \quad \text{(by (2.4))} \\
&= \frac{1}{4}\varphi^3(q)\varphi(q^3)+\frac{1}{4}\varphi^3(-q)\varphi(-q^3)+\frac{1}{4}\Big(\varphi^3(q)\varphi(-q^3)+\varphi^3(-q)\varphi(q^3)\Big)-1 \\
&= \frac{1}{4}\Big(1+\sum_{n=1}^{\infty}\big(6A(n)-2B(n)+3C(n)-D(n)\big)q^n\Big) \\
&\quad +\frac{1}{4}\Big(1+\sum_{n=1}^{\infty}\big(6A(n)-2B(n)+3C(n)-D(n)\big)(-q)^n\Big) \\
&\quad +\frac{1}{4}\Big(\varphi^3(q)\varphi(-q^3)+\varphi^3(-q)\varphi(q^3)\Big)-1 \quad \text{(by Proposition 2.5(b))} \\
&= \sum_{\substack{n=1\\ n \text{ even}}}^{\infty}\Big(3A(n)-B(n)+\frac{3}{2}C(n)-\frac{1}{2}D(n)\Big)q^n \\
&\quad +\frac{1}{4}\Big(\varphi^3(q)\varphi(-q^3)+\varphi^3(-q)\varphi(q^3)\Big)-\frac{1}{2}.
\end{aligned}
$$

Equating the two expressions for $\sum_{\substack{n=1\\ n \text{ even}}}^{\infty} N(1,1,1,1,12;n)q^n$, we obtain

$$
\begin{aligned}
-\frac{1}{2}\sum_{\substack{n=1\\ n \text{ even}}}^{\infty} & (-1)^{n/2}\big(3A(n)-B(n)-3C(n)+D(n)\big)q^n \\
&= \frac{1}{4}\Big(\varphi^3(q)\varphi(-q^3)+\varphi^3(-q)\varphi(q^3)\Big)-\frac{1}{2}
\end{aligned}
$$

so

$$
\begin{aligned}
&\varphi^3(q)\varphi(-q^3)+\varphi^3(-q)\varphi(q^3) \\
&\qquad = 2-2\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}(-1)^{n/2}\big(3A(n)-B(n)-3C(n)+D(n)\big)q^n.
\end{aligned}
$$

From Theorem 2.3(b) we have

$$
\varphi^3(q)\varphi(-q^3)-\varphi^3(-q)\varphi(q^3)=12\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}F(n)q^n.
$$

Adding these two equations, and dividing by 2, we obtain

$$\varphi^3(q)\varphi(-q^3) = 1 - \sum_{\substack{n=1\\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(3A(n) - B(n) - 3C(n) + D(n)\big)q^n + 6\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} F(n)q^n,$$

as asserted. ∎

4 The power series of $\varphi^2(q)\varphi(-q)\varphi(q^3)$

In this section we determine the power series expansion of $\varphi^2(q)\varphi(-q)\varphi(q^3)$ in powers of q.

Theorem 4.1.

$$\varphi^2(q)\varphi(-q)\varphi(q^3) = 1 + \sum_{\substack{n=1\\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(3A(n) - B(n) + 3C(n) - D(n)\big)q^n + 2\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} E(n)q^n.$$

We require a number of lemmas. For $n \in \mathbb{N}$ we set

$$R(n) := \text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ \ z \not\equiv t \pmod 2\right\}.$$

Lemma 4.1. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ z \equiv t \equiv 1 \pmod 2,\ \ z \equiv t \pmod 4\right\} = R(n/4).$$

Proof. Let $n \in \mathbb{N}$ satisfy $n \equiv 0 \pmod 4$. Set

$$T(n) = \left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ z \equiv t \equiv 1 \pmod 2,\ z \equiv t \pmod 4\right\}$$

and

$$U(n) = \left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n/4 = x^2 + y^2 + z^2 + 3t^2,\ \ z \not\equiv t \pmod 2\right\}.$$

The mapping $f : T(n) \longrightarrow U(n)$ given by

$$f((x,y,z,t)) = \big(x/2, y/2, (z+3t)/4, (z-t)/4\big)$$

is a bijection. Thus

$$\text{card } T(n) = \text{card } U(n) = R(n/4),$$

as asserted. ■

Lemma 4.2. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$N(1,1,1,3;n) = N(1,1,1,3;n/4) + 6R(n/4).$$

Proof. If $x^2+y^2+z^2+3t^2 = n \equiv 0 \pmod 4$ then

$$(x,y,z,t) \equiv (0,0,0,0), (0,0,1,1), (0,1,0,1) \text{ or } (1,0,0,1) \pmod 2.$$

Thus

$$\begin{aligned} N(1,1,1,3;n) &= N(1,1,1,3;n/4) \\ &\quad +3 \text{ card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+y^2+z^2+3t^2,\ \ z \equiv t \equiv 1 \ (\text{mod } 2)\right\} \\ &= N(1,1,1,3;n/4) + 6 \text{ card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+y^2+z^2+3t^2,\right. \\ &\qquad\qquad \left. z \equiv t \equiv 1 \ (\text{mod } 2), z \equiv t \ (\text{mod } 4)\right\} \\ &= N(1,1,1,3;n/4) + 6R(n/4), \end{aligned}$$

by Lemma 4.1. ■

Lemma 4.3. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$N(1,1,3,4;n) = N(1,1,1,3;n/4) + 4R(n/4).$$

Proof. If $x^2+y^2+3z^2+4t^2 = n \equiv 0 \pmod 4$ then $x^2+y^2+3z^2 \equiv 0 \pmod 4$ so

$$(x,y,z) \equiv (0,0,0), (1,0,1) \text{ or } (0,1,1) \pmod 2.$$

Thus

$$\begin{aligned} N(1,1,3,4;n) &= N(1,1,1,3;n/4) \\ &\quad +2 \text{ card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+y^2+3z^2+4t^2,\ x \equiv 1 \ (\text{mod } 2),\right. \\ &\qquad \left. y \equiv 0 \ (\text{mod } 2),\ z \equiv 1 \ (\text{mod } 2)\right\}. \end{aligned}$$

Now

$$\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+y^2+3z^2+4t^2,\ (x,y,z) \equiv (1,0,1) \ (\text{mod } 2)\right\}$$

$$
\begin{aligned}
&= \text{card}\left\{(x_1, y_1, z_1, t_1) \in \mathbb{Z}^4 \mid n = x_1^2 + y_1^2 + z_1^2 + 3t_1^2,\right.\\
&\qquad\qquad \left.(x_1, y_1, z_1, t_1) \equiv (0,0,1,1) \pmod 2\right\}\\
&= \text{card}\left\{(x_1, y_1, z_1, t_1) \in \mathbb{Z}^4 \mid n = x_1^2 + y_1^2 + z_1^2 + 3t_1^2,\right.\\
&\qquad\qquad \left.(z_1, t_1) \equiv (1,1) \pmod 2\right\}\\
&= 2\ \text{card}\left\{(x_1, y_1, z_1, t_1) \in \mathbb{Z}^4 \mid n = x_1^2 + y_1^2 + z_1^2 + 3t_1^2,\right.\\
&\qquad\qquad \left.z_1 \equiv t_1 \equiv 1 \pmod 2, z_1 \equiv t_1 \pmod 4\right\}\\
&= 2R(n/4),
\end{aligned}
$$

by Lemma 4.1. The asserted result now follows. ■

Lemma 4.4. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$N(1,1,3,4;n) = \frac{1}{3}N(1,1,1,3;n/4) + \frac{2}{3}N(1,1,1,3;n).$$

Proof. This result follows by eliminating $R(n/4)$ from the formulae of Lemmas 4.2 and 4.3. ■

Lemma 4.4 was stated but not proved by Liouville [11, p. 184].

Lemma 4.5. *Let $n \in \mathbb{N}$. Then*

$$N(1,1,3,4;n) = \frac{9}{2}A(n) - \frac{3}{2}B(n) + 3C(n) - D(n), \text{ if } n \equiv 0 \pmod 4.$$

Proof. By [1, Theorem 4.1] we have

$$N(1,1,1,3;n) = 6A(n) - 2B(n) + 3C(n) - D(n), \ n \in \mathbb{N}.$$

Hence, by Proposition 2.3, we obtain for $n \equiv 0 \pmod 4$

$$
\begin{aligned}
N(1,1,1,3;n/4) &= 6A(n/4) - 2B(n/4) + 3C(n/4) - D(n/4)\\
&= \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n).
\end{aligned}
$$

Then, by Lemma 4.4, we deduce

$$
\begin{aligned}
N(1,1,3,4;n) &= \frac{1}{3}\Big(\frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n)\Big)\\
&\quad + \frac{2}{3}\big(6A(n) - 2B(n) + 3C(n) - D(n)\big)\\
&= \frac{9}{2}A(n) - \frac{3}{2}B(n) + 3C(n) - D(n),
\end{aligned}
$$

as claimed. ■

Lemma 4.6. *Let* $n \in \mathbb{N}$. *Then*

$$N(1,1,3,4;n) = \frac{3}{2}A(n) - \frac{1}{2}B(n), \ \textit{if } n \equiv 2 \pmod 4.$$

Proof. If $x^2+y^2+3z^2+4t^2 = n \equiv 2 \pmod 4$ then $x^2+y^2+3z^2 \equiv 2 \pmod 4$ so $x \equiv y \pmod 2$ and $z \equiv 0 \pmod 2$. Hence

$$\frac{n}{2} = \Big(\frac{x+y}{2}\Big)^2 + \Big(\frac{x-y}{2}\Big)^2 + 2t^2 + 6\Big(\frac{z}{2}\Big)^2$$

so that by [1, Theorem 5.1] we have

$$N(1,1,3,4;n) = N(1,1,2,6;n/2) = 3A(n/2) + B(n/2) = \frac{3}{2}A(n) - \frac{1}{2}B(n),$$

if $n \equiv 2 \pmod 4$. ∎

Lemma 4.7. *Let* $n \in \mathbb{N}$. *If* $n \equiv 0 \pmod 2$ *then*

$$\begin{aligned} N(1,1,3,4;n) &= 3\Big(1+\frac{1}{2}(-1)^{n/2}\Big)A(n) - \Big(1+\frac{1}{2}(-1)^{n/2}\Big)B(n) \\ &\quad + \frac{3}{2}\Big(1+(-1)^{n/2}\Big)C(n) - \frac{1}{2}\Big(1+(-1)^{n/2}\Big)D(n). \end{aligned}$$

Proof. This follows from Lemmas 4.5 and 4.6. ∎

Proof of Theorem 4.1. We have by Lemma 4.7

$$\begin{aligned} &\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} N(1,1,3,4;n)q^n \\ &= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \Big(3\Big(1+\frac{1}{2}(-1)^{n/2}\Big)A(n) - \Big(1+\frac{1}{2}(-1)^{n/2}\Big)B(n) \\ &\qquad + \frac{3}{2}\Big(1+(-1)^{n/2}\Big)C(n) - \frac{1}{2}\Big(1+(-1)^{n/2}\Big)D(n)\Big)q^n \\ &= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \Big(3A(n) - B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n)\Big)q^n \\ &\qquad + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\Big(\frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n)\Big)q^n. \end{aligned}$$

On the other hand we have by (2.4) and Proposition 2.5(b)

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} N(1,1,3,4;n)q^n = \frac{1}{2}\sum_{n=1}^{\infty} N(1,1,3,4;n)q^n + \frac{1}{2}\sum_{n=1}^{\infty} N(1,1,3,4;n)(-q)^n$$

$$= \frac{1}{2}\Big(\sum_{n=0}^{\infty} N(1,1,3,4;n)q^n - 1\Big) + \frac{1}{2}\Big(\sum_{n=0}^{\infty} N(1,1,3,4;n)(-q)^n - 1\Big)$$

$$= \frac{1}{2}\varphi^2(q)\varphi(q^3)\varphi(q^4) + \frac{1}{2}\varphi^2(-q)\varphi(-q^3)\varphi(q^4) - 1$$

$$= \frac{1}{4}\varphi^2(q)\varphi(q^3)(\varphi(q)+\varphi(-q)) + \frac{1}{4}\varphi^2(-q)\varphi(-q^3)(\varphi(q)+\varphi(-q)) - 1$$

$$= \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi^3(-q)\varphi(-q^3)$$

$$+\frac{1}{4}\big(\varphi^2(q)\varphi(-q)\varphi(q^3) + \varphi^2(-q)\varphi(q)\varphi(-q^3)\big) - 1$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\Big(\frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{4}C(n) - \frac{1}{4}D(n)\Big)q^n$$

$$+\frac{1}{4} + \sum_{n=1}^{\infty}\Big(\frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{4}C(n) - \frac{1}{4}D(n)\Big)(-q)^n$$

$$+\frac{1}{4}\Big(\varphi^2(q)\varphi(-q)\varphi(q^3) + \varphi(q)\varphi^2(-q)\varphi(-q^3)\Big) - 1$$

$$= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \Big(3A(n) - B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n)\Big)q^n$$

$$+\frac{1}{4}\big(\varphi^2(q)\varphi(-q)\varphi(q^3) + \varphi(q)\varphi^2(-q)\varphi(-q^3)\big) - \frac{1}{2}.$$

Equating the two expressions for $\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} N(1,1,3,4;n)q^n$, we deduce

$$\frac{1}{4}\big(\varphi^2(q)\varphi(-q)\varphi(q^3) + \varphi(q)\varphi^2(-q)\varphi(-q^3)\big) - \frac{1}{2}$$
$$= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\Big(\frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n)\Big)q^n,$$

from which we obtain

$$\varphi^2(q)\varphi(-q)\varphi(q^3) + \varphi(q)\varphi^2(-q)\varphi(-q^3) = 2 + 2 \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(3A(n) - B(n) + 3C(n) - D(n)\big) q^n.$$

From Theorem 2.3(a) we have

$$\varphi^2(q)\varphi(-q)\varphi(q^3) - \varphi(q)\varphi^2(-q)\varphi(-q^3) = 4 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n) q^n.$$

Adding these two equations, and dividing by 2, we deduce

$$\varphi^2(q)\varphi(-q)\varphi(q^3) = 1 + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(3A(n) - B(n) + 3C(n) - D(n)\big) q^n + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n) q^n,$$

as asserted. ■

5 The power series of $\varphi(q)\varphi^2(q^3)\varphi(-q^3)$

In this section we determine the power series expansion of $\varphi(q)\varphi^2(q^3)\varphi(-q^3)$ in powers of q. The method of proof follows that of Theorem 4.1.

Theorem 5.1.

$$\varphi(q)\varphi^2(q^3)\varphi(-q^3) = 1 + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(A(n) + B(n) - C(n) - D(n)\big) q^n + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} F(n) q^n.$$

We require a number of lemmas before giving the proof of Theorem 5.1. For $n \in \mathbb{N}$ we set

$$S(n) := \text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + 3y^2 + 3z^2 + 3t^2,\ x \not\equiv y \pmod 2\right\}.$$

Lemma 5.1. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+12t^2,\right. \\ \left. x \equiv y \equiv 1 \pmod 2,\ x \equiv y \pmod 4\right\} = S(n/4).$$

Proof. Let $n \in \mathbb{N}$ satisfy $n \equiv 0 \pmod 4$. Set

$$V(n) = \left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+12t^2,\right. \\ \left. x \equiv y \equiv 1 \pmod 2,\ x \equiv y \pmod 4\right\}$$

and

$$W(n) = \left\{(x,y,z,t) \in \mathbb{Z}^4 \mid \frac{n}{4} = x^2+3y^2+3z^2+3t^2,\ x \not\equiv y \pmod 2\right\}.$$

The mapping $g : V(n) \longrightarrow W(n)$ given by

$$g((x,y,z,t)) = \left(\frac{x+3y}{4}, \frac{x-y}{4}, \frac{z}{2}, t\right)$$

is a bijection. Thus

$$\text{card}\ C(n) = \text{card}\ D(n) = S(n/4),$$

as claimed. ■

Lemma 5.2. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$N(1,3,3,3;n) = N(1,3,3,3;n/4) + 6S(n/4).$$

Proof. If $x^2+3y^2+3z^2+3t^2 = n \equiv 0 \pmod 4$ then

$$(x,y,z,t) \equiv (0,0,0,0), (1,1,0,0), (1,0,1,0) \text{ or } (1,0,0,1) \pmod 2.$$

Thus

$$\begin{aligned} N(1,3,3,3;n) &= N(1,3,3,3;n/4) \\ &\quad +3\,\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+3t^2,\ x \equiv y \equiv 1 \pmod 2\right\} \\ &= N(1,3,3,3;n/4) + 6\,\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+3t^2,\right. \\ &\qquad \left. x \equiv y \equiv 1 \pmod 2, x \equiv y \pmod 4\right\} \\ &= N(1,3,3,3;n/4) + 6S(n/4), \end{aligned}$$

by Lemma 5.1. ■

Lemma 5.3. *Let $n \in \mathbb{N}$ be such that $n \equiv 0 \pmod 4$. Then*

$$N(1,3,3,12;n) = N(1,3,3,3;n/4) + 4S(n/4).$$

Proof. If $x^2+3y^2+3z^2+12t^2=n\equiv 0 \pmod 4$ then $x^2+3y^3+3z^2\equiv 0 \pmod 4$ so

$$(x,y,z)\equiv(0,0,0),(1,1,0) \text{ or } (1,0,1) \pmod 2.$$

Thus

$$\begin{aligned} N(1,3,3,12;n) &= N(1,3,3,3;n/4) \\ &+2\,\mathrm{card}\left\{(x,y,z,t)\in\mathbb{Z}^4 \mid n=x^2+3y^2+3z^2+12t^2,\ x\equiv 1 \pmod 2,\right. \\ &\left. y\equiv 1 \pmod 2,\ z\equiv 0 \pmod 2\right\}. \end{aligned}$$

Now

$$\begin{aligned} &\mathrm{card}\left\{(x,y,z,t)\in\mathbb{Z}^4 \mid n=x^2+3y^2+3z^2+12t^2,\right. \\ &\qquad\qquad \left.(x,y,z)\equiv(1,1,0) \pmod 2\right\} \\ &= \mathrm{card}\left\{(x,y,z,t)\in\mathbb{Z}^4 \mid n=x^2+3y^2+3z^2+12t^2,\right. \\ &\qquad\qquad \left.(x,y)\equiv(1,1) \pmod 2\right\} \\ &= 2\,\mathrm{card}\left\{(x,y,z,t)\in\mathbb{Z}^4 \mid n=x^2+3y^2+3z^2+12t^2,\right. \\ &\qquad\qquad \left.x\equiv y\equiv 1 \pmod 2,\ x\equiv y \pmod 4\right\} \\ &= 2S(n/4), \end{aligned}$$

by Lemma 5.1. ∎

Lemma 5.4. *Let $n\in\mathbb{N}$ be such that $n\equiv 0 \pmod 4$. Then*

$$N(1,3,3,12;n)=\frac{1}{3}N(1,3,3,3;n/4)+\frac{2}{3}N(1,3,3,3;n).$$

Proof. From Lemmas 5.2 and 5.3 we obtain

$$\begin{aligned} N(1,3,3,12;n) &= N(1,3,3,3;n/4)+4S(n/4) \\ &= N(1,3,3,3;n/4)+\frac{4}{6}\big(N(1,3,3,3;n)-N(1,3,3,3;n/4)\big) \\ &= \frac{1}{3}N(1,3,3,3;n/4)+\frac{2}{3}N(1,3,3,3;n), \end{aligned}$$

as asserted. ∎

Although Lemma 5.4 is an analogue of Lemma 4.4, it was not stated by Liouville.

Lemma 5.5. *Let $n\in\mathbb{N}$. Then*

$$N(1,3,3,12;n)=\frac{3}{2}A(n)+\frac{3}{2}B(n)-C(n)-D(n), \quad \textit{if } n\equiv 0 \pmod 4.$$

Proof. By [1, Theorem 8.1] we have

$$N(1,3,3,3;n) = 2A(n) + 2B(n) - C(n) - D(n), \ \ n \in \mathbb{N}.$$

Hence for $n \equiv 0 \pmod 4$ we have by Proposition 2.3

$$\begin{aligned} N(1,3,3,3;n/4) &= 2A(n/4) + 2B(n/4) - C(n/4) - D(n/4) \\ &= \frac{1}{2}A(n) + \frac{1}{2}B(n) - C(n) - D(n). \end{aligned}$$

Then, by Lemma 5.4, we obtain

$$\begin{aligned} N(1,3,3,12;n) &= \frac{1}{3}\Big(\frac{1}{2}A(n) + \frac{1}{2}B(n) - C(n) - D(n)\Big) \\ &\quad + \frac{2}{3}\big(2A(n) + 2B(n) - C(n) - D(n)\big) \\ &= \frac{3}{2}A(n) + \frac{3}{2}B(n) - C(n) - D(n) \end{aligned}$$

for $n \equiv 0 \pmod 4$. ■

Lemma 5.6. *Let* $n \in \mathbb{N}$. *Then*

$$N(1,3,3,12;n) = \frac{1}{2}A(n) + \frac{1}{2}B(n), \ \ \textit{if}\ n \equiv 2 \pmod 4.$$

Proof. If $x^2 + 3y^2 + 3z^2 + 12t^2 = n \equiv 2 \pmod 4$ then $x^2 + 3y^2 + 3z^2 \equiv 2 \pmod 4$ so $(x,y,z) \equiv (0,1,1) \pmod 2$. Hence

$$\frac{n}{2} = 2\Big(\frac{x}{2}\Big)^2 + 3\Big(\frac{y+z}{2}\Big)^2 + 3\Big(\frac{y-z}{2}\Big)^2 + 6t^2.$$

Then, by [1, Theorem 10.1] and Proposition 2.3, we have for $n \equiv 2 \pmod 4$

$$N(1,3,3,12;n) = N(2,3,3,6;n/2) = A(n/2) - B(n/2) = \frac{1}{2}A(n) + \frac{1}{2}B(n),$$

as asserted. ■

Lemma 5.7. *Let* $n \in \mathbb{N}$. *Then*

$$\begin{aligned} N(1,3,3,12;n) &= \Big(1 + \frac{1}{2}(-1)^{n/2}\Big)A(n) + \Big(1 + \frac{1}{2}(-1)^{n/2}\Big)B(n) \\ &\quad - \frac{1}{2}\Big(1 + (-1)^{n/2}\Big)C(n) - \frac{1}{2}\Big(1 + (-1)^{n/2}\Big)D(n), \end{aligned}$$

if $n \equiv 0 \pmod 2$.

Proof. This follows from Lemmas 5.5 and 5.6. ∎

Proof of Theorem 5.1. By Lemma 5.7 we have

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} N(1,3,3,12;n)q^n$$

$$= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \Big(\Big(1+\frac{1}{2}(-1)^{n/2}\Big)A(n) + \Big(1+\frac{1}{2}(-1)^{n/2}\Big)B(n)$$

$$-\frac{1}{2}\Big(1+(-1)^{n/2}\Big)C(n) - \frac{1}{2}\Big(1+(-1)^{n/2}\Big)D(n)\Big)q^n$$

$$= \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \Big(A(n)+B(n)-\frac{1}{2}C(n)-\frac{1}{2}D(n)\Big)q^n$$

$$+\frac{1}{2}\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\Big(A(n)+B(n)-C(n)-D(n)\Big)q^n.$$

On the other hand we have by (2.4) and Proposition 2.5(a)

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} N(1,3,3,12;n)q^n = \frac{1}{2}\sum_{n=1}^{\infty} N(1,3,3,12;n)q^n + \frac{1}{2}\sum_{n=1}^{\infty} N(1,3,3,12;n)(-q)^n$$

$$= \frac{1}{2}\Big(\sum_{n=0}^{\infty} N(1,3,3,12;n)q^n - 1\Big) + \frac{1}{2}\Big(\sum_{n=0}^{\infty} N(1,3,3,12;n)(-q)^n - 1\Big)$$

$$= \frac{1}{2}\varphi(q)\varphi^2(q^3)\varphi(q^{12}) + \frac{1}{2}\varphi(-q)\varphi^2(-q^3)\varphi(q^{12}) - 1$$

$$= \frac{1}{4}\varphi(q)\varphi^2(q^3)(\varphi(q^3)+\varphi(-q^3)) + \frac{1}{4}\varphi(-q)\varphi^2(-q^3)(\varphi(q^3)+\varphi(-q^3)) - 1$$

$$= \frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi(-q)\varphi^3(-q^3)$$

$$+\frac{1}{4}\big(\varphi(q)\varphi^2(q^3)\varphi(-q^3) + \varphi(-q)\varphi^2(-q^3)\varphi(q^3)\big) - 1$$

$$= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}\big(2A(n)+2B(n)-C(n)-D(n)\big)q^n$$

$$+\frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}\big(2A(n)+2B(n)-C(n)-D(n)\big)(-q)^n$$

$$+\frac{1}{4}\big(\varphi(q)\varphi^2(q^3)\varphi(-q^3)+\varphi(-q)\varphi(q^3)\varphi^2(-q^3)\big)-1$$

$$=-\frac{1}{2}+\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}\Big(A(n)+B(n)-\frac{1}{2}C(n)-\frac{1}{2}D(n)\Big)q^n$$

$$+\frac{1}{4}\big(\varphi(q)\varphi(q^3)\varphi(-q^3)+\varphi(-q)\varphi(q^3)\varphi^2(-q^3)\big).$$

Equating the two expressions for $\sum_{\substack{n=1\\ n \text{ even}}}^{\infty} N(1,3,3,12;n)q^n$, we obtain

$$\varphi(q)\varphi^2(q^3)\varphi(-q^3)+\varphi(-q)\varphi(q^3)\varphi^2(-q^3)$$
$$=2+2\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}(-1)^{n/2}\big(A(n)+B(n)-C(n)-D(n)\big)q^n.$$

From Theorem 2.3(b) we have

$$\varphi(q)\varphi^2(q^3)\varphi(-q^3)-\varphi(-q)\varphi(q^3)\varphi^2(-q^3)=4\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}F(n)q^n.$$

Adding these two identities, and dividing by 2, we obtain

$$\varphi(q)\varphi^2(q^3)\varphi(-q^3)=1+\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}(-1)^{n/2}\big(A(n)+B(n)-C(n)-D(n)\big)q^n$$
$$+2\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}F(n)q^n,$$

as asserted.

6 The power series of $\varphi(q)\varphi^3(-q^3)$

In this section we determine the power series expansion of $\varphi(q)\varphi^3(-q^3)$ in powers of q. The proof is similar to that of Theorem 3.1.

Theorem 6.1.

$$\varphi(q)\varphi^3(-q^3) = 1-\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}(-1)^{n/2}\big(A(n)+B(n)+C(n)+D(n)\big)q^n$$

$$+2\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} E(n)q^n.$$

We need a lemma.

Lemma 6.1. *Let $n \in \mathbb{N}$. Then*

$$N(3,3,3,4;n) = \begin{cases} \frac{1}{2}A(n) + \frac{1}{2}B(n) - C(n) - D(n), & \text{if } n \equiv 0 \pmod 4, \\ \frac{3}{2}A(n) + \frac{3}{2}B(n), & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. If $3x^2+3y^2+3z^2+4t^2 = n \equiv 0 \pmod 4$ then $x^2+y^2+z^2 \equiv 0 \pmod 4$ so $x \equiv y \equiv z \equiv 0 \pmod 2$. Thus

$$\begin{aligned} N(3,3,3,4;n) &= N(1,3,3,3;n/4) \\ &= 2A(n/4) + 2B(n/4) - C(n/4) - D(n/4) \\ &= \frac{1}{2}A(n) + \frac{1}{2}B(n) - C(n) - D(n) \end{aligned}$$

by [1, eq. (8.1)] and Proposition 2.3.

If $3x^2+3y^2+3z^2+4t^2 = n \equiv 2 \pmod 4$ then $x^2+y^2+z^2 \equiv 2 \pmod 4$ so $(x,y,z) \equiv (1,1,0), (1,0,1)$ or $(0,1,1) \pmod 2$. Hence, for $n \equiv 2 \pmod 4$, we have

$$\begin{aligned} &N(3,3,3,4;n) \\ &\quad = 3\,\text{card}\,\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = 3x^2+3y^2+3z^2+4t^2, \\ &\qquad\qquad x \equiv y \equiv 1 \pmod 2,\ z \equiv 0 \pmod 2\} \\ &\quad = 3\,\text{card}\,\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = 3x^2+3y^2+12z^2+4t^2, \\ &\qquad\qquad x \equiv y \equiv 1 \pmod 2\} \\ &\quad = 3\,\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid \frac{n}{2} = 3\Big(\frac{x+y}{2}\Big)^2 + 3\Big(\frac{x-y}{2}\Big)^2 + 2t^2 + 6z^2\right\} \\ &\quad = 3\,\text{card}\left\{(x,y,z,t) \in \mathbb{Z}^4 \mid \frac{n}{2} = 2x^2+3y^2+3z^2+6t^2\right\} \\ &\quad = 3\big(A(n/2) - B(n/2)\big) \\ &\quad = \frac{3}{2}A(n) + \frac{3}{2}B(n) \end{aligned}$$

by [1, Theorem 10.1] and Proposition 2.3. ■

Proof of Theorem 6.1. We have by (2.4) and Propostion 2.5(a)

$$\sum_{\substack{n=0\\ n \text{ even}}}^{\infty} N(3,3,3,4;n)q^n$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=0}^{\infty} N(3,3,3,4;n)q^n + \frac{1}{2}\sum_{n=0}^{\infty} N(3,3,3,4;n)(-q)^n \\
&= \frac{1}{2}\varphi^3(q^3)\varphi(q^4) + \frac{1}{2}\varphi^3(-q^3)\varphi(q^4) \\
&= \frac{1}{4}\varphi^3(q^3)\big(\varphi(q)+\varphi(-q)\big) + \frac{1}{4}\varphi^3(-q^3)\big(\varphi(q)+\varphi(-q)\big) \\
&= \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3)+\varphi(-q)\varphi^3(q^3)\big) + \frac{1}{4}\big(\varphi(q)\varphi^3(q^3)+\varphi(-q)\varphi^3(-q^3)\big) \\
&= \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3)+\varphi(-q)\varphi^3(q^3)\big) \\
&\quad + \frac{1}{4}\Big(1+\sum_{n=1}^{\infty}\big(2A(n)+2B(n)-C(n)-D(n)\big)q^n \\
&\qquad + 1+\sum_{n=1}^{\infty}\big(2A(n)+2B(n)-C(n)-D(n)\big)(-q)^n\Big) \\
&= \frac{1}{2} + \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3)+\varphi(-q)\varphi^3(q^3)\big) \\
&\quad + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \big(A(n)+B(n)-\frac{1}{2}C(n)-\frac{1}{2}D(n)\big)q^n.
\end{aligned}
$$

On the other hand we have by Lemma 6.1

$$
\begin{aligned}
&\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} N(3,3,3,4;n)q^n \\
&\qquad = 1 + \sum_{\substack{n=1 \\ n \equiv 0 \,(\mathrm{mod}\ 4)}}^{\infty} \Big(\frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n)\Big)q^n \\
&\qquad\quad + \sum_{\substack{n=1 \\ n \equiv 2 \,(\mathrm{mod}\ 4)}}^{\infty} \Big(\frac{3}{2}A(n)+\frac{3}{2}B(n)\Big)q^n \\
&\qquad = 1 + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \big(A(n)+B(n)-\frac{1}{2}C(n)-\frac{1}{2}D(n)\big)q^n
\end{aligned}
$$

$$-\frac{1}{2} \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(A(n)+B(n)+C(n)+D(n)\big)q^n.$$

Thus, equating the two expressions for $\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} N(3,3,3,4;n)q^n$, we have

$$\varphi(q)\varphi^3(-q^3)+\varphi(-q)\varphi^3(q^3) = 2-2\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(A(n)+B(n)+C(n)+D(n)\big)q^n.$$

By Theorem 2.3(a) we have

$$\varphi(q)\varphi^3(-q^3)-\varphi(-q)\varphi^3(q^3) = 4\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n)q^n.$$

Adding these two identities, and dividing by 2, we obtain

$$\varphi(q)\varphi^3(-q^3) = 1-\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\big(A(n)+B(n)+C(n)+D(n)\big)q^n +2\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n)q^n,$$

as asserted.

7 Sixteen quaternary forms

Our first theorem of this section shows that the generating functions of the sixteen forms listed in Section 1, namely,

$$\begin{array}{l} \varphi^3(q)\varphi(q^{12}),\quad \varphi^2(q)\varphi(q^3)\varphi(q^4), \\ \varphi^2(q)\varphi(q^4)\varphi(q^{12}),\quad \varphi(q)\varphi^2(q^2)\varphi(q^{12}), \\ \varphi(q)\varphi^2(q^3)\varphi(q^{12}),\quad \varphi(q)\varphi(q^3)\varphi^2(q^4), \\ \varphi(q)\varphi(q^3)\varphi^2(q^{12}),\quad \varphi(q)\varphi^2(q^4)\varphi(q^{12}), \\ \varphi(q)\varphi^2(q^6)\varphi(q^{12}),\quad \varphi(q)\varphi^3(q^{12}), \end{array} \tag{7.1}$$

$$\varphi^2(q^2)\varphi(q^3)\varphi(q^4), \quad \varphi^3(q^3)\varphi(q^4),$$
$$\varphi^2(q^3)\varphi(q^4)\varphi(q^{12}), \quad \varphi(q^3)\varphi^3(q^4),$$
$$\varphi(q^3)\varphi(q^4)\varphi^2(q^6), \quad \varphi(q^3)\varphi(q^4)\varphi^2(q^{12}),$$

can all be expressed as linear combinations of the eight products

$$\begin{aligned} &\varphi(q)\varphi^3(q^3),\\ &\varphi^3(q)\varphi(q^3),\\ &\varphi^2(q)\varphi(-q)\varphi(-q^3),\\ &\varphi(-q)\varphi^2(q^3)\varphi(-q^3),\\ &\varphi^3(q)\varphi(-q^3),\\ &\varphi^2(q)\varphi(-q)\varphi(q^3),\\ &\varphi(q)\varphi^3(-q^3),\\ &\varphi(q)\varphi^2(q^3)\varphi(-q^3), \end{aligned} \tag{7.2}$$

and the eight products formed from them by replacing q by $-q$.

Theorem 7.1.

(a) $\varphi^3(q)\varphi(q^{12}) = \frac{1}{2}\varphi^3(q)\varphi(q^3) + \frac{1}{2}\varphi^3(q)\varphi(-q^3)$.

(b) $\varphi^2(q)\varphi(q^3)\varphi(q^4) = \frac{1}{2}\varphi^3(q)\varphi(q^3) + \frac{1}{2}\varphi^2(q)\varphi(-q)\varphi(q^3)$.

(c) $$\varphi^2(q)\varphi(q^4)\varphi(q^{12}) = \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(q^3) + \frac{1}{4}\varphi^3(q)\varphi(-q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3).$$

(d) $$\varphi(q)\varphi^2(q^2)\varphi(q^{12}) = \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi^3(q)\varphi(-q^3) + \frac{1}{4}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^2(-q)\varphi(-q^3).$$

(e) $\varphi(q)\varphi^2(q^3)\varphi(q^{12}) = \frac{1}{2}\varphi(q)\varphi^3(q^3) + \frac{1}{2}\varphi(q)\varphi^2(q^3)\varphi(-q^3)$.

(f) $$\varphi(q)\varphi(q^3)\varphi^2(q^4) = \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{1}{2}\varphi^2(q)\varphi(-q)\varphi(q^3).$$

(g) $$\varphi(q)\varphi(q^3)\varphi^2(q^{12}) = \frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi(q)\varphi(q^3)\varphi^2(-q^3) + \frac{1}{2}\varphi(q)\varphi^2(q^3)\varphi(-q^3).$$

(h) $\varphi(q)\varphi^2(q^4)\varphi(q^{12}) = \frac{1}{8}\varphi^3(q)\varphi(q^3) + \frac{1}{8}\varphi^3(q)\varphi(-q^3) + \frac{1}{8}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{1}{8}\varphi(q)\varphi^2(-q)\varphi(-q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3).$

(i) $\varphi(q)\varphi^2(q^6)\varphi(q^{12}) = \frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi(q)\varphi^2(q^3)\varphi(-q^3) + \frac{1}{4}\varphi(q)\varphi^2(-q^3)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^3(-q^3).$

(j) $\varphi(q)\varphi^3(q^{12}) = \frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{3}{8}\varphi(q)\varphi^2(q^3)\varphi(-q^3) + \frac{3}{8}\varphi(q)\varphi(q^3)\varphi^2(-q^3) + \frac{1}{8}\varphi(q)\varphi^3(-q^3).$

(k) $\varphi^2(q^2)\varphi(q^3)\varphi(q^4) = \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{1}{4}\varphi(q^3)\varphi^3(-q).$

(l) $\varphi^3(q^3)\varphi(q^4) = \frac{1}{2}\varphi(q)\varphi^3(q^3) + \frac{1}{2}\varphi(-q)\varphi^3(q^3).$

(m) $\varphi^2(q^3)\varphi(q^4)\varphi(q^{12}) = \frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi(q)\varphi^2(q^3)\varphi(-q^3) + \frac{1}{4}\varphi(-q)\varphi^3(q^3) + \frac{1}{4}\varphi(-q)\varphi^2(q^3)\varphi(-q^3).$

(n) $\varphi(q^3)\varphi^3(q^4) = \frac{1}{8}\varphi^3(q)\varphi(q^3) + \frac{1}{8}\varphi^3(-q)\varphi(q^3) + \frac{3}{8}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{3}{8}\varphi^2(q)\varphi(-q)\varphi(q^3).$

(o) $\varphi(q^3)\varphi(q^4)\varphi^2(q^6) = \frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi(-q)\varphi^3(q^3) + \frac{1}{4}\varphi(q)\varphi(q^3)\varphi^2(-q^3) + \frac{1}{4}\varphi(-q)\varphi(q^3)\varphi^2(-q^3).$

(p) $\varphi(q^3)\varphi(q^4)\varphi^2(q^{12}) = \frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi(-q)\varphi^3(q^3) + \frac{1}{8}\varphi(q)\varphi(q^3)\varphi^2(-q^3) + \frac{1}{8}\varphi(-q)\varphi(q^3)\varphi^2(-q^3) + \frac{1}{4}\varphi(q)\varphi^2(q^3)\varphi(-q^3) + \frac{1}{4}\varphi(-q)\varphi^2(q^3)\varphi(-q^3).$

Proof. We just give the proof of formula (n) as the rest can be proved similarly. We have by (2.4)

$$\varphi(q^3)\varphi^3(q^4) = \varphi(q^3)\Big(\frac{1}{2}\varphi(q) + \frac{1}{2}\varphi(-q)\Big)^3$$

$$\begin{aligned} &= \frac{1}{8}\varphi^3(q)\varphi(q^3) + \frac{3}{8}\varphi^2(q)\varphi(-q)\varphi(q^3) \\ &\quad + \frac{3}{8}\varphi(q)\varphi^2(-q)\varphi(q^3) + \frac{1}{8}\varphi^3(-q)\varphi(q^3) \end{aligned}$$

as claimed. ■

The power series expansions of the products listed in (7.2) are given in Proposition 2.5 and Theorems 3.1, 4.1, 5.1 and 6.1. Using these in Theorem 7.1 we obtain our main result.

Theorem 7.2. *Let $n \in \mathbb{N}$. Then*

$$\text{(a)}\quad N(1,1,1,12;n) = \begin{cases} 3A(n) - B(n) + \frac{3}{2}C(n) \\ -\frac{1}{2}D(n) + 3F(n), & \text{if } n \equiv 1 \pmod 2, \\ \frac{9}{2}A(n) - \frac{3}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(b)}\quad N(1,1,3,4;n) = \begin{cases} 3A(n) - B(n) + \frac{3}{2}C(n) \\ -\frac{1}{2}D(n) + E(n), & \text{if } n \equiv 1 \pmod 2, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{9}{2}A(n) - \frac{3}{2}B(n) + 3C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(c)}\quad N(1,1,4,12;n) = \begin{cases} \frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{2}C(n) \\ -\frac{1}{2}D(n) + \frac{1}{2}E(n) + \frac{3}{2}F(n), & \text{if } n \equiv 1 \pmod 2, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(d)}\quad N(1,2,2,12;n) = \begin{cases} \frac{3}{2}A(n) - \frac{1}{2}B(n) - \frac{1}{2}E(n) + \frac{3}{2}F(n), & \text{if } n \equiv 1 \pmod 2, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n) - \frac{1}{2}B(n) + 3C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(e)}\quad N(1,3,3,12;n) = \begin{cases} A(n) + B(n) - \frac{1}{2}C(n) \\ -\frac{1}{2}D(n) + F(n), & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{2}A(n) + \frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n) + \frac{3}{2}B(n) - C(n) - D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

$$\text{(f)}\quad N(1,3,4,4;n)=\begin{cases}\frac{3}{2}A(n)-\frac{1}{2}B(n)+E(n), & \text{if } n\equiv 1 \pmod 2,\\ 0, & \text{if } n\equiv 2 \pmod 4,\\ 3A(n)-B(n)+3C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(g)}\quad N(1,3,12,12;n)=\begin{cases}\frac{1}{2}A(n)+\frac{1}{2}B(n)+F(n), & \text{if } n\equiv 1 \pmod 2,\\ 0, & \text{if } n\equiv 2 \pmod 4,\\ A(n)+B(n)-C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(h)}\quad N(1,4,4,12;n)=\begin{cases}\frac{1}{4}\Big(3A(n)-B(n)+3C(n) & \\ \quad -D(n)+E(n)+3F(n)\Big), & \text{if } n\equiv 1 \pmod 2,\\ 0, & \text{if } n\equiv 2 \pmod 4,\\ \frac{3}{2}A(n)-\frac{1}{2}B(n)+3C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(i)}\quad N(1,6,6,12;n)=\begin{cases}\frac{1}{2}\Big(A(n)+B(n)+E(n)+F(n)\Big), & \text{if } n\equiv 1 \pmod 2,\\ \frac{1}{2}A(n)+\frac{1}{2}B(n), & \text{if } n\equiv 2 \pmod 4,\\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(j)}\quad N(1,12,12,12;n)=\begin{cases}\frac{1}{4}\Big(A(n)+B(n)+C(n) & \\ \quad +D(n)+E(n)+3F(n)\Big), & \text{if } n\equiv 1 \pmod 2,\\ 0, & \text{if } n\equiv 2 \pmod 4,\\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(k)}\quad N(2,2,3,4;n)=\begin{cases}\frac{3}{2}A(n)-\frac{1}{2}B(n)+\frac{1}{2}E(n)-\frac{3}{2}F(n), & \text{if } n\equiv 1 \pmod 2,\\ \frac{3}{2}A(n)-\frac{1}{2}B(n), & \text{if } n\equiv 2 \pmod 4,\\ \frac{3}{2}A(n)-\frac{1}{2}B(n)+3C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

$$\text{(l)}\quad N(3,3,3,4;n)=\begin{cases}A(n)+B(n)-\frac{1}{2}C(n) & \\ -\frac{1}{2}D(n)-E(n), & \text{if } n\equiv 1 \pmod 2,\\ \frac{3}{2}A(n)+\frac{3}{2}B(n), & \text{if } n\equiv 2 \pmod 4,\\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

(m) $$N(3,3,4,12;n) = \begin{cases} \frac{1}{2}\Big(A(n)+B(n)-C(n) \\ \quad -D(n)-E(n)+F(n)\Big), & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{2}A(n)+\frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(n) $$N(3,4,4,4;n) = \begin{cases} \frac{1}{4}\Big(3A(n)-B(n)-3C(n) \\ \quad +D(n)+3E(n)-3F(n)\Big), & \text{if } n \equiv 1 \pmod 2, \\ 0, & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}A(n)-\frac{1}{2}B(n)+3C(n)-D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(o) $$N(3,4,6,6;n) = \begin{cases} \frac{1}{2}\Big(A(n)+B(n)-E(n)-F(n)\Big), & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{2}A(n)+\frac{1}{2}B(n), & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(p) $$N(3,4,12,12;n) = \begin{cases} \frac{1}{4}\Big(A(n)+B(n)-C(n) \\ \quad -D(n)-E(n)+F(n)\Big), & \text{if } n \equiv 1 \pmod 2, \\ 0, & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{2}A(n)+\frac{1}{2}B(n)-C(n)-D(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Appealing to Theorem 7.2 and Proposition 2.3, and using the identity

$$A(n)+sB(n)+tC(n)+stD(n)$$

$$= (2^\alpha + t(-1)^{\alpha+\beta+(N-1)/2})(3^\beta + s(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big))A(N),$$

we obtain the following corollary, see Liouville [6]-[19].

Corollary 7.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

(a) $N(1,1,1,12;n)$

$$= \begin{cases} \frac{1}{2}(2+(-1)^{\beta+\frac{N-1}{2}})\Big(3^{\beta+1}-(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N)+3F(n), & \text{if } \alpha = 0, \\ 3\Big(3^{\beta+1}+(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N), & \text{if } \alpha = 1, \\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \text{if } \alpha \geq 2. \end{cases}$$

(b) $N(1,1,3,4;n)$

$$= \begin{cases} \frac{1}{2}\Big(2+(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+E(n), & \textit{if } \alpha=0, \\ \Big(3^{\beta+1}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(3\cdot 2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\ge 2. \end{cases}$$

(c) $N(1,1,4,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(1+(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N) & \\ +\frac{1}{2}E(n)+\frac{3}{2}F(n), & \textit{if } \alpha=0, \\ \Big(3^{\beta+1}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\ge 2. \end{cases}$$

(d) $N(1,2,2,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)-\frac{1}{2}E(n)+\frac{3}{2}F(n), & \textit{if } \alpha=0, \\ \Big(3^{\beta+1}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\ge 2. \end{cases}$$

(e) $N(1,3,3,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(2-(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+F(n), & \textit{if } \alpha=0, \\ \Big(3^{\beta}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(3\cdot 2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\ge 2. \end{cases}$$

(f) $N(1,3,4,4;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+E(n), & \textit{if } \alpha=0, \\ 0, & \textit{if } \alpha=1, \\ \Big(2^{\alpha}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\ge 2. \end{cases}$$

(g) $N(1,3,12,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^{\beta}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+F(n), & \textit{if } \alpha=0,\\ 0, & \textit{if } \alpha=1,\\ \Big(2^{\alpha}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(h) $N(1,4,4,12;n)$

$$= \begin{cases} \frac{1}{4}\Big(1+(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N) & \\ +\frac{1}{4}E(n)+\frac{3}{4}F(n), & \textit{if } \alpha=0,\\ 0, & \textit{if } \alpha=1,\\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(i) $N(1,6,6,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^{\beta}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+\frac{1}{2}E(n)+\frac{1}{2}F(n), & \textit{if } \alpha=0,\\ \Big(3^{\beta}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1,\\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(j) $N(1,12,12,12;n)$

$$= \begin{cases} \frac{1}{4}\Big(1+(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N) & \\ +\frac{1}{4}E(n)+\frac{3}{4}F(n), & \textit{if } \alpha=0,\\ 0, & \textit{if } \alpha=1,\\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta}+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(k) $N(2,2,3,4;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^{\beta+1}-(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N)+\frac{1}{2}E(n)-\frac{3}{2}F(n), & \textit{if } \alpha=0,\\ \Big(3^{\beta+1}+(-1)^{\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1,\\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(l) $N(3,3,3,4;n)$

$$= \begin{cases} \frac{1}{2}\Big(2-(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N)-E(n), & \textit{if } \alpha=0, \\ 3\Big(3^\beta-(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(m) $N(3,3,4,12;n)$

$$= \begin{cases} \frac{1}{2}\Big(1-(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N) \\ -\frac{1}{2}E(n)+\frac{1}{2}F(n), & \textit{if } \alpha=0, \\ \Big(3^\beta-(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(n) $N(3,4,4,4;n)$

$$= \begin{cases} \frac{1}{4}\Big(1-(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N) \\ +\frac{3}{4}E(n)-\frac{3}{4}F(n), & \textit{if } \alpha=0, \\ 0, & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^{\beta+1}-(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(o) $N(3,4,6,6;n)$

$$= \begin{cases} \frac{1}{2}\Big(3^\beta+(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N)-\frac{1}{2}E(n)-\frac{1}{2}F(n), & \textit{if } \alpha=0, \\ \Big(3^\beta-(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

(p) $N(3,4,12,12;n)$

$$= \begin{cases} \frac{1}{4}\Big(1-(-1)^{\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^\beta\Big(\frac{N}{3}\Big)\Big)A(N) \\ -\frac{1}{4}E(n)+\frac{1}{4}F(n), & \textit{if } \alpha=0, \\ 0, & \textit{if } \alpha=1, \\ \Big(2^{\alpha-1}-(-1)^{\alpha+\beta+\frac{N-1}{2}}\Big)\Big(3^\beta+(-1)^{\alpha+\beta}\Big(\frac{N}{3}\Big)\Big)A(N), & \textit{if } \alpha\geq 2. \end{cases}$$

8 Conclusion

We conclude by noting that the quantities $R(n)$ and $S(n)$ used in Sections 4 and 5 respectively can be determined explicitly. Replacing n by $4n$ in Lemma 4.2 we obtain

$$R(n) = \frac{1}{6}\left(N(1,1,1,3;4n) - N(1,1,1,3;n)\right).$$

From [1, Theorem 4.1] we have

$$N(1,1,1,3;n) = 6A(n) - 2B(n) + 3C(n) - D(n).$$

Thus, by Proposition 2.3, we have

$$N(1,1,1,3;4n) = 24A(n) - 8B(n) + 3C(n) - D(n).$$

Hence

$$R(n) = 3A(n) - B(n) = 2^{\alpha}\left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N),$$

which is Theorem 13.2 of [1]. In a similar way starting from Lemma 5.2 and appealing to Theorem 8.1 of [1], we can obtain an explicit formula for $S(n)$. We leave this to the reader.

The methods of this paper can be used to determine explicit formulae for $N(a,b,c,d;n)$ (valid for all $n \in \mathbb{N}$) for other forms $ax^2 + by^2 + cz^2 + dt^2$, see for example [3], [4].

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