

# The Convolution Sum

$$\sum_{m < n/16} \sigma(m)\sigma(n - 16m)$$

Ayşe Alaca, Şaban Alaca, and Kenneth S. Williams

*Abstract.* The convolution sum  $\sum_{m < n/16} \sigma(m)\sigma(n - 16m)$  is evaluated for all  $n \in \mathbb{N}$ . This evaluation is used to determine the number of representations of  $n$  by the quadratic form  $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 4x_5^2 + 4x_6^2 + 4x_7^2 + 4x_8^2$ .

## 1 Introduction

Let  $\mathbb{N}$  denote the set of natural numbers and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Let  $\mathbb{Z}$  denote the set of all integers. For  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$  we set

$$\sigma_k(n) = \sum_{d|n} d^k,$$

where  $d$  runs through the positive integers dividing  $n$ . If  $n \notin \mathbb{N}$ , we set  $\sigma_k(n) = 0$ . We write  $\sigma(n)$  for  $\sigma_1(n)$ . Let  $\mathbb{R}$  and  $\mathbb{C}$  denote the fields of real and complex numbers respectively. In [10] the third author showed that

$$(1.1) \quad \sum_{m < n/8} \sigma(m)\sigma(n - 8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) \\ + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{64}c_8(n),$$

where  $c_8(n)$  ( $n \in \mathbb{N}$ ) are integers defined by

$$(1.2) \quad q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 := \sum_{n=1}^{\infty} c_8(n)q^n, \quad q \in \mathbb{C}, |q| < 1.$$

We note that  $c_8(1) = 1$  and  $c_8(2n) = 0$ ,  $n \in \mathbb{N}$ . In this paper we evaluate the convolution sum

$$\sum_{m < n/16} \sigma(m)\sigma(n - 16m)$$

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Received by the editors October 11, 2005.

The research of the third author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

AMS subject classification: Primary: 11A25; secondary: 11E20, 11E25.

Keywords: divisor functions, convolution sums, Eisenstein series.

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for all  $n \in \mathbb{N}$ . In order to do this, it is convenient to define, for  $k \in \{1, 2, 3, 4, 5, 6, 7\}$  and  $q \in \mathbb{C}, |q| < 1$ ,

$$(1.3) \quad A_k(q) := \prod_{n=1}^{\infty} (1+q^n)^{24-4k} (1-q^n)^8 (1-q^{4n-2})^{16-2k}.$$

It is easy to show that

$$(1.4) \quad A_k(-q) = A_{8-k}(q).$$

Next we define rational numbers  $c_{16}(n)$  ( $n \in \mathbb{N}$ ) by

$$(1.5) \quad \sum_{n=1}^{\infty} c_{16}(n)q^n := \frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) \\ - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q),$$

so that  $c_{16}(1) = 1$ ,  $c_{16}(2) = 12/7$ ,  $c_{16}(3) = 4/7$ ,  $c_{16}(4) = 0$ ,  $c_{16}(5) = -2$ ,  $c_{16}(6) = -48/7$ , etc. In Section 2 we show that

$$(1.6) \quad 7c_{16}(n) \in \mathbb{Z}, \quad n \in \mathbb{N},$$

and prove the following result.

**Theorem 1.1** For all  $n \in \mathbb{N}$  we have

$$\sum_{m < n/16} \sigma(m)\sigma(n-16m) = \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{64}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{8}\right) \\ + \frac{1}{3}\sigma_3\left(\frac{n}{16}\right) + \left(\frac{1}{24} - \frac{n}{64}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{16}\right) - \frac{7}{256}c_{16}(n).$$

As an application of Theorem 1.1, we prove the following result in Section 3.

**Theorem 1.2** For all  $n \in \mathbb{N}$  the number  $N(n)$  of  $(x_1, \dots, x_8) \in \mathbb{Z}^8$  such that

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 4x_5^2 + 4x_6^2 + 4x_7^2 + 4x_8^2$$

is given by

$$N(n) = \sigma_3(n) + 3\sigma_3\left(\frac{n}{2}\right) - 68\sigma_3\left(\frac{n}{4}\right) + 48\sigma_3\left(\frac{n}{8}\right) + 256\sigma_3\left(\frac{n}{16}\right) + 7c_{16}(n).$$

## 2 Proof of Theorem 1.1

For  $z \in \mathbb{C}$  with  $|z| < 1$  we set

$$(2.1) \quad w(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where  ${}_2F_1$  is the Gaussian hypergeometric function and  $(a)_n$  is the Pochhammer symbol, see for example [4, p. 247], [8, p. 45]. Clearly  $w(0) = 1$ . The infinite series (2.1) diverges at  $z = 1$  [4, p. 249], so that  $w(1) = +\infty$ . For  $x \in \mathbb{R}$  with  $0 \leq x < 1$ , we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{2n!^2}{n!^4 2^{4n}} x^n \geq 1,$$

so that

$$(2.2) \quad w(x) \neq 0, \quad 0 \leq x < 1.$$

The derivative with respect to  $x$  of the function

$$y(x) := \pi \frac{w(1-x)}{w(x)}, \quad 0 < x < 1,$$

is [1, p. 87]

$$(2.3) \quad y'(x) = \frac{-1}{x(1-x)w(x)^2}, \quad 0 < x < 1.$$

Thus, by (2.2) and (2.3), we have  $y'(x) < 0$ ,  $0 < x < 1$ . Hence, as  $x$  increases from 0 to 1,  $y(x)$  strictly decreases from  $y(0) = \pi \frac{w(1)}{w(0)} = +\infty$  to  $y(1) = \pi \frac{w(0)}{w(1)} = 0$ . Now restrict  $q$  so that  $q \in \mathbb{R}$  and  $0 < q < 1$ . Thus  $0 < -\log q < +\infty$ . Hence there is a unique value of  $x$  between 0 and 1 such that  $y(x) = -\log q$ . The Eisenstein series  $L(q)$  and  $M(q)$  are defined by

$$(2.4) \quad L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n,$$

$$(2.5) \quad M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.$$

The discriminant function  $\Delta(q)$  is defined by

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

It is proved in [2] that

$$(2.6) \quad L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx},$$

$$(2.7) \quad M(q) = (1 + 14x + x^2)w^4,$$

$$(2.8) \quad \Delta(q) = \frac{x(1 - x)^4 w^{12}}{2^4}.$$

Applying the duplication principle [2, p. 125],

$$q \rightarrow q^2, \quad x \rightarrow \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2, \quad w \rightarrow \left( \frac{1 + \sqrt{1 - x}}{2} \right) w,$$

to (2.6), (2.7) and (2.8), we obtain

$$(2.9) \quad L(q^2) = (1 - 2x)w^2 + 6x(1 - x)w \frac{dw}{dx},$$

$$(2.10) \quad M(q^2) = (1 - x + x^2)w^4,$$

$$(2.11) \quad \Delta(q^2) = \frac{x^2(1 - x)^2 w^{12}}{2^8}.$$

Formulae (2.9) and (2.10) are given in [2, p. 122, p. 126]. Applying the duplication principle to (2.9), (2.10) and (2.11), we obtain

$$(2.12) \quad L(q^4) = \left( 1 - \frac{5}{4}x \right) w^2 + 3x(1 - x)w \frac{dw}{dx},$$

$$(2.13) \quad M(q^4) = \left( 1 - x + \frac{1}{16}x^2 \right) w^4,$$

$$(2.14) \quad \Delta(q^4) = \frac{x^4(1 - x)w^{12}}{2^{16}}.$$

Applying the duplication principle to (2.12) and (2.13), we have

$$(2.15) \quad L(q^8) = \left( \frac{5}{8} - \frac{11}{16}x + \frac{3}{8}\sqrt{1 - x} \right) w^2 + \frac{3}{2}x(1 - x)w \frac{dw}{dx},$$

$$(2.16) \quad M(q^8) = \left( \frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1 - x} - \frac{15}{64}x\sqrt{1 - x} \right) w^4.$$

Applying the duplication principle again, this time to (2.15) and (2.16), we obtain

$$(2.17) \quad L(q^{16}) = \left( \frac{11}{32} - \frac{23}{64}x + \frac{9}{32}\sqrt{1 - x} + \frac{3}{16}(1 - x)^{1/4} \right. \\ \left. + \frac{3}{16}(1 - x)^{3/4} \right) w^2 + \frac{3}{4}x(1 - x)w \frac{dw}{dx},$$

$$(2.18) \quad M(q^{16}) = \left( \frac{137}{512} - \frac{137}{512}x + \frac{1}{4096}x^2 + \frac{135}{512}\sqrt{1-x} \right. \\ \left. - \frac{135}{1024}x\sqrt{1-x} + \frac{15}{64}(1-x)^{1/4} - \frac{105}{512}x(1-x)^{1/4} \right. \\ \left. + \frac{15}{64}(1-x)^{3/4} - \frac{15}{512}x(1-x)^{3/4} \right) w^4.$$

From (2.7), (2.10) and (2.13) we obtain

$$(2.19) \quad w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4),$$

$$(2.20) \quad xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2),$$

$$(2.21) \quad x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4).$$

Next, for  $k \in \{1, 2, 3, 4, 5, 6, 7\}$ , we see from (2.8), (2.11) and (2.14) that

$$\Delta(q)^{\frac{k-4}{6}} \Delta(q^2)^{\frac{20-3k}{12}} \Delta(q^4)^{\frac{k-8}{12}} = (1-x)^{k/4} w^4 = A^k w^4,$$

where for convenience we have set

$$(2.22) \quad A = (1-x)^{1/4}.$$

Note that  $x = 1 - A^4$ . Further

$$\Delta(q)^{\frac{k-4}{6}} \Delta(q^2)^{\frac{20-3k}{12}} \Delta(q^4)^{\frac{k-8}{12}} \\ = \prod_{n=1}^{\infty} (1-q^n)^{4k-16} (1-q^{2n})^{40-6k} (1-q^{4n})^{2k-16} \\ = \prod_{n=1}^{\infty} (1-q^n)^{4k-16} (1-q^{2n})^{24-4k} (1-q^{4n-2})^{16-2k} \\ = \prod_{n=1}^{\infty} (1-q^n)^{4k-16} (1-q^n)^{24-4k} (1+q^n)^{24-4k} (1-q^{4n-2})^{16-2k} \\ = \prod_{n=1}^{\infty} (1+q^n)^{24-4k} (1-q^n)^8 (1-q^{4n-2})^{16-2k}.$$

Hence

$$(2.23) \quad A_k(q) = A^k w^4 = \Delta(q)^{\frac{k-4}{6}} \Delta(q^2)^{\frac{20-3k}{12}} \Delta(q^4)^{\frac{k-8}{12}}.$$

From (2.6) and (2.17) we obtain

$$(2.24) \quad L(q) - 16L(q^{16}) = \left( -\frac{9}{2} + \frac{3}{4}x - 3A - \frac{9}{2}A^2 - 3A^3 \right) w^2.$$

Squaring (2.24), we obtain

$$(2.25) \quad (L(q) - 16L(q^{16}))^2 = \frac{117}{2}w^4 - 45xw^4 + \frac{9}{16}x^2w^4 + \frac{45}{2}Aw^4 \\ + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

Appealing to (2.19), (2.20), (2.21) and (2.25), we obtain

$$(2.26) \quad (L(q) - 16L(q^{16}))^2 = \frac{9}{10}M(q) - \frac{21}{5}M(q^2) + \frac{309}{5}M(q^4) + \frac{45}{2}Aw^4 \\ + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

From (2.16), (2.19), (2.20), (2.21) and (2.22), we deduce

$$(2.27) \quad M(q^8) = -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{64}A^2w^4 + \frac{15}{64}A^6w^4.$$

From (2.18), (2.19), (2.20), (2.21), (2.22), (2.23) we obtain

$$(2.28) \quad M(q^{16}) = -\frac{9}{512}M(q^2) + \frac{73}{256}M(q^4) + \frac{15}{512}Aw^4 + \frac{135}{1024}A^2w^4 \\ + \frac{105}{512}A^3w^4 + \frac{105}{512}A^5w^4 + \frac{135}{1024}A^6w^4 + \frac{15}{512}A^7w^4.$$

From (1.5) and (2.23) we have

$$(2.29) \quad \sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{1}{32}Aw^4 + \frac{3}{112}A^2w^4 + \frac{1}{224}A^3w^4 \\ - \frac{1}{32}A^5w^4 - \frac{3}{112}A^6w^4 - \frac{1}{224}A^7w^4.$$

Then, from (2.27), (2.28) and (2.29), we deduce

$$(2.30) \quad \frac{24}{5}M(q^8) - \frac{1152}{5}M(q^{16}) - 504 \sum_{n=1}^{\infty} c_{16}(n)q^n \\ = \frac{45}{2}Aw^4 + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

We see from (2.26) and (2.30) that

$$(2.31) \quad (L(q) - 16L(q^{16}))^2 = \frac{9}{10}M(q) - \frac{21}{5}M(q^2) + \frac{309}{5}M(q^4) \\ + \frac{24}{5}M(q^8) - \frac{1152}{5}M(q^{16}) - 504 \sum_{n=1}^{\infty} c_{16}(n)q^n.$$

Then

$$\begin{aligned} 576 \sum_{n=1}^{\infty} \sum_{m < n/16} \sigma(m)\sigma(n-16m)q^n &= \left(24 \sum_{l=1}^{\infty} \sigma(l)q^l\right) \left(24 \sum_{m=1}^{\infty} \sigma(m)q^{16m}\right) \\ &= 1 - L(q) - L(q^{16}) + L(q)L(q^{16}), \end{aligned}$$

so that

$$\begin{aligned} (2.32) \quad \sum_{n=1}^{\infty} \sum_{m < n/16} \sigma(m)\sigma(n-16m)q^n &= \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^{16}) \\ &\quad + \frac{1}{18432}L(q)^2 + \frac{1}{72}L(q^{16})^2 - \frac{1}{18432}(L(q) - 16L(q^{16}))^2. \end{aligned}$$

The following result is classical

$$(2.33) \quad L(q)^2 = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n,$$

see for example [5, 6]. Letting  $q \rightarrow q^{16}$  in (2.4) and (2.33), we obtain

$$(2.34) \quad L(q^{16}) = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{16}\right)q^n,$$

$$(2.35) \quad L(q^{16})^2 = 1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{16}\right) - 18n\sigma\left(\frac{n}{16}\right)\right)q^n.$$

Putting (2.4), (2.31), (2.33) and (2.34) into (2.32), we obtain, on equating the coefficients of  $q^n$ , the asserted formula of Theorem 1.1.  $\blacksquare$

By Theorem 1.1 we have

$$\begin{aligned} 7c_{16}(n) &= \frac{1}{3}\sigma_3(n) + \sigma_3\left(\frac{n}{2}\right) + 4\sigma_3\left(\frac{n}{4}\right) + 16\sigma_3\left(\frac{n}{8}\right) + \frac{256}{3}\sigma_3\left(\frac{n}{16}\right) \\ &\quad + \left(\frac{32}{3} - 4n\right)\sigma(n) + \left(\frac{32}{3} - 64n\right)\sigma\left(\frac{n}{16}\right) \\ &\quad - 256 \sum_{m < n/16} \sigma(m)\sigma(n-16m) \\ &\equiv \frac{1}{3}\sigma_3(n) - \frac{1}{3}\sigma(n) + \frac{1}{3}\sigma_3\left(\frac{n}{16}\right) - \frac{1}{3}\sigma\left(\frac{n}{16}\right) \pmod{1}. \end{aligned}$$

As  $\sigma_3(k) \equiv \sigma(k) \pmod{3}$ ,  $k \in \mathbb{N}$ , we deduce (1.6). Replacing  $n$  by  $2n$  in Theorem 1.1, and making use of the elementary identities

$$(2.36) \quad \sigma(2k) = 3\sigma(k) - 2\sigma\left(\frac{k}{2}\right), \quad k \in \mathbb{N},$$

$$(2.37) \quad \sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3\left(\frac{k}{2}\right), \quad k \in \mathbb{N},$$

we obtain

$$\begin{aligned}
(2.38) \quad \sum_{m < n/8} \sigma(m)\sigma(2n - 16m) &= \frac{9}{768}\sigma_3(n) - \frac{5}{768}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{64}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{8}\right) \\
&+ \frac{1}{3}\sigma_3\left(\frac{n}{16}\right) + \left(\frac{1}{8} - \frac{3n}{32}\right)\sigma(n) \\
&- \left(\frac{1}{12} - \frac{n}{16}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{2}\right)\sigma\left(\frac{n}{8}\right) - \frac{7}{256}c_{16}(2n).
\end{aligned}$$

Next, by (2.36) we have

$$\sum_{m < n/8} \sigma(m)\sigma(2n - 16m) = 3 \sum_{m < n/8} \sigma(m)\sigma(n - 8m) - 2 \sum_{m < n/8} \sigma(m)\sigma\left(\frac{n}{2} - 4m\right).$$

By [7, Theorem 4], we have

$$\begin{aligned}
(2.39) \quad \sum_{m < n/4} \sigma(m)\sigma(n - 4m) &= \frac{1}{48}\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) \\
&+ \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{4}\right).
\end{aligned}$$

Appealing to (1.1), (1.2), and (2.39) with  $n$  replaced by  $n/2$ , we obtain

$$\begin{aligned}
(2.40) \quad \sum_{m < n/8} \sigma(m)\sigma(2n - 16m) &= \frac{1}{64}\sigma_3(n) + \frac{1}{192}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) \\
&+ \left(\frac{1}{8} - \frac{3}{32}n\right)\sigma(n) - \left(\frac{1}{12} - \frac{n}{16}\right)\sigma\left(\frac{n}{2}\right) \\
&+ \left(\frac{1}{24} - \frac{n}{2}\right)\sigma\left(\frac{n}{8}\right) - \frac{3}{64}c_8(n).
\end{aligned}$$

Equating (2.38) and (2.40), we obtain

**Corollary 2.1**  $c_{16}(2n) = \frac{12}{7}c_8(n)$ ,  $n \in \mathbb{N}$ .

By Corollary 2.1, we see that  $c_{16}(4n) = \frac{12}{7}c_8(2n) = 0$ ,  $n \in \mathbb{N}$ . Appealing to (1.4)



and (1.5), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} c_{16}(2n)q^{2n} &= \frac{1}{2} \left( \sum_{n=1}^{\infty} c_{16}(n)q^n + \sum_{n=1}^{\infty} c_{16}(n)(-q)^n \right) \\
&= \frac{1}{2} \left( \frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) \right. \\
&\quad \left. - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q) \right. \\
&\quad \left. + \frac{1}{32}A_7(q) + \frac{3}{112}A_6(q) + \frac{1}{224}A_5(q) \right. \\
&\quad \left. - \frac{1}{32}A_3(q) - \frac{3}{112}A_2(q) - \frac{1}{224}A_1(q) \right) \\
&= \frac{3}{224}(A_1(q) - A_3(q) - A_5(q) + A_7(q)).
\end{aligned}$$

Hence, by Corollary 2.1, we have

$$(2.41) \quad A_1(q) - A_3(q) - A_5(q) + A_7(q) = 128 \sum_{n=1}^{\infty} c_8(n)q^{2n}.$$

Using (1.2) and (1.3) in (2.41), we obtain the following identity.

**Corollary 2.2**

$$\begin{aligned}
&\prod_{n=1}^{\infty} (1+q^n)^{20}(1-q^n)^8(1-q^{4n-2})^{14} - \prod_{n=1}^{\infty} (1+q^n)^{12}(1-q^n)^8(1-q^{4n-2})^{10} \\
&\quad - \prod_{n=1}^{\infty} (1+q^n)^4(1-q^n)^8(1-q^{4n-2})^6 + \prod_{n=1}^{\infty} (1+q^n)^{-4}(1-q^n)^8(1-q^{4n-2})^2 \\
&\quad = 128q^2 \prod_{n=1}^{\infty} (1-q^{4n})^4(1-q^{8n})^4.
\end{aligned}$$

From [7, Lemma] we have

$$\begin{aligned}
\sum_{m < n/16} \sigma(m)\sigma(n-16m) &= -\frac{1}{24}\sigma_3(n) + \frac{1}{4}\sigma_3\left(\frac{n}{16}\right) + \frac{1}{24}\sigma(n) - \frac{n}{4}\sigma\left(\frac{n}{16}\right) \\
&\quad + \frac{1}{4} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n \\ x \equiv -y \pmod{16}}} ab + \frac{1}{4} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n \\ x \equiv y \pmod{16}}} ab.
\end{aligned}$$

Combining this result with Theorem 1.1, we obtain the following finite formula for  $c_{16}(n)$  ( $n \in \mathbb{N}$ ).

**Corollary 2.3**

$$\begin{aligned} c_{16}(n) &= \frac{11}{7}\sigma_3(n) + \frac{1}{7}\sigma_3\left(\frac{n}{2}\right) + \frac{4}{7}\sigma_3\left(\frac{n}{4}\right) + \frac{16}{7}\sigma_3\left(\frac{n}{8}\right) + \frac{64}{21}\sigma_3\left(\frac{n}{16}\right) \\ &\quad - \frac{4}{7}n\sigma(n) + \frac{32}{21}\sigma\left(\frac{n}{16}\right) - \frac{64}{7} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n \\ x \equiv -y \pmod{16}}} ab - \frac{64}{7} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n \\ x \equiv y \pmod{16}}} ab. \end{aligned}$$

### 3 Proof of Theorem 1.2

For  $l \in \mathbb{N}_0$  we set  $r_4(l) = \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = l\}$ , so that  $r_4(0) = 1$ . It is a classical result of Jacobi (see for example [9]) that

$$(3.1) \quad r_4(n) = 8 \sum_{\substack{d|n \\ 4 \nmid d}} d = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right), \quad n \in \mathbb{N}.$$

Then the number  $N(n)$  of representations of  $n \in \mathbb{N}$  by the quadratic form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 4x_5^2 + 4x_6^2 + 4x_7^2 + 4x_8^2$$

is given by

$$N(n) = \sum_{\substack{l, m \in \mathbb{N}_0 \\ l+4m=n}} r_4(l)r_4(m) = r_4(n) + r_4\left(\frac{n}{4}\right) + \sum_{\substack{l, m \in \mathbb{N} \\ m < n/4}} r_4(m)r_4(n-4m).$$

Appealing to (3.1), we obtain

$$\begin{aligned} N(n) &= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{4}\right) - 32\sigma\left(\frac{n}{16}\right) \\ &\quad + \sum_{m < n/4} \left(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)\right) \left(8\sigma(n-4m) - 32\sigma\left(\frac{n}{4} - m\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} N(n) &- \left(8\sigma(n) - 24\sigma\left(\frac{n}{4}\right) - 32\sigma\left(\frac{n}{16}\right)\right) \\ &= 64 \sum_{m < n/4} \sigma(m)\sigma(n-4m) - 256 \sum_{m < n/4} \sigma\left(\frac{m}{4}\right)\sigma(n-4m) \\ &\quad - 256 \sum_{m < n/4} \sigma(m)\sigma\left(\frac{n}{4} - m\right) + 1024 \sum_{m < n/4} \sigma\left(\frac{m}{4}\right)\sigma\left(\frac{n}{4} - m\right). \end{aligned}$$

The value of the first sum is given in (2.39). The second sum is

$$\sum_{m < n/4} \sigma\left(\frac{m}{4}\right) \sigma(n - 4m) = \sum_{m < n/16} \sigma(m) \sigma(n - 16m),$$

whose value is given in Theorem 1.1. The third sum is

$$\sum_{m < n/4} \sigma(m) \sigma\left(\frac{n}{4} - m\right) = \frac{5}{12} \sigma_3\left(\frac{n}{4}\right) + \left(\frac{1}{12} - \frac{n}{8}\right) \sigma\left(\frac{n}{4}\right),$$

see [3, 7]. The fourth sum is (by (2.39))

$$\begin{aligned} \sum_{m < n/4} \sigma\left(\frac{m}{4}\right) \sigma\left(\frac{n}{4} - m\right) &= \sum_{m < n/16} \sigma(m) \sigma\left(\frac{n}{4} - 4m\right) \\ &= \frac{1}{48} \sigma_3\left(\frac{n}{4}\right) + \frac{1}{16} \sigma_3\left(\frac{n}{8}\right) + \frac{1}{3} \sigma_3\left(\frac{n}{16}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{64}\right) \sigma\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{n}{16}\right) \sigma\left(\frac{n}{16}\right). \end{aligned}$$

Putting these results together we obtain the assertion of Theorem 1.2. ■

The values of  $N(n)$  for  $n = 1, 2, \dots, 20$  are given in the following table.

$n$	$N(n)$	$\sigma_3(n)$	$c_{16}(n)$	$\sigma_3(n) + 3\sigma_3(n/2) - 68\sigma_3(n/4) + 48\sigma_3(n/8) + 256\sigma_3(n/16) + 7c_{16}(n)$
1	8	1	1	8
2	24	9	12/7	24
3	32	28	4/7	32
4	32	73	0	32
5	112	126	-2	112
6	288	252	-48/7	288
7	320	344	-24/7	320
8	240	585	0	240
9	680	757	-11	680
10	1488	1134	-24/7	1488
11	1376	1332	44/7	1376
12	896	2044	0	896
13	2352	2198	22	2352
14	4416	3096	288/7	4416
15	3520	3528	-8/7	3520
16	2160	4681	0	2160
17	5264	4914	50	5264
18	8952	6813	-132/7	8952
19	6816	6860	-44/7	6816
20	4032	9198	0	4032

We conclude by remarking that the table suggests that  $c_{16}(4k + 1) \in \mathbb{Z}$  for  $k \in \mathbb{N}_0$ . We have verified this for  $0 \leq k \leq 100$ .

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Centre for Research in Algebra and Number Theory  
 School of Mathematics and Statistics  
 Carleton University  
 Ottawa, ON  
 K1S 5B6  
 e-mail: aalaca@math.carleton.ca  
           salaca@math.carleton.ca  
           williams@math.carleton.ca