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Lambert series and Liouville's identities

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CONTENTS

Abstract

The relationship between Liouville's arithmetic identities and products of Lambert series is investigated. For example it is shown that Liouville's arithmetic formula for the sum

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (F(a-b)-F(a+b)),$$

where $n\in\mathbb{N}$ and $F:\mathbb{Z}\to\mathbb{C}$ is an even function, is equivalent to the Lambert series for

$$\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\right)^2 \qquad (\theta\in\mathbb{R},\ |q|<1)$$

given by Ramanujan.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $F : \mathbb{Z} \to \mathbb{C}$ we adopt the convention that $F(x) = 0$ for $x \notin \mathbb{Z}$. Throughout this paper θ denotes a real number and q denotes a complex number satisfying $|q| < 1$. We also set

$$u_n := \frac{q^n}{1-q^n}, \qquad v_n := \frac{u_n}{1+q^n} = q^{-n}u_{2n} = \frac{q^n}{1-q^{2n}} \qquad (n \in \mathbb{N}).$$

Let $f : \mathbb{Z} \to \mathbb{C}$. We are interested in Lambert series of the form

$$\sum_{n=1}^{\infty} u_n f(n) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}, \qquad \sum_{n=1}^{\infty} v_n f(n) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}$$

and

$$\sum_{n=0}^{\infty} v_{2n+1} f(n) \begin{Bmatrix} \sin(2n+1)\theta \\ \cos(2n+1)\theta \end{Bmatrix}.$$

As usual we denote the number of (positive) divisors of $n \in \mathbb{N}$ by $d(n)$ and the sum of the divisors of n by $\sigma(n)$. In a famous series [8] of eighteen articles published between 1858 and 1865 Liouville stated but did not prove a number of arithmetic identities. Typical of these is the following identity:

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (F(a-b) - F(a+b)) = F(0)(\sigma(n) - d(n)) + \sum_{d|n}\left(1 - d + \frac{2n}{d}\right)F(d) - 2\sum_{d|n}\sum_{1\le k\le d} F(k).$$

In 1916 Ramanujan [23, eqn. (17)], [24, p. 139] determined the square of the Lambert series $\sum_{n=1}^{\infty} u_n \sin n\theta$. He proved

$$\left(\frac{1}{4}\cot\frac{1}{2}\theta + \sum_{n=1}^{\infty} u_n \sin n\theta\right)^2 = \left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^2 + \sum_{n=1}^{\infty}(u_n + u_n^2)\cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty} n u_n (1 - \cos n\theta).$$

Although these two equalities appear to be totally unrelated, we show the surprising result that they are in fact equivalent, that is, we can deduce Ramanujan's analytic identity from Liouville's arithmetic formula and conversely.

In this paper we examine the relationship between products of Lambert series and arithmetic identities of Liouville type. Our investigation is based upon three elementary analytic theorems, which evaluate products of pairs of Lambert series of the type mentioned at the beginning of the Introduction. These theorems (Theorems 2.1, 2.2 and 2.3) are stated in Section 2 and proved in Section 8. In Section 3 we determine ten products of pairs of particular Lambert series (Theorems 3.1–3.10), all of which can be deduced from the three basic theorems. For example we show that

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty} v_n \sum_{k=1}^{n-1}(1-\cos k\theta) - \frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2+3u_n-4v_n^2-2nv_n-v_n)(1-\cos n\theta)$$

(see Theorem 3.4(a)). The proofs of these ten evaluations are given in Section 9. In Section 4 we give the cubes and fourth powers of the three Lambert series

$$\sum_{n=1}^{\infty} u_n \sin n\theta, \qquad \sum_{n=1}^{\infty} v_n \sin n\theta, \qquad \sum_{n=0}^{\infty} v_{2n+1} \sin{(2n+1)\theta}$$

(see Theorems 4.1–4.6). These evaluations are proved in Section 10. They are all accomplished by repeated applications of the three basic theorems. In Section 5 we give three arithmetic formulae, which were first stated by Liouville and later proved by others, as well as seven new arithmetic formulae of Liouville type. In Section 6 we give six more arithmetic formulae, two of which are due to McAfee [11] and the remaining four are new. In preparation for the proofs of our results we give in Section 7 some basic properties of u_n and v_n that we shall need, as well as some trigonometric identities which will be useful.

In Section 11 we prove a theorem which we use in Sections 12, 13 and 14, where we prove the equivalence of the analytic theorems (Theorems 3.1–3.10 and 4.1–4.6) with the arithmetic theorems (Theorems 5.1–5.10 and 6.1–6.6). In Section 15 we give alternative forms of Theorems 3.1, 4.1 and 4.4. The first of these (Theorem 15.1) is due to Ramanujan [23, eqn. (17)], [24, p. 139], the second (Theorem 15.2) to Ramamani [21, p. 104], and the third (Theorem 15.3) is new. In this section we also prove another identity of Ramanujan (Theorem 15.4) as well as an identity due to Liu (Theorem 15.5) from Theorems 2.1 and 2.2. In Section 16 we give eleven arithmetic identities which follow from our results. Finally, in Section 17, we give eighteen new identities which follow from our three basic theorems.

The ideas in this paper owe their origins to a classical paper of Humbert [7], a research note of Ou [17], and a thesis of McAfee [11]. The authors are grateful to George Andrews for bringing Humbert's paper to their attention.

2. Three basic theorems for products of Lambert series

In this section we state our three basic theorems giving the product of two Lambert series. These theorems are proved in Section 8.

It is convenient to assume that the functions $f : \mathbb{Z} \to \mathbb{C}$ and $g : \mathbb{Z} \to \mathbb{C}$ in Theorems 2.1–2.3 satisfy the inequalities

$$|f(n)| \le An^r, \qquad |g(n)| \le Bn^s, \qquad n \in \mathbb{Z}, \tag{2.1}$$

where $A > 0$, $B > 0$, $r \ge 0$, $s \ge 0$ do not depend on n, so that all series occurring in the theorems converge absolutely.

THEOREM 2.1. *Let $f : \mathbb{Z} \to \mathbb{C}$ and $g : \mathbb{Z} \to \mathbb{C}$ satisfy* (2.1).

(a) *If f and g are both even then*

$$\Big(\sum_{n=1}^{\infty} u_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n) \sin n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)g(n)u_n^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k \cos k\theta,$$

where for $k \in \mathbb{N}$,

$$\begin{aligned} c_k = {} & u_k \sum_{n=1}^{\infty} [f(n+k)g(n) + f(n)g(n+k)]u_n - u_k \sum_{n=1}^{\infty} [f(n)g(n-k) + f(n-k)g(n)]u_n \\ & - \sum_{n=k+1}^{\infty} [f(n)g(n-k) + f(n-k)g(n)]u_n - u_k \sum_{n=1}^{k-1} f(n)g(k-n) \\ & + u_k^2 [f(k)g(0) + f(0)g(k)]. \end{aligned}$$

(b) *If f is even and g is odd then*

$$\Big(\sum_{n=1}^{\infty} u_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n) \cos n\theta\Big) = \frac{1}{2}\sum_{k=1}^{\infty} c_k \sin k\theta,$$

where for $k \in \mathbb{N}$,

$$\begin{aligned} c_k = {} & u_k \sum_{n=1}^{\infty} [f(n+k)g(n) - f(n)g(n+k)]\, u_n + u_k \sum_{n=1}^{\infty} [f(n-k)g(n) - f(n)g(n-k)]\, u_n \\ & + \sum_{n=k+1}^{\infty} [f(n-k)g(n) - f(n)g(n-k)]\, u_n + u_k \sum_{n=1}^{k-1} f(n)g(k-n) - u_k^2 f(0)g(k). \end{aligned}$$

(c) *If f and g are both odd then*

$$\Big(\sum_{n=1}^{\infty} u_n f(n) \cos n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n) \cos n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)g(n)u_n^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k \cos k\theta,$$

where for $k \in \mathbb{N}$,

$$\begin{aligned} c_k = {} & u_k \sum_{n=1}^{\infty} [f(n+k)g(n) + f(n)g(n+k)]u_n + u_k \sum_{n=1}^{\infty} [f(n)g(k-n) + f(k-n)g(n)]\, u_n \\ & + \sum_{n=k+1}^{\infty} [f(n)g(k-n) + f(k-n)g(n)]\, u_n + u_k \sum_{n=1}^{k-1} f(n)g(k-n). \end{aligned}$$

Our second basic theorem is the following result.

THEOREM 2.2. *Let $f:\mathbb{Z}\to\mathbb{C}$ and $g:\mathbb{Z}\to\mathbb{C}$ satisfy* (2.1).

(a) *If f and g are both even then*

$$\Big(\sum_{n=1}^{\infty} v_n f(n)\sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n)\sin n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)g(n)v_n^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k v_k \cos k\theta,$$

where for $k\in\mathbb{N}$,

$$\begin{aligned} c_k = {} & \sum_{n=1}^{\infty}(f(n)g(n+k)+f(n+k)g(n))u_{2n} - \sum_{n=1}^{\infty}(f(n)g(n-k)+f(n-k)g(n))u_{2n} \\ & - \sum_{n=1}^{k-1} f(n)g(k-n) + u_{2k}(f(0)g(k)+f(k)g(0)). \end{aligned}$$

(b) *If f is odd and g is even then*

$$\Big(\sum_{n=1}^{\infty} v_n f(n)\cos n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n)\sin n\theta\Big) = \frac{1}{2}\sum_{k=1}^{\infty} c_k v_k \sin k\theta,$$

where for $k\in\mathbb{N}$,

$$\begin{aligned} c_k = {} & \sum_{n=1}^{\infty}(f(n)g(n+k)-f(n+k)g(n))u_{2n} + \sum_{n=1}^{\infty}(f(n)g(n-k)-f(n-k)g(n))u_{2n} \\ & + \sum_{n=1}^{k-1} f(n)g(k-n) - u_{2k}f(k)g(0). \end{aligned}$$

(c) *If f and g are both odd then*

$$\Big(\sum_{n=1}^{\infty} v_n f(n)\cos n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n)\cos n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)g(n)v_n^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k v_k \cos k\theta,$$

where for $k\in\mathbb{N}$,

$$\begin{aligned} c_k = {} & \sum_{n=1}^{\infty}(f(n)g(n+k)+f(n+k)g(n))u_{2n} \\ & - \sum_{n=1}^{\infty}(f(n)g(n-k)+f(n-k)g(n))u_{2n} + \sum_{n=1}^{k-1} f(n)g(k-n). \end{aligned}$$

Our third and final basic theorem places no restrictions on the parities of the functions f and g.

THEOREM 2.3. *Let $f:\mathbb{N}_0\to\mathbb{C}$ and $g:\mathbb{N}_0\to\mathbb{C}$ satisfy* (2.1).

(a)

$$\begin{aligned} & \Big(\sum_{n=0}^{\infty} v_{2n+1} f(n)\sin(2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} g(n)\sin(2n+1)\theta\Big) \\ & \qquad = \frac{1}{2}\sum_{n=0}^{\infty} f(n)g(n)v_{2n+1}^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k v_{2k}\cos 2k\theta, \end{aligned}$$

where for $k \in \mathbb{N}$,

$$c_k = \sum_{n=0}^{\infty} [f(n)g(n+k) + f(n+k)g(n)] u_{4n+2} - \sum_{n=k}^{\infty} [f(n)g(n-k) + f(n-k)g(n)] u_{4n+2}$$
$$- \sum_{n=0}^{k-1} [f(n)g(k-n-1) + f(k-n-1)g(n)] u_{4n+2} - \sum_{n=0}^{k-1} f(n)g(k-n-1).$$

(b) $$\Big(\sum_{n=0}^{\infty} v_{2n+1} f(n) \cos(2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} g(n) \sin(2n+1)\theta\Big) = \frac{1}{2}\sum_{k=1}^{\infty} c_k v_{2k} \sin 2k\theta,$$

where for $k \in \mathbb{N}$,

$$c_k = \sum_{n=0}^{\infty} [f(n)g(n+k) - f(n+k)g(n)] u_{4n+2} + \sum_{n=k}^{\infty} [f(n)g(n-k) - f(n-k)g(n)] u_{4n+2}$$
$$+ \sum_{n=0}^{k-1} [f(n)g(k-n-1) + f(k-n-1)g(n)] u_{4n+2} + \sum_{n=0}^{k-1} f(n)g(k-n-1).$$

(c) $$\Big(\sum_{n=0}^{\infty} v_{2n+1} f(n) \cos(2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} g(n) \cos(2n+1)\theta\Big)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} f(n)g(n) v_{2n+1}^2 + \frac{1}{2}\sum_{k=1}^{\infty} c_k v_{2k} \cos 2k\theta,$$

where for $k \in \mathbb{N}$,

$$c_k = \sum_{n=0}^{\infty} [f(n)g(n+k) + f(n+k)g(n)] u_{4n+2} - \sum_{n=k}^{\infty} [f(n)g(n-k) + f(n-k)g(n)] u_{4n+2}$$
$$+ \sum_{n=0}^{k-1} [f(n)g(k-n-1) + f(k-n-1)g(n)] u_{4n+2} + \sum_{n=0}^{k-1} f(n)g(k-n-1).$$

In all likelihood there are similar formulae for "mixed" products such as

$$\Big(\sum_{n=1}^{\infty} u_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n) \sin n\theta\Big).$$

However we have not pursued finding such formulae as we have been able to determine products such as

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$

(see Theorem 3.4), from our three basic theorems with suitable choices of the functions $f(n)$ and $g(n)$.

3. Ten products of two Lambert series

In this section we give ten important products of pairs of particular Lambert series (Theorems 3.1–3.10), all of which can be deduced from Theorems 2.1–2.3. Proofs are given in Section 9.

THEOREM 3.1.

(a) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty}(nu_n - u_n - 2u_n^2)(1-\cos n\theta) + \sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}(1-\cos k\theta).$$

(b) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} nu_n + \sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta - \frac{1}{2}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n \sin n\theta.$$

Variants of Theorem 3.1 can be found for example in [21, p. 17], [22, p. 287].

THEOREM 3.2.

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty}(2v_n^2 + nv_n - u_n^2 - u_n)(1-\cos n\theta).$$

A slightly different formulation of Theorem 3.2 is given by Ramamani [21, p. 17]. Her proof uses elliptic functions.

THEOREM 3.3.

$$\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} nv_{2n}(1-\cos 2n\theta).$$

Replacing q by $q^{1/2}$ in Theorem 3.3, we obtain an identity due to Liu [9, Theorem 10]. Liu proves his result using Cauchy's residue theorem applied to elliptic functions. We show in Section 9 that it is an immediate consequence of Theorem 2.3(a) and in Section 12 that it is a simple consequence of an arithmetic formula of Liouville (Theorem 5.3).

THEOREM 3.4.

(a) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} v_n \sum_{k=1}^{n-1}(1-\cos k\theta) - \frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2 + 3u_n - 4v_n^2 - 2nv_n - v_n)(1-\cos n\theta).$$

(b) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nu_n + \frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2 + 3u_n - 4v_n^2 - 2nv_n)\cos n\theta - \frac{1}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} v_n \sin n\theta.$$

THEOREM 3.5.

(a) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nu_n(1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty} u_{2n+1}(1-\cos(2n+1)\theta)$$
$$- \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}^2(1-\cos(2n+1)\theta) + \frac{1}{2}\sum_{n=1}^{\infty} u_{2n}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta)$$
$$+ \sum_{n=0}^{\infty} u_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta) + \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta).$$

(b) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin(2n+1)\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} nu_n + \sum_{n=0}^{\infty} nu_{2n+1} - \frac{1}{4}\sum_{n=1}^{\infty} nu_n \cos n\theta + \frac{1}{4}\sum_{n=0}^{\infty} u_{2n+1}\cos(2n+1)\theta$$
$$- \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}^2(1-\cos(2n+1)\theta) - \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}\cos 2n\theta$$
$$- \frac{1}{4}\csc\theta \sum_{n=1}^{\infty} u_{2n}\sin 2n\theta - \frac{1}{4}(\csc\theta + 2\cot\theta)\sum_{n=0}^{\infty} u_{2n+1}\sin 2n\theta.$$

Theorem 3.6.

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nv_n(1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty}(u_{2n+1}^2 + u_{2n+1} - 2v_{2n+1}^2)(1-\cos(2n+1)\theta).$$

Theorem 3.7.

(a) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} nu_{2n}(1-\cos 2n\theta) + \sum_{n=0}^{\infty} u_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta).$$

(b) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)^2 = \frac{1}{4}\sum_{n=1}^{\infty} nu_n + \frac{1}{4}\sum_{n=0}^{\infty}(2n+1)u_{2n+1}$$
$$- \frac{1}{2}\sum_{n=1}^{\infty} nu_{2n}\cos 2n\theta - \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}\cos 2n\theta$$
$$- \frac{1}{2}\cot\theta\sum_{n=0}^{\infty} u_{2n+1}\sin 2n\theta.$$

Theorem 3.8.

(a) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nv_n(1-\cos n\theta)$$
$$+ \frac{1}{4}\sum_{n=0}^{\infty}(4v_{2n+1}^2 + v_{2n+1} - 2u_{2n+1}^2 - 2u_{2n+1})(1-\cos(2n+1)\theta)$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{2n}(1-\cos k\theta).$$

(b) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)v_{2n+1} - \frac{1}{2}\sum_{n=0}^{\infty}(u_{2n+1}^2 + u_{2n+1} - 2v_{2n+1}^2)(1-\cos(2n+1)\theta)$$
$$+ \frac{1}{4}\sum_{n=1}^{\infty} nu_n(1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1}\cos(2n+1)\theta$$
$$- \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1}\cos 2n\theta - \frac{1}{4}\cot\frac{\theta}{2}\sum_{n=0}^{\infty} v_{2n+1}\sin 2n\theta.$$

Theorem 3.9.

(a) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nv_n(1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty}(u_{2n+1} + u_{2n+1}^2)(1-\cos(2n+1)\theta)$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta) + \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta).$$

(b) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} nv_n - \frac{1}{4}\sum_{n=0}^{\infty}(u_{2n+1}^2 + u_{2n+1})(1-\cos(2n+1)\theta) - \frac{1}{4}\sum_{n=1}^{\infty} nv_n\cos n\theta$$
$$- \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1}\cos 2n\theta - \frac{1}{4}\cot\theta\sum_{n=0}^{\infty} v_{2n+1}\sin 2n\theta - \frac{1}{4}\csc\theta\sum_{n=1}^{\infty} v_{2n}\sin 2n\theta.$$

Theorem 3.10.

(a) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} nv_{2n}(1-\cos 2n\theta) + \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta).$$

(b) $$\Big(\sum_{n=0}^{\infty} u_{2n+1}\sin(2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} nv_n - \frac{1}{2}\sum_{n=1}^{\infty} nv_{2n}\cos 2n\theta - \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1}\cos 2n\theta - \frac{1}{4}\cot\theta\sum_{n=0}^{\infty} v_{2n+1}\sin 2n\theta.$$

There are many other interesting products of pairs of Lambert series that could be considered. For example the two products

(3.1) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n \cos n\theta\Big) = \frac{1}{2}\sum_{n=1}^{\infty}(n-1)u_n\sin n\theta + \sum_{n=1}^{\infty} u_n\sin n\theta\sum_{k=1}^{n-1} u_k$$

and

$$\left(\sum_{n=1}^{\infty} u_n \cos n\theta\right)^2 = \frac{1}{2}\sum_{n=1}^{\infty} u_n^2 + 2\sum_{n=1}^{\infty} u_n \cos n\theta \sum_{n=1}^{n-1} u_n - \sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} \cos k\theta + \sum_{n=1}^{\infty} u_n^2 \cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty}(n-1)u_n \cos n\theta \tag{3.2}$$

can be deduced from our three basic theorems. For the former take

$$f(n) = 1, \quad g(n) = \mathrm{sgn}(n), \qquad n \in \mathbb{Z},$$

in Theorem 2.1(b), and for the latter take

$$f(n) = g(n) = \mathrm{sgn}(n), \qquad n \in \mathbb{Z},$$

in Theorem 2.1(c). Equating coefficients of θ and θ^3 in the former, we obtain the identities

$$\left(\sum_{n=1}^{\infty} u_n\right)\left(\sum_{n=1}^{\infty} nu_n\right) = \frac{1}{2}\sum_{n=1}^{\infty}(n^2-n)u_n + \sum_{n=1}^{\infty} nu_n \sum_{k=1}^{n-1} u_k$$

and

$$\left(\sum_{n=1}^{\infty} u_n\right)\left(\sum_{n=1}^{\infty} n^3u_n\right) + 3\left(\sum_{n=1}^{\infty} nu_n\right)\left(\sum_{n=1}^{\infty} n^2u_n\right) = \frac{1}{2}\sum_{n=1}^{\infty}(n^4-n^3)u_n + \sum_{n=1}^{\infty} n^3u_n \sum_{k=1}^{n-1} u_k,$$

respectively. Taking $\theta = 0$ in the latter, we obtain the identity

$$\left(\sum_{n=1}^{\infty} u_n\right)^2 = \sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}(1+2u_k).$$

4. Third and fourth powers of some Lambert series

In this section we give formulae for the cubes and fourth powers of the Lambert series

$$\sum_{n=1}^{\infty} u_n \sin n\theta, \qquad \sum_{n=1}^{\infty} v_n \sin n\theta, \qquad \sum_{n=0}^{\infty} v_{2n+1} \sin{(2n+1)\theta}$$

(see Theorems 4.1–4.6). See also the comment by Ramamani [21, p. 17]. All of these results are obtained by repeated application of Theorems 2.1, 2.2 and 2.3. The proofs are given in Section 10.

THEOREM 4.1.

(a)
$$\left(\sum_{n=1}^{\infty} u_n \sin n\theta\right)^3 = \frac{3}{2}\left(\sum_{n=1}^{\infty} nu_n\right)\left(\sum_{n=1}^{\infty} u_n \sin n\theta\right) - \frac{3}{2}\sum_{n=1}^{\infty} u_n^3 \sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty}(n-2)u_n^2 \sin n\theta - \frac{1}{8}\sum_{n=1}^{\infty}(n^2-3n+2)u_n \sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k\sin k\theta + \frac{3}{2}\sum_{n=1}^{\infty} u_n^2 \sum_{k=1}^{n-1} \sin k\theta - \frac{3}{2}\sum_{n=1}^{\infty}(n-1)u_n \sum_{k=1}^{n-1} \sin k\theta.$$

(b) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^3 = \frac{3}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big) - \frac{13}{16}\sum_{n=1}^{\infty} u_n \sin n\theta$$
$$+ \frac{3}{4}\sum_{n=1}^{\infty} nu_n \sin n\theta - \frac{9}{4}\sum_{n=1}^{\infty} u_n^2 \sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty} nu_n^2 \sin n\theta$$
$$- \frac{1}{8}\sum_{n=1}^{\infty} n^2 u_n \sin n\theta - \frac{3}{2}\sum_{n=1}^{\infty} u_n^3 \sin n\theta$$
$$- \frac{3}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}\big(1 - \frac{n}{2} + u_n\big)u_n \cos n\theta + \frac{3}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} u_n \sin n\theta.$$

Theorem 4.1(a) is due to McAfee and Williams [12].

THEOREM 4.2.

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^3 = \frac{3}{2}\Big(\sum_{n=1}^{\infty} v_n^2\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) - \frac{3}{2}\sum_{n=1}^{\infty} v_n^3 \sin n\theta$$
$$- \frac{1}{8}\sum_{n=1}^{\infty}(n^2 - 3n + 2)v_n \sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty} nv_n u_{2n} \sin n\theta.$$

Theorem 4.2 is due to Ramamani [21, p. 121].

THEOREM 4.3.

$$\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)^3 = \frac{3}{4}\Big(\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)$$
$$- \frac{1}{8}\sum_{n=0}^{\infty}(n^2 + n)v_{2n+1} \sin(2n+1)\theta.$$

Theorem 4.3 is a simpler version of a theorem of Liu [9, Theorem 11, p. 148].

THEOREM 4.4.

(a) $$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^4 = \frac{1}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty}(-3 + 2n - 6u_n)u_n(1 - \cos n\theta)\Big)$$
$$+ 3\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}(1 - \cos k\theta)\Big) - \frac{1}{4}\sum_{n=1}^{\infty}(7 - 12n + 5n^2)u_n \sum_{k=1}^{n-1}(1 - \cos k\theta)$$
$$+ \frac{3}{2}\sum_{n=1}^{\infty}(2n - 3)u_n^2 \sum_{k=1}^{n-1}(1 - \cos k\theta) + \frac{1}{4}\sum_{n=1}^{\infty}(5n - 6)u_n \sum_{k=1}^{n-1} k(1 - \cos k\theta)$$
$$- \frac{1}{4}\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k^2(1 - \cos k\theta) - \frac{3}{2}\sum_{n=1}^{\infty} u_n^2 \sum_{k=1}^{n-1} k(1 - \cos k\theta)$$
$$- 3\sum_{n=1}^{\infty} u_n^3 \sum_{k=1}^{n-1}(1 - \cos k\theta) + \frac{1}{48}\sum_{n=1}^{\infty}(6 - 11n + 6n^2 - n^3)u_n(1 - \cos n\theta)$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}(7-6n+n^2)u_n^2(1-\cos n\theta)$$

$$+\frac{3}{2}\sum_{n=1}^{\infty}(3-n)u_n^3(1-\cos n\theta)+3\sum_{n=1}^{\infty}u_n^4(1-\cos n\theta).$$

(b)
$$\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)^4=\frac{1}{16}\sum_{n=1}^{\infty}(3n^3-4n^2)u_n-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n^2$$

$$+\frac{1}{48}\sum_{n=1}^{\infty}(n^3-12n^2+38n-30)u_n\cos n\theta-\frac{1}{8}\sum_{n=1}^{\infty}(2n^2-18n+29)u_n^2\cos n\theta$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}(3n-12)u_n^3\cos n\theta-3\sum_{n=1}^{\infty}u_n^4\cos n\theta$$

$$+\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}(3u_n^2+3u_n-nu_n)\cos n\theta\Big)$$

$$+\frac{1}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}(2n^2-12n+13)u_n\sin n\theta$$

$$-\frac{3}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}(n-3)u_n^2\sin n\theta+\frac{3}{2}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}u_n^3\sin n\theta$$

$$-\frac{3}{2}\cot\frac{\theta}{2}\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)-\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n$$

$$-\frac{3}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}(n-2)u_n\cos n\theta+\frac{3}{8}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}u_n^2\cos n\theta-\frac{1}{16}\cot^3\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta.$$

Theorem 4.5.

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}v_n\sin n\theta\Big)^4&=\frac{1}{8}\sum_{n=1}^{\infty}\big(3u_n^4+6u_n^3+(4+n^2)u_n^2+(1+n^2)u_n\big)(1-\cos n\theta)\\&\quad-\frac{1}{48}\sum_{n=1}^{\infty}(144v_n^4+72nv_n^3+(30+12n^2)v_n^2+(11+n^3)v_n)(1-\cos n\theta)\\&\quad-\frac{1}{2}\left(\sum_{n=1}^{\infty}v_n^2\right)\sum_{n=1}^{\infty}(3u_n^2+3u_n-2nv_n-6v_n^2)(1-\cos n\theta).\end{aligned}$$

Theorem 4.6.

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}v_{2n+1}\sin(2n+1)\theta\Big)^4&=\frac{1}{2}\Big(\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big)\Big(\sum_{n=1}^{\infty}nv_{2n}(1-\cos 2n\theta)\Big)\\&\quad-\frac{1}{48}\sum_{n=1}^{\infty}(n^3-n)v_{2n}(1-\cos 2n\theta).\end{aligned}$$

Theorems 4.4–4.6 appear to be new.

5. Liouville's arithmetic identities

In 1858 Liouville [8] stated but did not prove the following three arithmetic identities.

THEOREM 5.1 ([8, article 5, p. 275]). *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (F(a-b)-F(a+b)) = F(0)(\sigma(n)-d(n)) + \sum_{d\mid n} F(d) - \sum_{d\mid n} dF(d) + 2\sum_{d\mid n}\frac{n}{d}F(d) - 2\sum_{d\mid n}\sum_{1\le k\le d} F(k).$$

Proofs of Theorem 5.1 have been given by McAfee [11], Meissner [15], Pepin [19, p. 93] and Piuma [20].

THEOREM 5.2 ([8, article 1, p. 144, and article 2, p. 194]). *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x,y \text{ odd}}} (F(a-b)-F(a+b)) = F(0)\sigma(n/2) - \sum_{d\mid n} dF(d) + \sum_{d\mid n}\frac{n}{d}F(d) + \sum_{d\mid n/2} dF(d) - \sum_{d\mid n/2}\frac{n}{d}F(d).$$

Proofs of Theorem 5.2 have been given by McAfee [11] and Pepin [18, p. 159].

THEOREM 5.3 ([8, article 4, p. 242]). *Let $n \in \mathbb{N}$ be even. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a,b,x,y \text{ odd}}} (F(a-b)-F(a+b)) = F(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}\sigma(n/2)\right) - \frac{1}{2}\sum_{d\mid n} dF(d) + \frac{1}{2}\sum_{d\mid n/2} dF(d).$$

Proofs of Theorem 5.3 have been given by Baskakov [1, p. 344], Bugaev [2, p. 9], Deltour [4, p. 123], Humbert [7], Mathews [10], McAfee [11], Pepin [19, p. 94] and Smith [25, pp. 346–348], [26, Vol. 1, p. 348].

The following seven theorems were not stated by Liouville but can be proved arithmetically by the method given in McAfee [11]. It is convenient in places to abbreviate $a \equiv b \pmod{m}$ by $a \equiv b\ (m)$.

THEOREM 5.4. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x \text{ odd}}} (F(a-b)-F(a+b)) = F(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}d(n) + \frac{1}{2}d(n/2)\right) + \frac{1}{2}\sum_{d\mid n} F(d) - \sum_{d\mid n} dF(d) + \frac{3}{2}\sum_{d\mid n}\frac{n}{d}F(d) - \frac{1}{2}\sum_{d\mid n/2} F(d) + \sum_{d\mid n/2} dF(d) - \sum_{d\mid n/2}\frac{n}{d}F(d) - \sum_{d\mid n}\sum_{k\le d} F(k) + \sum_{d\mid n/2}\sum_{k\le d} F(k).$$

THEOREM 5.5. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a \text{ odd}}} (F(a-b) - F(a+b)) = F(0)(\sigma(n) - \sigma(n/2) - d(n) + d(n/2)) + \frac{1}{2} \sum_{\substack{d|n \\ d\equiv 1\,(2)}} F(d) - \frac{1}{2}\sum_{d|n} dF(d) + \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \frac{n}{d} F(d) - \sum_{d|n} \sum_{\substack{k\le d \\ k\equiv 1\,(2)}} F(k) - 2 \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k).$$

THEOREM 5.6. *Let $n \in \mathbb{N}$. Define $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$, m odd, by $n = 2^r m$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{ax+by=n \\ a,x,y \text{ odd}}} (F(a-b) - F(a+b)) = \begin{cases} 2^{r-1} \sum_{d|m} d(F(0) - F(2^r d)), & n \text{ even}, \\ \frac{1}{2} \sum_{d|n} \left(\frac{n}{d} - d\right) F(d), & n \text{ odd}. \end{cases}$$

THEOREM 5.7. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a,b \text{ odd}}} (F(a-b) - F(a+b)) = F(0)(\sigma(n) - \sigma(n/2) - d(n) + d(n/2)) - \frac{1}{2} \sum_{\substack{d|n \\ d\equiv 0\,(2)}} dF(d) - 2 \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k).$$

THEOREM 5.8. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\begin{aligned} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a,x \text{ odd}}} (F(a-b) - F(a+b)) &= F(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}\sigma(n/2) - \frac{1}{2}d(n) + d(n/2) - \frac{1}{2}d(n/4)\right) \\ &+ \frac{1}{2} \sum_{\substack{d|n \\ d\equiv 1(2)}} F(d) - \frac{1}{2}\sum_{d|n} dF(d) + \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \frac{n}{d} F(d) \\ &- \frac{1}{2} \sum_{\substack{d|n/2 \\ d\equiv 1(2)}} F(d) + \frac{1}{2} \sum_{d|n/2} dF(d) - \sum_{\substack{d|n/2 \\ d\equiv 1\,(2)}} \frac{n}{d} F(d) \\ &- \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \sum_{k\le d} F(k) + \sum_{\substack{d|n/2 \\ d\equiv 1(2)}} \sum_{k\le d} F(k). \end{aligned}$$

THEOREM 5.9. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a,y \text{ odd}}} (F(a-b) - F(a+b)) = F(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}\sigma(n/2) - \frac{1}{2}d(n) + d(n/2) - \frac{1}{2}d(n/4)\right) - \frac{1}{2}\sum_{d|n} dF(d) + \frac{1}{2}\sum_{d|n/2} dF(d) + \frac{1}{2}\sum_{\substack{d|n \\ d\equiv 1\,(2)}} \frac{n}{d}F(d) - \sum_{\substack{d|n \\ d\equiv 0\,(2)}} \sum_{\substack{k\le d \\ k\equiv 1\,(2)}} F(k) - \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k) + \sum_{\substack{d|n/2 \\ d\equiv 0\,(2)}} \sum_{\substack{k\le d \\ k\equiv 1\,(2)}} F(k) + \sum_{\substack{d|n/2 \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k).$$

THEOREM 5.10. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a,b,x \text{ odd}}} (F(a-b) - F(a+b)) = F(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}\sigma(n/2) - \frac{1}{2}d(n) + d(n/2) - \frac{1}{2}d(n/4)\right) - \frac{1}{2}\sum_{\substack{d|n \\ d\equiv 0\,(2)}} dF(d) + \frac{1}{2}\sum_{\substack{d|n/2 \\ d\equiv 0\,(2)}} dF(d) - \sum_{\substack{d|n \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k) + \sum_{\substack{d|n/2 \\ d\equiv 1\,(2)}} \sum_{\substack{k\le d \\ k\equiv 0\,(2)}} F(k).$$

We close this section by noting that other products of Lambert series also give rise to Liouville type identities. For example it can be shown (although we will not do so) that the identity (3.1) is equivalent to the following arithmetic identity:

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (F(a+b) + F(a-b)) - 2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a<b}} F(b) = \sum_{d|n} (d-1)F(d).$$

Taking $F(x) = x$ we deduce that

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a\le b}} a = \frac{1}{2}(\sigma_2(n) - \sigma(n)).$$

This result can also be deduced from Theorem 5.1 by taking $F(x) = |x|$.

In this paper we show in Section 12 the suprising result that the arithmetic identity in Theorem 5.m ($m = 1, \ldots, 10$) is equivalent to the Lambert series given in Theorem 3.m

$(m = 1, \ldots, 10)$. Details are only provided of the equivalence of Theorems 3.1 and 5.1 as the other equivalences can be proved similarly.

6. Further arithmetic identities of Liouville type

We begin by giving three arithmetic identities of Liouville type, which involve the equation $ax + by + cz = n$ instead of $ax + by = n$.

THEOREM 6.1. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n}} (F(a+b+c) - F(a-b+c) - F(a+b-c) + F(a-b-c))$$
$$= \sum_{d|n}\left(\frac{d^2-3d+2}{2} + 3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right)F(d)$$
$$+ 3\sum_{d|n}\sum_{1\le k<d}\left(2d - \frac{2n}{d} - k\right)F(k) - 6\sum_{\substack{(n_1,n_2)\in\mathbb{N}^2 \\ n=n_1+n_2}} \sigma(n_1)\sum_{d|n_2}F(d).$$

Theorem 6.1 was first stated and proved by McAfee [11]. Her proof is completely arithmetic. Theorem 6.1 has been used to give a completely arithmetic proof of the formula for the number of representations of a positive integer as the sum of six squares [11], [14].

THEOREM 6.2. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n \\ x,y,z \text{ odd}}} (F(a+b+c) - F(a-b+c) - F(a+b-c) + F(a-b-c))$$
$$= \frac{1}{4}\sum_{\substack{d|n \\ n/d \text{ odd}}}\left(2d^2 - 6n + 3\left(\frac{n}{d}\right)^2 + 1\right)F(d) - 6\sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ a\equiv n \pmod 2 \\ x \text{ odd}}} \sigma\left(\frac{n-ax}{2}\right)F(a).$$

Theorem 6.2 was not given by McAfee [11] and appears to be new.

THEOREM 6.3. *Let $n \in \mathbb{N}$ be odd. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) - F(a-b+c) - F(a+b-c) + F(a-b-c))$$
$$= \frac{1}{8}\sum_{d|n}(d^2-1)F(d) - 3\sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a),$$

where $o(k)$ denotes the odd part of $k \in \mathbb{N}$.

Theorem 6.3 was first stated by Liouville [8, sixth article, p. 331]. Nazimov [16, p. 110] indicates how a proof can be given using elliptic functions but does not give the details. The first proof was given by McAfee [11], see also McAfee and Williams [13]. The proof given in [11], [13] is completely arithmetic in nature.

In Section 13 we show that Theorems 6.1, 6.2 and 6.3 are equivalent to the Lambert series for

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^3, \quad \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^3, \quad \Big(\sum_{n=1}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)^3,$$

given in Theorems 4.1, 4.2 and 4.3 respectively. Details are just given for the equivalence of Theorems 4.2 and 6.2 as the other two equivalences can be proved similarly.

In Section 14 we show that the following three new arithmetic identities (Theorems 6.4–6.6) are equivalent to the Lambert series for

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^4, \quad \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^4, \quad \Big(\sum_{n=1}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big)^4,$$

given in Theorems 4.4, 4.5 and 4.6. Details are just given for the equivalence of Theorems 4.4 and 6.4.

THEOREM 6.4. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be even. Then*

$$\begin{aligned}
\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n}} & (F(a+b+c+d) - F(a-b+c+d) - F(a+b-c+d) \\
& - F(a+b+c-d) + F(a-b-c+d) + F(a-b+c-d) \\
& + F(a+b-c-d) - F(a-b-c-d)) \\
= F(0)\Big(& d(n) + \frac{1}{2}\sigma(n) + 11\sigma_2(n) + \frac{3}{2}\sigma_3(n) - 12nd(n) - 2n\sigma(n) - 12 \sum_{\substack{(r,s)\in\mathbb{N}^2 \\ r+s=n}} \sigma(r)d(s)\Big) \\
& + \frac{1}{6}\sum_{d|n}\Big(6 + 11d + 6d^2 + d^3 - 24\frac{n}{d} + 36\left(\frac{n}{d}\right)^2 - 24\Big(\frac{n}{d}\Big)^3 - 12nd - 36n + 36\frac{n^2}{d}\Big)F(d) \\
& + 2\sum_{d|n}\Big(1 - 12n + 5d^2 + 6\Big(\frac{n}{d}\Big)^2\Big)\sum_{k=1}^{d-1}F(k) - 2\sum_{d|n}\Big(5d - 6\frac{n}{d}\Big)\sum_{k=1}^{d-1}kF(k) \\
& + 2\sum_{d|n}\sum_{k=1}^{d-1}k^2F(k) - 4\sum_{\substack{(r,s)\in\mathbb{N}^2 \\ r+s=n}}\sigma(r)\sum_{d|s}\Big(2d - 6\frac{s}{d} + 3\Big)F(d) - 24\sum_{\substack{(r,s)\in\mathbb{N}^2 \\ r+s=n}}\sigma(r)\sum_{d|s}\sum_{k=1}^{d-1}F(k).
\end{aligned}$$

For $k, n \in \mathbb{N}$ we set

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \sigma_k^*(n) = \sum_{\substack{d|n \\ n/d\ \mathrm{odd}}} d^k.$$

Recall that $o(n)$ denotes the odd part of $n \in \mathbb{N}$ and $d(n)$ denotes the number of (positive) divisors of $n \in \mathbb{N}$. If $n = 2^\alpha N$, where $N = o(n)$ and $\alpha \in \mathbb{N}_0$, then

$$\sigma_k^*(n) = \sum_{\substack{d|n \\ n/d\,\text{odd}}} d^k = \sum_{e|N} (2^\alpha e)^k = 2^{\alpha k}\sigma_k(N)$$

and

$$\sum_{\substack{d|n \\ n/d\,\text{odd}}} \left(\frac{n}{d}\right)^k = \sum_{e|N}\left(\frac{n}{2^\alpha e}\right)^k = \sum_{e|N}\left(\frac{N}{e}\right)^k = \sigma_k(N).$$

Theorem 6.5. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be even. Then*

$$\begin{aligned}
\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n \\ x,y,z,w\ odd}} & (F(a+b+c+d) - F(a-b+c+d) - F(a+b-c+d) \\
& - F(a+b+c-d) + F(a-b-c+d) + F(a-b+c-d) \\
& + F(a+b-c-d) - F(a-b-c-d)) \\
= F(0)\Big(& -\frac{1}{6}\,\sigma_3^*(n) - 3\sigma_3(o(n)) + 2\sigma^*(n) - 6n\sigma^*(n) + 9n\sigma(o(n)) - 3\sigma(o(n)) \\
& + 4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x\ odd \\ a\equiv n \pmod 2}} (2a-3x)\sigma\left(\frac{n-ax}{2}\right)\Big) \\
& + \frac{1}{6} \sum_{\substack{d|n \\ n/d\ odd}} \left(d^3 - 6nd + 2d + 9\,\frac{n^2}{d} - 3\left(\frac{n}{d}\right) - 3\left(\frac{n}{d}\right)^3\right) F(d) \\
& - 4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x\ odd \\ a\equiv n \pmod 2}} (2a-3x)\sigma\left(\frac{n-ax}{2}\right) F(a).
\end{aligned}$$

Theorem 6.6. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be even. Then*

$$\begin{aligned}
\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w\ odd}} & (F(a+b+c+d) - F(a-b+c+d) - F(a+b-c+d) \\
& - F(a+b+c-d) + F(a-b-c+d) + F(a-b+c-d) \\
& + F(a+b-c-d) - F(a-b-c-d)) \\
& = F(0)\Big(-\frac{1}{6}\,\sigma_3^*(n) - \sigma^*(n) + 4 \sum_{\substack{ax<n \\ x\ odd}} a\sigma(o(n-ax))\Big) \\
& \quad + \frac{1}{6} \sum_{\substack{d|n \\ n/d\ odd}} (d^3-d)F(2d) - 4 \sum_{\substack{ax<n \\ x\ odd}} a\sigma(o(n-ax))F(2a).
\end{aligned}$$

It may be possible to prove Theorems 6.4–6.6 by the elementary arithmetic methods used by McAfee in [11]. However, it seems that the calculations would be extremely complicated.

7. Basic properties of u_n and v_n, and some trigonometric identities

In this section we state without proof the basic properties of u_n and v_n that we shall use (often without comment) in the proofs in the following sections. All can be proved in a straightforward manner. Let $k, n \in \mathbb{N}$. Then:

(7.1) $q^{-n} u_n = 1 + u_n.$

(7.2) $q^{-n} v_n = 1 + q^n v_n.$

(7.3) $u_{2n} = u_n - v_n.$

(7.4) $2u_n u_{2n} = u_n^2 - u_n + v_n.$

(7.5) $2u_n v_n = u_n^2 + u_n - v_n.$

(7.6) $2u_{2n} v_{2n} = v_n^2 - v_{2n}.$

(7.7) $v_n^2 = u_{2n}^2 + u_{2n}.$

(7.8) $u_n u_{n+k} = u_k(u_n - u_{n+k}) - u_{n+k}.$

(7.9) $u_n u_{n+k} = q^n u_k(u_n - u_{n+k}).$

(7.10) $u_n u_{k-n} = u_k(1 + u_n + u_{k-n}), \quad n < k.$

(7.11) $v_n v_{n+k} = v_k(q^n v_n - q^{n+k} v_{n+k}).$

(7.12) $v_n v_{k-n} = v_k(1 + q^n v_n + q^{k-n} v_{k-n}), \quad n < k.$

(7.13) $v_{2n+1} v_{2(n+k)+1} = v_{2k}(q^{2n+1} v_{2n+1} - q^{2(n+k)+1} v_{2(n+k)+1}).$

(7.14) $v_{2n+1} v_{2(k-n)+1} = v_{2k+2}(1 + q^{2n+1} v_{2n+1} + q^{2(k-n)+1} v_{2(k-n)+1}), \quad n < k.$

(7.15) $$\sum_{n=1}^{k-1} u_n u_{k-n} = u_k\Big(k - 1 + 2\sum_{n=1}^{k-1} u_n\Big), \quad n < k.$$

(7.16) $$\sum_{n=1}^{\infty} u_n u_{n+k} = u_k \sum_{n=1}^{k} u_n - \sum_{n=k+1}^{\infty} u_n.$$

(7.17) $$\sum_{n=1}^{k-1} v_n v_{k-n} = v_k\Big(k - 1 + 2\sum_{n=1}^{k-1} u_{2n}\Big), \quad n < k.$$

(7.18) $$\sum_{n=1}^{\infty} v_n v_{n+k} = v_k \sum_{n=1}^{k} u_{2n}.$$

(7.19) $$\sum_{n=1}^{\infty} v_n^2 = \sum_{n=1}^{\infty} n u_{2n}.$$

(7.20) $$\sum_{n=0}^{\infty} u_{4n+2}^2 = \sum_{n=1}^{\infty} (n-1) v_n = \sum_{n=1}^{\infty} v_n^2 + 2\sum_{n=1}^{\infty} u_{2n} v_n.$$

(7.21) $$\sum_{n=1}^{\infty} u_n^2 = \sum_{n=1}^{\infty} (n-1) u_n.$$

(7.22) $$\sum_{n=1}^{\infty} v_{2n+1}v_{2(n+k)+1} = v_{2k}\sum_{n=1}^{k} u_{4n+2}.$$

(7.23) $$\sum_{n=1}^{k-1} v_{2n+1}v_{2(k-n)+1} = v_{2k+2}\Big(k-1+2\sum_{n=1}^{k-1} u_{4n+2}\Big).$$

(7.24) $$\sum_{n=0}^{\infty} v_{2n+1}^2 = \sum_{n=1}^{\infty} nv_{2n}.$$

(7.25) $$\sum_{n=1}^{\infty} nv_{2n+1} = \sum_{n=1}^{\infty} u_{4n+2}v_{2n+1}.$$

Next we state some simple trigonometric identities, which will be useful in what follows:

(7.26) $$\sum_{k=1}^{n-1} \sin k\theta = -\frac{1}{2}\sin n\theta + \frac{1}{2}\,(1-\cos n\theta)\cot\frac{\theta}{2}.$$

(7.27) $$\sum_{k=1}^{n-1} \cos k\theta = -\frac{1}{2}(1+\cos n\theta) + \frac{1}{2}\,\sin n\theta\cot\frac{\theta}{2}.$$

(7.28) $$\sum_{k=0}^{n-1} \cos(2k+1)\theta = \frac{1}{2}\csc\theta\sin 2n\theta.$$

(7.29) $$\sum_{k=1}^{n-1} k\sin k\theta = \frac{1}{4}\,(1-2n)\sin n\theta - \frac{n}{2}\cos n\theta\cot\frac{\theta}{2} + \frac{1}{4}\sin n\theta\cot^2\frac{\theta}{2}.$$

(7.30) $$\sum_{k=1}^{n-1} k\cos k\theta = \frac{1}{4}\,((1-2n)\cos n\theta - 1) + \frac{n}{2}\sin n\theta\cot\frac{\theta}{2} + \frac{1}{4}\,(\cos n\theta - 1)\cot^2\frac{\theta}{2}.$$

(7.31) $$\sum_{k=1}^{n-1} k^2\sin k\theta = \frac{1}{2}\,n(1-n)\sin n\theta + \frac{1}{4}\,((1-2n^2)\cos n\theta - 1)\cot\frac{\theta}{2} + \frac{n}{2}\sin n\theta\cot^2\frac{\theta}{2} + \frac{1}{4}\,(\cos n\theta - 1)\cot^3\frac{\theta}{2}.$$

(7.32) $$\sum_{k=1}^{n-1} k^2\cos k\theta = \frac{1}{2}\,n(1-n)\cos n\theta + \frac{1}{4}\,(2n^2-1)\sin n\theta\cot\frac{\theta}{2} + \frac{n}{2}\cos n\theta\cot^2\frac{\theta}{2} - \frac{1}{4}\sin n\theta\cot^3\frac{\theta}{2}.$$

From these we immediately obtain the following fourteen useful identities:

(7.33) $$\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n\cos n\theta = \cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n - \sum_{n=1}^{\infty} u_n\sin n\theta - 2\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1}\sin k\theta.$$

(7.34) $$\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n^2\cos n\theta = \cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n^2 - \sum_{n=1}^{\infty} u_n^2\sin n\theta - 2\sum_{n=1}^{\infty} u_n^2\sum_{k=1}^{n-1}\sin k\theta.$$

(7.35) $$\cot\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n\cos n\theta = \cot\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n - \sum_{n=1}^{\infty} nu_n\sin n\theta - 2\sum_{n=1}^{\infty} nu_n\sum_{k=1}^{n-1}\sin k\theta.$$

$$(7.36)\quad \cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta = 2\cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n\cos n\theta + 2\sum_{n=1}^{\infty}nu_n\sin n\theta - \sum_{n=1}^{\infty}u_n\sin n\theta + 4\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}\sin k\theta.$$

$$(7.37)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta = \sum_{n=1}^{\infty}u_n + \sum_{n=1}^{\infty}u_n\cos n\theta + 2\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.38)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n\sin n\theta = \sum_{n=1}^{\infty}nu_n + \sum_{n=1}^{\infty}nu_n\cos n\theta + 2\sum_{n=1}^{\infty}nu_n\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.39)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}n^2u_n\sin n\theta = \sum_{n=1}^{\infty}n^2u_n + \sum_{n=1}^{\infty}n^2u_n\cos n\theta + 2\sum_{n=1}^{\infty}n^2u_n\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.40)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}u_n^2\sin n\theta = \sum_{n=1}^{\infty}u_n^2 + \sum_{n=1}^{\infty}u_n^2\cos n\theta + 2\sum_{n=1}^{\infty}u_n^2\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.41)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n^2\sin n\theta = \sum_{n=1}^{\infty}nu_n^2 + \sum_{n=1}^{\infty}nu_n^2\cos n\theta + 2\sum_{n=1}^{\infty}nu_n^2\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.42)\quad \cot\frac{\theta}{2}\sum_{n=1}^{\infty}u_n^3\sin n\theta = \sum_{n=1}^{\infty}u_n^3 + \sum_{n=1}^{\infty}u_n^3\cos n\theta + 2\sum_{n=1}^{\infty}u_n^3\sum_{k=1}^{n-1}\cos k\theta.$$

$$(7.43)\quad \cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\cos n\theta = \left(1+\cot^2\frac{\theta}{2}\right)\sum_{n=1}^{\infty}u_n - 2\cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n\sin n\theta + 2\sum_{n=1}^{\infty}nu_n\cos n\theta - \sum_{n=1}^{\infty}u_n\cos n\theta + 4\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}k\cos k\theta.$$

$$(7.44)\quad \cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n\cos n\theta = \left(1+\cot^2\frac{\theta}{2}\right)\sum_{n=1}^{\infty}nu_n - 2\cot\frac{\theta}{2}\sum_{n=1}^{\infty}n^2u_n\sin n\theta + 2\sum_{n=1}^{\infty}n^2u_n\cos n\theta - \sum_{n=1}^{\infty}nu_n\cos n\theta + 4\sum_{n=1}^{\infty}nu_n\sum_{k=1}^{n-1}k\cos k\theta.$$

$$(7.45)\quad \cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}u_n^2\cos n\theta = \left(1+\cot^2\frac{\theta}{2}\right)\sum_{n=1}^{\infty}u_n^2 - 2\cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n^2\sin n\theta + 2\sum_{n=1}^{\infty}nu_n^2\cos n\theta - \sum_{n=1}^{\infty}u_n^2\cos n\theta + 4\sum_{n=1}^{\infty}u_n^2\sum_{k=1}^{n-1}k\cos k\theta.$$

$$(7.46)\quad \cot^3\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta = 2\left(1+\cot^2\frac{\theta}{2}\right)\sum_{n=1}^{\infty}nu_n\cos n\theta + 2\cot\frac{\theta}{2}\sum_{n=1}^{\infty}n^2u_n\sin n\theta - \cot\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta - 2\sum_{n=1}^{\infty}n^2u_n\cos n\theta - 4\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}k^2\cos k\theta.$$

8. Proofs of Theorems 2.1–2.3

We make use of ideas of Ramanujan [23]. We have

$$\begin{aligned}\Big(\sum_{m=1}^{\infty} u_m f(m)\sin m\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n)\sin n\theta\Big)\\ &= \sum_{m,n=1}^{\infty} u_m u_n f(m)g(n)\sin m\theta \sin n\theta\\ &= \frac{1}{2}\sum_{m,n=1}^{\infty} u_m u_n f(m)g(n)[\cos(m-n)\theta - \cos(m+n)\theta] = \frac{1}{2}\sum_{k=0}^{\infty} c_k \cos k\theta,\end{aligned}$$

where

$$c_0 = \sum_{\substack{m,n=1\\ m=n}}^{\infty} u_m u_n f(m)g(n) = \sum_{n=1}^{\infty} u_n^2 f(n)g(n)$$

and c_k $(k \in \mathbb{N})$ is given by

$$\begin{aligned}c_k &= \sum_{m-n=k} u_m u_n f(m)g(n) + \sum_{m-n=-k} u_m u_n f(m)g(n) - \sum_{m+n=k} u_m u_n f(m)g(n)\\ &= \sum_{n=1}^{\infty} u_n u_{n+k} f(n+k)g(n) + \sum_{n=1}^{\infty} u_n u_{n+k} f(n)g(n+k) - \sum_{n=1}^{k-1} u_n u_{k-n} f(n)g(k-n)\\ &= \sum_{n=1}^{\infty} [f(n+k)g(n) + f(n)g(n+k)]\, u_n u_{n+k} - \sum_{n=1}^{k-1} f(k-n)g(n)u_n u_{k-n}.\end{aligned}$$

Set

$$A(k,n) = f(n+k)g(n) + f(n)g(n+k)$$

so that

$$A(k,n-k) = f(n)g(n-k) + f(n-k)g(n) \quad \text{and} \quad A(k,0) = f(k)g(0) + f(0)g(k).$$

Using (7.9) and (7.10), we obtain

$$\begin{aligned}c_k = u_k \sum_{n=1}^{\infty} A(k,n)q^n u_n - u_k \sum_{n=1}^{\infty} A(k,n)q^n u_{n+k} - u_k \sum_{n=1}^{k-1} f(k-n)g(n)\\ - u_k \sum_{n=1}^{k-1} f(k-n)g(n)u_n - u_k \sum_{n=1}^{k-1} f(k-n)g(n)u_{k-n}.\end{aligned}$$

Replacing n by $n-k$ in the second sum, and n by $k-n$ in the last sum, we obtain

$$\begin{aligned}c_k = u_k \sum_{n=1}^{\infty} A(k,n)q^n u_n - u_k q^{-k} \sum_{n=k+1}^{\infty} A(k,n-k)q^n u_n\\ - u_k \sum_{n=1}^{k-1} f(k-n)g(n) - u_k \sum_{n=1}^{k-1} f(k-n)g(n)u_n - u_k \sum_{n=1}^{k-1} f(n)g(k-n)u_n\end{aligned}$$

$$
\begin{aligned}
&= u_k \sum_{n=1}^{\infty} A(k,n) q^n u_n - (1+u_k) \sum_{n=k+1}^{\infty} A(k,n-k) q^n u_n \\
&\quad - u_k \sum_{n=1}^{k-1} A(k,n-k) u_n - u_k \sum_{n=1}^{k-1} f(k-n) g(n) \\
&= u_k \sum_{n=1}^{\infty} A(k,n) q^n u_n - \sum_{n=k+1}^{\infty} A(k,n-k) q^n u_n - u_k \sum_{n=k+1}^{\infty} A(k,n-k) q^n u_n \\
&\quad - u_k \sum_{n=1}^{k-1} A(k,n-k)(q^n u_n + q^n) - u_k \sum_{n=1}^{k-1} f(k-n) g(n).
\end{aligned}
$$

Combining the third sum and the first half of the fourth sum, we obtain (remembering that f and g are even)

$$
\begin{aligned}
c_k &= u_k \sum_{n=1}^{\infty} A(k,n) q^n u_n - \sum_{n=k+1}^{\infty} A(k,n-k) q^n u_n - u_k \sum_{n=1}^{\infty} A(k,n-k) q^n u_n \\
&\quad - u_k \sum_{n=1}^{k-1} A(k,n-k) q^n + {u_k}^2 A(k,0) q^k - u_k \sum_{n=1}^{k-1} f(n-k) g(n).
\end{aligned}
$$

Then by (7.1) we have $q^n u_n = u_n - q^n$, so

$$
\begin{aligned}
c_k &= u_k \sum_{n=1}^{\infty} A(k,n) u_n - \sum_{n=k+1}^{\infty} A(k,n-k) u_n - u_k \sum_{n=1}^{\infty} A(k,n-k) u_n \\
&\quad - u_k \sum_{n=1}^{k-1} f(n-k) g(n) + S(k),
\end{aligned}
$$

where

$$
\begin{aligned}
S(k) &= -u_k \sum_{n=1}^{\infty} A(k,n) q^n + \sum_{n=k+1}^{\infty} A(k,n-k) q^n + u_k \sum_{n=1}^{\infty} A(k,n-k) q^n \\
&\quad - u_k \sum_{n=1}^{k-1} A(k,n-k) q^n + u_k^2 A(k,0) q^k \\
&= -u_k \sum_{n=1}^{\infty} A(k,n) q^n + \sum_{n=k+1}^{\infty} A(k,n-k) q^n + u_k \sum_{n=k}^{\infty} A(k,n-k) q^n + u_k^2 A(k,0) q^k \\
&= -u_k \sum_{n=1}^{\infty} A(k,n) q^n + (1+u_k) \sum_{n=k+1}^{\infty} A(k,n-k) q^n + u_k (1+u_k) A(k,0) q^k \\
&= -\frac{1}{1-q^k} \sum_{n=1}^{\infty} A(k,n) q^{n+k} + (1+u_k) \sum_{n=k+1}^{\infty} A(k,n-k) q^n + u_k^2 A(k,0) \\
&= -\frac{1}{1-q^k} \sum_{n=k+1}^{\infty} A(k,n-k) q^n + \frac{1}{1-q^k} \sum_{n=k+1}^{\infty} A(k,n-k) q^n + u_k^2 A(k,0) \\
&= u_k^2 A(k,0).
\end{aligned}
$$

This completes the proof of part (a) of Theorem 2.1. Parts (b) and (c) of Theorem 2.1 as well as Theorems 2.2 and 2.3 can be proved in a similar manner. ∎

9. Proofs of Theorems 3.1–3.10

We just give a sketch of each of these proofs.

Proof of Theorem 3.1. Taking $\theta = 0$ in Theorem 2.1(a), we obtain

$$\sum_{k=1}^{\infty} c_k = -\sum_{k=1}^{\infty} f(k)g(k)u_k^2.$$

Thus Theorem 2.1(a) can be reformulated as

$$\Big(\sum_{n=1}^{\infty} u_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n) \sin n\theta\Big) = -\frac{1}{2}\sum_{k=1}^{\infty} c_k(1-\cos k\theta).$$

Taking $f(n) \equiv g(n) \equiv 1$ $(n \in \mathbb{Z})$ in Theorem 2.1(a), we find

$$c_k = -2\sum_{n=k+1}^{\infty} u_n - (k-1)u_k + 2u_k^2.$$

Hence

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty}(nu_n - u_n - 2u_n^2)(1-\cos n\theta) + \sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}(1-\cos k\theta),$$

which is Theorem 3.1. ∎

Proof of Theorem 3.2. Taking $\theta = 0$ in Theorem 2.2(a), we deduce that

$$\sum_{n=1}^{\infty} v_n^2 f(n)g(n) = -\sum_{n=1}^{\infty} c_n v_n.$$

Thus Theorem 2.2(a) can be rewritten as

$$\Big(\sum_{n=1}^{\infty} v_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n) \sin n\theta\Big) = -\frac{1}{2}\sum_{k=1}^{\infty} c_k v_k(1-\cos k\theta).$$

Taking $f(n) = g(n) = 1$ $(n \in \mathbb{Z})$ in Theorem 2.2(a), we find

$$c_k = -(k-1) + 2u_{2k}, \qquad k \in \mathbb{N}.$$

Hence

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty}(2v_n^2 + nv_n - u_n^2 - u_n)(1-\cos n\theta),$$

which is Theorem 3.2. ∎

Proof of Theorem 3.3. Taking $\theta = 0$ in Theorem 2.3(a), we obtain

$$\sum_{k=0}^{\infty} f(k)g(k)v_{2k+1}^2 = -\sum_{k=1}^{\infty} c_k v_{2k}.$$

Thus Theorem 2.3(a) can be rewritten as

$$\Big(\sum_{n=0}^{\infty} v_{2n+1} f(n)\sin{(2n+1)\theta}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} g(n)\sin{(2n+1)\theta}\Big) = -\frac{1}{2}\sum_{k=1}^{\infty} c_k v_{2k}(1-\cos 2k\theta).$$

Taking $f(n)\equiv g(n)\equiv 1$ ($n\in\mathbb{Z}$) in Theorem 2.3(a), we obtain

$$c_k = -k, \qquad k\in\mathbb{N}.$$

Hence

$$\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin{(2n+1)\theta}\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} n v_{2n}(1-\cos 2n\theta),$$

which is Theorem 3.3. ■

Proof of Theorem 3.4. Let

$$f(n)=1,\quad n\in\mathbb{Z},\qquad g(n)=\begin{cases}1+q^n & \text{if } n\ge 0,\\ 1+q^{-n} & \text{if } n\le -1.\end{cases}$$

Using Theorem 2.2(a), we obtain

$$c_k = -v_k^{-1}\sum_{n=k+1}^{\infty} v_n + (1-k) + v_k + 3u_{2k}, \qquad k\ge 1,$$

and thus

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n\sin n\theta\Big) &= \Big(\sum_{n=1}^{\infty} v_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}(1+q^n)v_n\sin n\theta\Big)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(v_n^2+v_nu_{2n}) + \frac{1}{2}\sum_{n=1}^{\infty}(1-n)v_n\cos n\theta\\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty}(v_n^2+3v_nu_{2n})\cos n\theta - \frac{1}{2}\sum_{n=1}^{\infty} v_n\sum_{k=1}^{n-1}\cos k\theta.\end{aligned}$$

Using the identities $u_n = v_n + u_{2n}$ and $2u_nv_n = u_n^2 + u_n - v_n$, this equation can be rewritten as

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n\sin n\theta\Big)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(v_n^2+v_nu_{2n}) + \frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2+3u_n-4v_n^2-2nv_n-v_n)\cos n\theta - \frac{1}{2}\sum_{n=1}^{\infty} v_n\sum_{k=1}^{n-1}\cos k\theta.\end{aligned}$$

Taking $\theta=0$, we obtain

$$\frac{1}{2}\sum_{n=1}^{\infty}(v_n^2+v_nu_{2n}) = -\frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2+3u_n-4v_n^2-2nv_n-v_n) + \frac{1}{2}\sum_{n=1}^{\infty}(n-1)v_n.$$

Thus

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n\sin n\theta\Big)\\ &\qquad = -\frac{1}{4}\sum_{n=1}^{\infty}(3u_n^2+3u_n-4v_n^2-2nv_n-v_n)(1-\cos n\theta) + \frac{1}{2}\sum_{n=1}^{\infty} v_n\sum_{k=1}^{n-1}(1-\cos k\theta).\end{aligned}$$ ■

Proof of Theorem 3.5. Let

$$f(n)=1,\quad n\in\mathbb{Z},\qquad g(n)=\begin{cases}1 & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Using Theorem 2.1(a), we obtain

$$\begin{aligned} c_{2k} &= -2\sum_{n=k+1}^{\infty}u_{2n}-(k-1)u_{2k}+2u_{2k}^2, \qquad k\ge 1,\\ c_{2k+1} &= -\sum_{n=2k+2}^{\infty}u_n-ku_{2k+1}+u_{2k+1}^2, \qquad k\ge 0,\end{aligned}$$

and thus

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}u_{2n}\sin 2n\theta\Big)=\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}u_n g(n)\sin n\theta\Big)\\ &\quad=\frac12\sum_{n=1}^{\infty}u_{2n}^2-\frac14\sum_{n=1}^{\infty}(nu_n-u_n-2u_n^2)\cos n\theta+\frac14\sum_{n=1}^{\infty}(2u_{2n}^2+u_{2n})\cos 2n\theta\\ &\qquad-\sum_{n=1}^{\infty}u_{2n}\sum_{k=1}^{n-1}\cos 2k\theta-\frac12\sum_{n=1}^{\infty}u_{2n}\sum_{k=0}^{n-1}\cos(2k+1)\theta-\frac12\sum_{n=1}^{\infty}u_{2n+1}\sum_{k=0}^{n-1}\cos(2k+1)\theta.\end{aligned}$$

Taking $\theta=0$, we obtain

$$\begin{aligned}\frac12\sum_{n=1}^{\infty}u_{2n}^2&=\frac14\sum_{n=1}^{\infty}(nu_n-u_n-2u_n^2)-\frac14\sum_{n=1}^{\infty}(2u_{2n}^2+u_{2n})\\ &\quad+\sum_{n=1}^{\infty}(n-1)u_{2n}+\frac12\sum_{n=1}^{\infty}nu_{2n}+\frac12\sum_{n=1}^{\infty}nu_{2n+1}.\end{aligned}$$

Thus

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}u_{2n}\sin 2n\theta\Big)&=\frac14\sum_{n=1}^{\infty}(nu_n-u_n-2u_n^2)(1-\cos n\theta)\\ &\quad-\frac14\sum_{n=1}^{\infty}(2u_{2n}^2+u_{2n})(1-\cos 2n\theta)+\sum_{n=1}^{\infty}u_{2n}\sum_{k=1}^{n-1}(1-\cos 2k\theta)\\ &\quad+\frac12\sum_{n=1}^{\infty}u_{2n}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta)+\frac12\sum_{n=1}^{\infty}u_{2n+1}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta).\end{aligned}$$

Then

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=0}^{\infty}u_{2n+1}\sin(2n+1)\theta\Big)\\ &\qquad=\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)^2-\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}u_{2n}\sin 2n\theta\Big)\end{aligned}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty} nu_n(1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty} u_{2n+1}(1-\cos(2n+1)\theta)$$
$$- \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}^2(1-\cos(2n+1)\theta) + \frac{1}{2}\sum_{n=1}^{\infty} u_{2n}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta)$$
$$+ \sum_{n=0}^{\infty} u_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta) + \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}\sum_{k=0}^{n-1}(1-\cos(2k+1)\theta). \blacksquare$$

Proof of Theorem 3.6. Let

$$f(n) = 1, \quad n \in \mathbb{Z}, \qquad g(n) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Using Theorem 2.2(a), we obtain

$$c_k = \begin{cases} -(k-2)/2 + 2u_{2k} & \text{if } k \text{ is even}, \\ -(k-1)/2 + u_{2k} & \text{if } k \text{ is odd} \end{cases}$$

and

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n}\sin 2n\theta\Big)$$
$$= \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n)\sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}^2 - \frac{1}{4}\sum_{n=1}^{\infty}(nv_n - v_n - 2u_{2n}v_n)\cos n\theta + \frac{1}{4}\sum_{n=1}^{\infty}(v_{2n} + 2u_{4n}v_{2n})\cos 2n\theta.$$

Using identities (7.3) and (7.5), we can rewrite this as

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n}\sin 2n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}^2 - \frac{1}{4}\sum_{n=1}^{\infty}(2v_n^2 + nv_n - u_n^2 - u_n)\cos n\theta + \frac{1}{4}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2)\cos 2n\theta.$$

Taking $\theta = 0$, we obtain

$$\frac{1}{2}\sum_{n=1}^{\infty} v_{2n}^2 = \frac{1}{4}\sum_{n=1}^{\infty}(2v_n^2 + nv_n - u_n^2 - u_n) - \frac{1}{4}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2).$$

Thus

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n}\sin 2n\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty}(2v_n^2 + nv_n - u_n^2 - u_n)(1-\cos n\theta) - \frac{1}{4}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2)(1-\cos 2n\theta).$$

Hence

$$
\begin{aligned}
&\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta\Big) \\
&\qquad = \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^2 - \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big) \\
&\qquad = \frac{1}{4}\sum_{n=1}^{\infty} n v_n (1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty}(u_{2n+1}^2 + u_{2n+1} - 2v_{2n+1}^2)(1-\cos(2n+1)\theta),
\end{aligned}
$$

using identities (7.3), (7.5), and Theorem 3.2. ∎

Proof of Theorem 3.7. For $n \in \mathbb{Z}$ let

$$
f(n) = g(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
$$

Then

$$
\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin(2n+1)\theta\Big)^2 = \Big(\sum_{n=1}^{\infty} u_n f(n) \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n g(n) \sin n\theta\Big).
$$

Using Theorem 2.1(a), we obtain

$$
c_{2k} = -2\sum_{n=k}^{\infty} u_{2n+1} - k u_{2k}, \quad k \geq 1, \qquad c_{2k+1} = 0, \qquad k \geq 0,
$$

and thus

$$
\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin(2n+1)\theta\Big)^2 = \frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}^2 - \frac{1}{2}\sum_{n=1}^{\infty} n u_{2n} \cos 2n\theta - \sum_{n=1}^{\infty} u_{2n+1} \sum_{k=1}^{n} \cos 2k\theta.
$$

Taking $\theta = 0$, we obtain

$$
\frac{1}{2}\sum_{n=0}^{\infty} u_{2n+1}^2 = \frac{1}{2}\sum_{n=1}^{\infty} n u_{2n} + \sum_{n=1}^{\infty} n u_{2n+1}.
$$

Thus

$$
\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin(2n+1)\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} n u_{2n}(1-\cos 2n\theta) + \sum_{n=1}^{\infty} u_{2n+1} \sum_{k=1}^{n}(1-\cos 2k\theta). \ \blacksquare
$$

Proof of Theorem 3.8. For $n \in \mathbb{Z}$ let

$$
f(n) = \begin{cases} 1+q^n & \text{if } n \geq 0, \\ 1+q^{-n} & \text{if } n \leq -1, \end{cases} \qquad g(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
$$

Then

$$
\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big) = \Big(\sum_{n=1}^{\infty}(1+q^n) v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n) \sin n\theta\Big).
$$

Using Theorem 2.2(a) and $q^n u_{2n} = v_n - q^n$, we obtain

$$c_{2k} = -v_{2k}^{-1} \sum_{n=k+1}^{\infty} v_{2n} - (k-1) + 2u_{4k} + u_{2k}, \qquad k \geq 1,$$

$$c_{2k+1} = -v_{2k+1}^{-1} \sum_{n=k+1}^{\infty} v_{2n} - k + u_{2k+1}, \qquad k \geq 0,$$

and thus

$$\begin{aligned}
&\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big) \\
&\quad = \frac{1}{2}\sum_{n=0}^{\infty}(v_{2n}^2 + u_{4n}v_{2n}) - \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \sum_{k=1}^{n-1} \cos 2k\theta - \frac{1}{4}\sum_{n=1}^{\infty} n v_n \cos n\theta \\
&\qquad + \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \cos 2n\theta + \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1} \cos(2n+1)\theta + \frac{1}{2}\sum_{n=1}^{\infty} u_n v_n \cos n\theta \\
&\qquad + \sum_{n=1}^{\infty} u_{4n} v_{2n} \cos 2n\theta - \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \sum_{k=0}^{n-1} \cos(2k+1)\theta.
\end{aligned}$$

Using identities (7.3) and (7.5), we can rewrite this as

$$\begin{aligned}
&\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big) \\
&= \frac{1}{2}\sum_{n=1}^{\infty}(v_{2n}^2 + u_{4n}v_{2n}) + \frac{1}{4}\sum_{n=1}^{\infty}(u_n^2 + u_n - n v_n - v_n) \cos n\theta \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2) \cos 2n\theta + \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1} \cos(2n+1)\theta - \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \sum_{k=1}^{2n-1} \cos k\theta.
\end{aligned}$$

Taking $\theta = 0$, we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty}(v_{2n}^2 + u_{4n}v_{2n}) &= -\frac{1}{4}\sum_{n=1}^{\infty}(u_n^2 + u_n - n v_n - v_n) - \frac{1}{2}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2) \\
&\quad - \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1} + \frac{1}{2}\sum_{n=1}^{\infty}(2n-1) v_{2n}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big) \\
&\quad = -\frac{1}{4}\sum_{n=1}^{\infty}(u_n^2 + u_n - n v_n - v_n)(1 - \cos n\theta) - \frac{1}{2}\sum_{n=1}^{\infty}(u_{2n}^2 + u_{2n} - 2v_{2n}^2)(1 - \cos 2n\theta) \\
&\qquad - \frac{1}{4}\sum_{n=0}^{\infty} v_{2n+1}(1 - \cos(2n+1)\theta) + \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \sum_{k=1}^{2n-1}(1 - \cos k\theta).
\end{aligned}$$

Hence

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin{(2n+1)\theta}\Big)$$
$$= \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) - \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} n v_n (1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty}(2u_{2n+1}^2 + 2u_{2n+1} - 4v_{2n+1}^2 - v_{2n+1})(1-\cos{(2n+1)\theta})$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{2n}(1-\cos k\theta). \ \blacksquare$$

Proof of Theorem 3.9. Let

$$f(n) = 1, \quad n \in \mathbb{Z}, \qquad g(n) = \begin{cases} q^n & \text{if } n \text{ is odd and } n \geq 1\ , \\ 0 & \text{if } n \text{ is even,} \\ q^{-n} & \text{if } n \text{ is odd and } n \leq -1. \end{cases}$$

Then

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} u_{4n+2} \sin{(2n+1)\theta}\Big) = \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} q^{2n+1} v_{2n+1} \sin{(2n+1)\theta}\Big)$$
$$= \Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n g(n) \sin n\theta\Big).$$

Using Theorem 2.2(a) and $q^n u_{2n} = v_n - q^n$, we obtain

$$c_{2k} = -v_{2k}^{-1}\sum_{n=k}^{\infty} v_{2n+1}, \quad k \geq 1, \qquad c_{2k+1} = v_{2k+1} - v_{2k+1}^{-1}\sum_{n=k+1}^{\infty} v_{2n}, \quad k \geq 0,$$

and thus

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} u_{4n+2} \sin{(2n+1)\theta}\Big)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}u_{4n+2} + \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}^2 \cos{(2n+1)\theta}$$
$$- \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}\sum_{k=0}^{n-1}\cos{(2k+1)\theta} - \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{n}\cos 2k\theta.$$

Taking $\theta = 0$, we obtain

$$\frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}u_{4n+2} = -\frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}^2 + \frac{1}{2}\sum_{n=1}^{\infty} n v_{2n} + \frac{1}{2}\sum_{n=0}^{\infty} n v_{2n+1}.$$

Thus

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} u_{4n+2} \sin (2n+1)\theta\Big)$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}^2 (1-\cos (2n+1)\theta) + \frac{1}{2}\sum_{n=1}^{\infty} v_{2n} \sum_{k=0}^{n-1}(1-\cos (2k+1)\theta)$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1} \sum_{k=1}^{n}(1-\cos 2k\theta).$$

Using $u_{2n+1} = v_{2n+1} + u_{4n+2}$, which follows from (7.3), we obtain

$$\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin (2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$= \Big(\sum_{n=0}^{\infty} v_{2n+1} \sin (2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) + \Big(\sum_{n=0}^{\infty} u_{4n+2} \sin (2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$=\frac{1}{4}\sum_{n=1}^{\infty} n v_n (1-\cos n\theta) - \frac{1}{4}\sum_{n=0}^{\infty}(u_{2n+1}^2 + u_{2n+1})(1-\cos (2n+1)\theta)$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}\sum_{k=0}^{n-1}(1-\cos (2k+1)\theta) + \frac{1}{2}\sum_{n=0}^{\infty} v_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta). \ \blacksquare$$

Proof of Theorem 3.10. Replacing q by q^2 and θ by 2θ in Theorem 3.4, we obtain

$$\Big(\sum_{n=1}^{\infty} u_{2n} \sin 2n\theta\Big)\Big(\sum_{n=1}^{\infty} v_{2n} \sin 2n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} v_{2n}\sum_{k=1}^{n-1}(1-\cos 2k\theta)$$
$$- \frac{1}{4}\sum_{n=1}^{\infty}(3u_{2n}^2 + 3u_{2n} - 4v_{2n}^2 - 2nv_{2n} - v_{2n})(1-\cos 2n\theta).$$

Hence

$$\Big(\sum_{n=0}^{\infty} u_{2n+1} \sin (2n+1)\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin (2n+1)\theta\Big)$$
$$= \Big(\sum_{n=0}^{\infty} u_{2n+1} \sin (2n+1)\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) - \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)$$
$$+ \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin (2n+1)\theta\Big) + \Big(\sum_{n=0}^{\infty} u_{2n} \sin 2n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n} \sin 2n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} n v_{2n}(1-\cos 2n\theta) + \frac{1}{2}\sum_{n=1}^{\infty} v_{2n+1}\sum_{k=1}^{n}(1-\cos 2k\theta). \ \blacksquare$$

10. Proofs of Theorems 4.1–4.6

All of these results are obtained by repeated application of Theorems 2.1–2.3. We just sketch the details for Theorems 4.1 and 4.4.

Proof of Theorem 4.1. Let

$$f(n)=1, \quad n\in\mathbb{Z}, \qquad g(n)=\begin{cases}1-\dfrac{n}{2}+u_n & \text{if } n\ge 1,\\ 0 & \text{if } n=0,\end{cases}$$

and

$$g(-n)=-g(n) \quad \text{if } n\ge 1.$$

By Theorem 2.1(b), and (7.15), (7.16) and (7.21), we obtain

$$\begin{aligned}c_k&=2u_k\sum_{n=1}^{\infty}nu_n+\sum_{n=k+1}^{\infty}u_n^2+\sum_{n=k+1}^{\infty}u_n\\&\quad-\frac{k}{2}\sum_{n=k+1}^{\infty}u_n-3u_k^3-4u_k^2+\frac{3}{2}ku_k^2-\frac{1}{4}k^2u_k+\frac{5}{4}ku_k-u_k,\end{aligned}$$

and thus

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty}u_nf(n)\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}u_ng(n)\cos n\theta\Big)\\&\quad=\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta\Big)\\&\quad=\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)+\frac{1}{2}\sum_{n=1}^{\infty}u_n^2\sum_{k=1}^{n-1}\sin k\theta+\frac{1}{2}\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}\sin k\theta\\&\qquad-\frac{1}{4}\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}k\sin k\theta-\frac{3}{2}\sum_{n=1}^{\infty}u_n^3\sin n\theta-2\sum_{n=1}^{\infty}u_n^2\sin n\theta+\frac{3}{4}\sum_{n=1}^{\infty}nu_n^2\sin n\theta\\&\qquad-\frac{1}{8}\sum_{n=1}^{\infty}n^2u_n\sin n\theta+\frac{5}{8}\sum_{n=1}^{\infty}nu_n\sin n\theta-\frac{1}{2}\sum_{n=1}^{\infty}u_n\sin n\theta.\end{aligned}$$

Appealing to (7.26) and (7.29), we obtain

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta\Big)\\&\quad=\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)-\frac{13}{16}\sum_{n=1}^{\infty}u_n\sin n\theta+\frac{3}{4}\sum_{n=1}^{\infty}nu_n\sin n\theta-\frac{9}{4}\sum_{n=1}^{\infty}u_n^2\sin n\theta\\&\qquad+\frac{3}{4}\sum_{n=1}^{\infty}nu_n^2\sin n\theta-\frac{1}{8}\sum_{n=1}^{\infty}n^2u_n\sin n\theta-\frac{3}{2}\sum_{n=1}^{\infty}u_n^3\sin n\theta\\&\qquad-\frac{1}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta+\frac{1}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n-\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta.\end{aligned}$$

Recalling Theorem 3.1(b) we have

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} nu_n - \frac{1}{2}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n \sin n\theta + \sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta.$$

Then

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^3 &= \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2\\
&= \frac{1}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big) - \frac{1}{2}\cot\frac{\theta}{2}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2\\
&\quad + \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta\Big)\\
&= \frac{3}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big) - \frac{13}{16}\sum_{n=1}^{\infty} u_n\sin n\theta\\
&\quad + \frac{3}{4}\sum_{n=1}^{\infty} nu_n \sin n\theta - \frac{9}{4}\sum_{n=1}^{\infty} u_n^2\sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty} nu_n^2\sin n\theta\\
&\quad - \frac{1}{8}\sum_{n=1}^{\infty} n^2u_n\sin n\theta - \frac{3}{2}\sum_{n=1}^{\infty} u_n^3\sin n\theta\\
&\quad + \frac{3}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} u_n\sin n\theta - \frac{3}{4}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}\Big(1-\frac{n}{2}+u_n\Big)u_n\cos n\theta. \ \blacksquare\end{aligned}$$

Proof of Theorem 4.4. This will be accomplished by means of a series of lemmas.

LEMMA 10.1.

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} nu_n \sin n\theta\Big)\\
&= \frac{1}{2}\sum_{n=1}^{\infty} nu_n^2 + \sum_{n=1}^{\infty} u_n\cos n\theta\sum_{k=1}^{n} ku_k + \sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1} ku_k\cos k\theta + \sum_{n=1}^{\infty} nu_n\sum_{k=1}^{n-1}\cos k\theta\\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1} k\cos k\theta - \frac{1}{4}\sum_{n=1}^{\infty}(n^2-n)u_n\cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty} nu_n^2\cos n\theta.\end{aligned}$$

Proof. For $n \in \mathbb{Z}$ let $f(n) = 1$ and $g(n) = |n|$. With this choice the value of c_k ($k \in \mathbb{N}$) in Theorem 2.1(a) is

$$c_k = 2u_k\sum_{n=1}^{k} nu_n + 2ku_k\sum_{n=k+1}^{\infty} u_n - 2\sum_{n=k+1}^{\infty} nu_n + k\sum_{n=k+1}^{\infty} u_n - \frac{k^2-k}{2}u_k + ku_k^2.$$

The lemma now follows from Theorem 2.1(a). ∎

Taking $\theta = 0$ in Lemma 10.1 we obtain

$$2\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1} ku_k = \sum_{n=1}^{\infty}(n^2-n)u_n - 2\sum_{n=1}^{\infty} nu_n^2.$$

LEMMA 10.2.

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} n^2 u_n \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} n^2 u_n^2 + 2\Big(\sum_{n=1}^{\infty} n u_n\Big)\Big(\sum_{n=1}^{\infty} n u_n \cos n\theta\Big) - \sum_{n=1}^{\infty} n^2 u_n \sum_{k=1}^{n-1} \cos k\theta + \sum_{n=1}^{\infty} n u_n \sum_{k=1}^{n-1} k \cos k\theta$$
$$- \frac{1}{2}\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k^2 \cos k\theta - \frac{1}{12}\sum_{n=1}^{\infty}(2n^3 - 3n^2 + n) u_n \cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty} n^2 u_n^2 \cos n\theta.$$

Proof. For $n \in \mathbb{Z}$ let $f(n) = 1$ and $g(n) = n^2$. With this choice in Theorem 2.1(a), the value of c_k $(k \in \mathbb{N})$ is

$$c_k = 4ku_k \sum_{n=1}^{\infty} n u_n - 2\sum_{n=k+1}^{\infty} n^2 u_n + 2k\sum_{n=k+1}^{\infty} n u_n - k^2 \sum_{n=k+1}^{\infty} u_n - u_k\,\frac{2k^3 - 3k^2 + k}{6} + k^2 u_k^2.$$

The lemma now follows from Theorem 2.1(a). ∎

Taking $\theta = 0$ in Lemma 10.2 we obtain

$$\Big(\sum_{n=1}^{\infty} n u_n\Big)^2 = \frac{1}{12}\sum_{n=1}^{\infty}(5n^3 - 6n^2 + n) u_n - \frac{1}{2}\sum_{n=1}^{\infty} n^2 u_n^2, \tag{10.1}$$

which can also be obtained by equating coefficients of θ^2 in Theorem 3.1.

LEMMA 10.3.

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} n u_n^2 \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} n u_n^3 - \sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k u_k \cos k\theta - \frac{1}{2}\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k \cos k\theta - \sum_{n=1}^{\infty} u_n \cos n\theta \sum_{k=1}^{n} k u_k$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty} n u_n \sum_{k=1}^{n-1} \cos k\theta - \frac{1}{2}\sum_{n=1}^{\infty} n u_n^2 \sum_{k=1}^{n-1} \cos k\theta + \sum_{n=1}^{\infty} n u_n^3 \cos n\theta$$
$$+ \frac{3}{4}\sum_{n=1}^{\infty} n u_n^2 \cos n\theta - \frac{1}{4}\sum_{n=1}^{\infty} n^2 u_n^2 \cos n\theta.$$

Proof. For $n \in \mathbb{Z}$ let $f(n) = 1$ and

$$g(n) = \begin{cases} n u_n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -n u_{-n} & \text{if } n \leq -1. \end{cases}$$

With this choice in Theorem 2.1(a), the value of c_k $(k \in \mathbb{N})$ is

$$c_k = -(2ku_k + k)\sum_{n=k+1}^{\infty} u_n - 2u_k \sum_{n=1}^{k} n u_n + \sum_{n=k+1}^{\infty} n u_n - \sum_{n=k+1}^{\infty} n u_n^2 + 2k u_k^3 + \frac{3}{2} k u_k^2 - \frac{k^2}{2} u_k^2.$$

The lemma now follows from Theorem 2.1(a). ∎

Taking $\theta = 0$ in Lemma 10.3 we obtain

$$2\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k u_k = \frac{3}{2}\sum_{n=1}^{\infty} n u_n^3 - \frac{3}{4}\sum_{n=1}^{\infty} n^2 u_n^2 + \frac{1}{4}\sum_{n=1}^{\infty} n u_n^2 + \frac{1}{4}\sum_{n=1}^{\infty}(n^2-n)u_n.$$

Equating our two expressions for $\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1} k u_k$ gives the identity

$$\sum_{n=1}^{\infty} n u_n^3 = \frac{1}{2}\sum_{n=1}^{\infty}(n^2-3n)u_n^2 + \frac{1}{2}\sum_{n=1}^{\infty}(n^2-n)u_n. \tag{10.2}$$

LEMMA 10.4.

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n^2 \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} u_n^3 - \sum_{n=1}^{\infty}(u_n^2+u_n)\cos n\theta \sum_{k=1}^{n} u_k - \frac{1}{2}\sum_{n=1}^{\infty}(u_n^2-u_n)\sum_{k=1}^{n-1}\cos k\theta$$
$$+ \frac{3}{2}\sum_{n=1}^{\infty} u_n^3 \cos n\theta + \sum_{n=1}^{\infty} u_n^2 \cos n\theta - \frac{1}{2}\sum_{n=1}^{\infty} n u_n^2 \cos n\theta.$$

Proof. For $n \in \mathbb{Z}$ let $f(n) = 1$ and

$$g(n) = \begin{cases} u_n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ u_{-n} & \text{if } n \leq -1. \end{cases}$$

With this choice in Theorem 2.1.(a), the value of c_k $(k \in \mathbb{N})$ is

$$c_k = -(2u_k^2 + u_k)\sum_{n=1}^{k} u_n - \sum_{n=k+1}^{\infty}(u_n^2 - u_n) + 3u_k^3 + (2-k)u_k^2.$$

The lemma now follows from Theorem 2.1(a). ∎

Taking $\theta = 0$ in Lemma 10.4, we obtain the identity

$$\sum_{n=1}^{\infty}(u_n^2+u_n)\sum_{k=1}^{n} u_k = 2\sum_{n=1}^{\infty}(u_n^3+u_n^2) - \sum_{n=1}^{\infty} n u_n^2.$$

LEMMA 10.5.

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n^3 \sin n\theta\Big)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} u_n^4 + \frac{3}{2}\sum_{n=1}^{\infty} u_n^3 \cos n\theta + 2\sum_{n=1}^{\infty} u_n^4 \cos n\theta - \frac{1}{2}\sum_{n=1}^{\infty} n u_n^3 \cos n\theta$$
$$- \frac{1}{2}\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}\cos k\theta - \Big(\sum_{n=1}^{\infty} n u_n\Big)\Big(\sum_{n=1}^{\infty}(u_n^2+u_n)\cos n\theta\Big)$$
$$+ \frac{3}{2}\sum_{n=1}^{\infty}(u_n^2+u_n)\cos n\theta \sum_{k=1}^{n} u_k - \frac{1}{2}\sum_{n=1}^{\infty} u_n^3 \sum_{k=1}^{n-1}\cos k\theta.$$

Proof. For $n \in \mathbb{Z}$ let $f(n) = 1$ and

$$g(n) = \begin{cases} u_n^2 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ u_{-n}^2 & \text{if } n \leq -1. \end{cases}$$

With this choice in Theorem 2.1(a), the value of c_k ($k \in \mathbb{N}$) is

$$c_k = -2(u_k^2 + u_k) \sum_{n=1}^{\infty} n u_n - \sum_{n=k+1}^{\infty} u_n + 3(u_k^2 + u_k) \sum_{n=1}^{k} u_n - \sum_{n=1}^{\infty} u_n^3 + (3-k) u_k^3 + 4u_k^4.$$

The lemma now follows from Theorem 2.1(a). ∎

Taking $\theta = 0$ in Lemma 10.5, and using equations (7.21) and (10.1), we obtain

$$3 \sum_{n=1}^{\infty} (u_n^2 + u_n) \sum_{k=1}^{n} u_k = 2 \sum_{n=1}^{\infty} (n-2) u_n^3 - 5 \sum_{n=1}^{\infty} u_n^4 + \frac{1}{6} \sum_{n=1}^{\infty} (5n^3 - 6n^2 + 7n - 6) u_n - \sum_{n=1}^{\infty} n^2 u_n^2.$$

Making use of the equation (10.2), we obtain

$$\sum_{n=1}^{\infty} (u_n^2 + u_n) \sum_{k=1}^{n} u_k = -\frac{5}{3} \sum_{n=1}^{\infty} u_n^4 - \frac{4}{3} \sum_{n=1}^{\infty} u_n^3 - \sum_{n=1}^{\infty} n u_n^2 + \frac{1}{18} \sum_{n=1}^{\infty} (5n^3 + n - 6) u_n.$$

Equating our two expressions for $\sum_{n=1}^{\infty} (u_n^2 + u_n) \sum_{k=1}^{n} u_k$, and appealing to (7.21), we obtain

$$\sum_{n=1}^{\infty} u_n^4 = -2 \sum_{n=1}^{\infty} u_n^3 + \frac{1}{6} \sum_{n=1}^{\infty} (n^3 - 7n + 6) u_n. \tag{10.3}$$

We are now ready to proceed with the proof of Theorem 4.4. Appealing to Theorem 4.1(b) we have

$$\begin{aligned} \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^4 &= \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^3 \\ &= \frac{3}{2}\Big(\sum_{n=1}^{\infty} n u_n\Big)\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 - \frac{13}{16}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 \\ &\quad + \frac{3}{4}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} n u_n \sin n\theta\Big) - \frac{9}{4}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n^2 \sin n\theta\Big) \\ &\quad + \frac{3}{4}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} n u_n^2 \sin n\theta\Big) - \frac{1}{8}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} n^2 u_n \sin n\theta\Big) \\ &\quad - \frac{3}{2}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} u_n^3 \sin n\theta\Big) + \frac{3}{16} \cot^2 \frac{\theta}{2}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2 \\ &\quad - \frac{3}{4} \cot \frac{\theta}{2}\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)\Big(\sum_{n=1}^{\infty} \Big(1 - \frac{n}{2} + u_n\Big) u_n \cos n\theta\Big) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{13}{32}\sum_{n=1}^{\infty} nu_n + \frac{3}{8}\sum_{n=1}^{\infty} nu_n^2 - \frac{1}{16}\sum_{n=1}^{\infty} n^2u_n^2 - \frac{9}{8}\sum_{n=1}^{\infty} u_n^3 + \frac{3}{8}\sum_{n=1}^{\infty} nu_n^3 \\
&\quad + \frac{3}{4}\Big(\sum_{n=1}^{\infty} nu_n\Big)^2 - \frac{3}{4}\sum_{n=1}^{\infty} u_n^4 + \Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty}(3u_n^2+3u_n-nu_n)\cos n\theta\Big) \\
&\quad - \frac{3}{2}\cot\frac{\theta}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big) - \frac{13}{16}\sum_{n=1}^{\infty} u_n\cos n\theta + \frac{29}{48}\sum_{n=1}^{\infty} nu_n\cos n\theta \\
&\quad - \frac{49}{16}\sum_{n=1}^{\infty} u_n^2\cos n\theta - \frac{7}{32}\sum_{n=1}^{\infty} n^2u_n\cos n\theta + \frac{33}{16}\sum_{n=1}^{\infty} nu_n^2\cos n\theta - \frac{1}{4}\sum_{n=1}^{\infty} n^2u_n^2\cos n\theta \\
&\quad + \frac{1}{48}\sum_{n=1}^{\infty} n^3u_n\cos n\theta - \frac{45}{8}\sum_{n=1}^{\infty} u_n^3\cos n\theta + \frac{3}{2}\sum_{n=1}^{\infty} nu_n^3\cos n\theta - 3\sum_{n=1}^{\infty} u_n^4\cos n\theta \\
&\quad + \frac{65}{64}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n\sin n\theta - \frac{9}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n\sin n\theta + \frac{27}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n^2\sin n\theta \\
&\quad - \frac{9}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n^2\sin n\theta + \frac{3}{32}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} n^2u_n\sin n\theta + \frac{9}{8}\cot\frac{\theta}{2}\sum_{n=1}^{\infty} u_n^3\sin n\theta \\
&\quad - \frac{3}{32}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n + \frac{3}{8}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} u_n\cos n\theta - \frac{3}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} nu_n\cos n\theta \\
&\quad + \frac{3}{8}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty} u_n^2\cos n\theta - \frac{3}{64}\cot^3\frac{\theta}{2}\sum_{n=1}^{\infty} u_n\sin n\theta \\
&\quad + \frac{1}{8}\sum_{n=1}^{\infty}(6u_n^3+9u_n^2-3nu_n^2-3nu_n+n^2u_n-3u_n)\sum_{k=1}^{n-1}\cos k\theta \\
&\quad - \frac{1}{8}\sum_{n=1}^{\infty} nu_n\sum_{k=1}^{n-1} k\cos k\theta + \frac{1}{16}\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1} k^2\cos k\theta.
\end{aligned}
$$

Using (7.22), (7.28), (7.31) and (7.33), we obtain

$$
\begin{aligned}
\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big)^4 &= -\frac{3}{4}\sum_{n=1}^{\infty} nu_n + \frac{9}{16}\sum_{n=1}^{\infty} nu_n^2 - \frac{1}{16}\sum_{n=1}^{\infty} n^2u_n^2 \\
&\quad - \frac{3}{2}\sum_{n=1}^{\infty} u_n^3 + \frac{3}{8}\sum_{n=1}^{\infty} nu_n^3 + \frac{3}{4}\Big(\sum_{n=1}^{\infty} nu_n\Big)^2 - \frac{3}{4}\sum_{n=1}^{\infty} u_n^4 - \frac{1}{16}\sum_{n=1}^{\infty} n^2u_n \\
&\quad + \frac{3}{4}\sum_{n=1}^{\infty} u_n + \Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty}(3u_n^2+3u_n-nu_n)\cos n\theta\Big) \\
&\quad - \frac{3}{2}\cot\frac{\theta}{2}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big) - \frac{5}{8}\sum_{n=1}^{\infty} u_n\cos n\theta
\end{aligned}
$$

$$+\frac{19}{24}\sum_{n=1}^{\infty}nu_n\cos n\theta-\frac{29}{8}\sum_{n=1}^{\infty}u_n^2\cos n\theta-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n\cos n\theta+\frac{9}{4}\sum_{n=1}^{\infty}nu_n^2\cos n\theta$$
$$-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n^2\cos n\theta+\frac{1}{48}\sum_{n=1}^{\infty}n^3u_n\cos n\theta-6\sum_{n=1}^{\infty}u_n^3\cos n\theta+\frac{3}{2}\sum_{n=1}^{\infty}nu_n^3\cos n\theta$$
$$-3\sum_{n=1}^{\infty}u_n^4\cos n\theta+\frac{1}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}(13u_n-12nu_n+2n^2u_n+36u_n^2-12nu_n^2+24u_n^3)\sin n\theta$$
$$-\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n+\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}(6u_n-3nu_n+6u_n^2)\cos n\theta-\frac{1}{16}\cot^3\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta.$$

By using (10.1)–(10.3), we obtain

$$-\frac{3}{4}\sum_{n=1}^{\infty}nu_n+\frac{9}{16}\sum_{n=1}^{\infty}nu_n^2-\frac{1}{16}\sum_{n=1}^{\infty}n^2u_n^2-\frac{3}{2}\sum_{n=1}^{\infty}u_n^3+\frac{3}{8}\sum_{n=1}^{\infty}nu_n^3+\frac{3}{4}\Big(\sum_{n=1}^{\infty}nu_n\Big)^2$$
$$-\frac{3}{4}\sum_{n=1}^{\infty}u_n^4-\frac{1}{16}\sum_{n=1}^{\infty}n^2u_n+\frac{3}{4}\sum_{n=1}^{\infty}u_n=\frac{1}{16}\sum_{n=1}^{\infty}(3n^3-4n^2)u_n-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n^2.$$

Thus

$$\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)^4$$
$$=\frac{1}{16}\sum_{n=1}^{\infty}(3n^3-4n^2)u_n-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n^2+\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}(3u_n^2+3u_n-nu_n)\cos n\theta\Big)$$
$$-\frac{3}{2}\cot\frac{\theta}{2}\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)+\frac{1}{48}\sum_{n=1}^{\infty}(n^3-12n^2+38n-30)u_n\cos n\theta$$
$$-\frac{1}{8}\sum_{n=1}^{\infty}(2n^2-18n+29)u_n^2\cos n\theta+\frac{1}{2}\sum_{n=1}^{\infty}(3n-12)u_n^3\cos n\theta$$
$$-3\sum_{n=1}^{\infty}u_n^4\cos n\theta-\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n$$
$$+\frac{1}{16}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}(13u_n-12nu_n+2n^2u_n+36u_n^2-12nu_n^2+24u_n^3)\sin n\theta$$
$$+\frac{1}{16}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}(6u_n-3nu_n+6u_n^2)\cos n\theta-\frac{1}{16}\cot^3\frac{\theta}{2}\sum_{n=1}^{\infty}u_n\sin n\theta.$$

Finally, making use of the identities (7.37)–(7.46), we obtain

$$\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big)^4=\frac{1}{2}\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}(-3+2n-6u_n)u_n(1-\cos n\theta)\Big)$$
$$+3\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}(1-\cos k\theta)\Big)$$
$$-\frac{1}{4}\sum_{n=1}^{\infty}(7-12n+5n^2)u_n\sum_{k=1}^{n-1}(1-\cos k\theta)$$

$$+\frac{3}{2}\sum_{n=1}^{\infty}(2n-3)u_n^2\sum_{k=1}^{n-1}(1-\cos k\theta)+\frac{1}{4}\sum_{n=1}^{\infty}(5n-6)u_n\sum_{k=1}^{n-1}k(1-\cos k\theta)$$

$$-\frac{1}{4}\sum_{n=1}^{\infty}u_n\sum_{k=1}^{n-1}k^2(1-\cos k\theta)-\frac{3}{2}\sum_{n=1}^{\infty}u_n^2\sum_{k=1}^{n-1}k(1-\cos k\theta)-3\sum_{n=1}^{\infty}u_n^3\sum_{k=1}^{n-1}(1-\cos k\theta)$$

$$+\frac{1}{48}\sum_{n=1}^{\infty}(6-11n+6n^2-n^3)u_n(1-\cos n\theta)+\frac{1}{4}\sum_{n=1}^{\infty}(7-6n+n^2)u_n^2(1-\cos n\theta)$$

$$+\frac{3}{2}\sum_{n=1}^{\infty}(3-n)u_n^3(1-\cos n\theta)+3\sum_{n=1}^{\infty}u_n^4(1-\cos n\theta). \blacksquare$$

11. A useful theorem

The following theorem will be used in the next section where the equivalence of Theorem 3.k with Theorem 5.k is established for $k=1,\ldots,10$.

THEOREM 11.1. *Let $n\in\mathbb{N}$. Let $F:\mathbb{Z}\to\mathbb{C}$. Define*

$$F_n(x)=\frac{1}{2n!}\sum_{i=-n}^{n}(-1)^{n-i}\binom{2n}{n+i}F(i)\prod_{\substack{j=-n\\ j\neq i}}^{n}(x-j).$$

Then

(i) $F_n(x)\in\mathbb{C}[x]$,
(ii) $\deg(F_n)\le 2n$,
(iii) F_n *is even if* F *is even*,
(iv) F_n *is odd if* F *is odd*,
(v) $F_n(k)=F(k)$ *for* $k=-n,-(n-1),\ldots,-1,0,1,\ldots,n$.

Proof. It is clear that $F_n(x)$ is a polynomial with coefficients in $\mathbb{C}$ of degree at most $2n$. Also

$$\begin{aligned}F_n(-x)&=\frac{1}{2n!}\sum_{i=-n}^{n}(-1)^{n-i}\binom{2n}{n+i}F(i)\prod_{\substack{j=-n\\ j\neq i}}^{n}(-x-j)\\
&=\frac{1}{2n!}\sum_{i=-n}^{n}(-1)^{n-i}\binom{2n}{n+i}F(i)\prod_{\substack{j=-n\\ j\neq i}}^{n}(x+j)\\
&=\frac{1}{2n!}\sum_{i=-n}^{n}(-1)^{n+i}\binom{2n}{n-i}F(-i)\prod_{\substack{j=-n\\ j\neq -i}}^{n}(x+j)\\
&=\frac{1}{2n!}\sum_{i=-n}^{n}(-1)^{n-i}\binom{2n}{n+i}F(-i)\prod_{\substack{j=-n\\ j\neq -i}}^{n}(x+j)\end{aligned}$$

$$= \frac{1}{2n!} \sum_{i=-n}^{n} (-1)^{n-i} \binom{2n}{n+i} F(-i) \prod_{\substack{j=-n \\ j \neq i}}^{n} (x-j)$$

$$= \alpha \frac{1}{2n!} \sum_{i=-n}^{n} (-1)^{n-i} \binom{2n}{n+i} F(i) \prod_{\substack{j=-n \\ j \neq i}}^{n} (x-j) = \alpha F_n(x),$$

where

$$\alpha = \begin{cases} 1 & \text{if } F \text{ is even}, \\ -1 & \text{if } F \text{ is odd}. \end{cases}$$

Thus

$$\begin{cases} F_n(x) \text{ is even if } F(x) \text{ is even}, \\ F_n(x) \text{ is odd if } F(x) \text{ is odd}. \end{cases}$$

Further, for $k = -n, \ldots, n$, we have

$$F_n(k) = \frac{1}{2n!} \sum_{i=-n}^{n} (-1)^{n-i} \binom{2n}{n+i} F(i) \prod_{\substack{j=-n \\ j \neq i}}^{n} (k-j)$$

$$= \frac{1}{2n!} (-1)^{n-k} \binom{2n}{n+k} F(k) \prod_{\substack{j=-n \\ j \neq k}}^{n} (k-j)$$

$$= \frac{1}{2n!} (-1)^{n-k} \frac{2n!}{(n+k)!(n-k)!} F(k)(-1)^{n-k}(n-k)!(k+n)! = F(k).$$

This completes the proof of Theorem 11.1. ■

12. Equivalence of Theorems 3.1–3.10 with Theorems 5.1–5.10

We just prove that Theorem 3.1 is equivalent to Theorem 5.1, as the remaining theorems can be treated similarly.

Proof that Theorem 5.1 implies Theorem 3.1. Let $n \in \mathbb{N}$. We apply Liouville's formula (Theorem 5.1) to the (even) function $F(x) = \cos x\theta$. We obtain

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (\cos(a-b)\theta - \cos(a+b)\theta)$$
$$= \sigma(n) - d(n) + \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) \cos d\theta - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \cos k\theta.$$

As

$$\cos(a-b)\theta - \cos(a+b)\theta = 2 \sin a\theta \sin b\theta,$$

we have

$$2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} \sin a\theta \sin b\theta = \sigma(n) - d(n) + \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) \cos d\theta - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \cos k\theta.$$

Multiplying both sides by q^n and summing over $n \in \mathbb{N}$ gives

$$2\sum_{n=1}^{\infty} q^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} \sin a\theta \sin b\theta$$
$$= \sum_{n=1}^{\infty} q^n(\sigma(n)-d(n)) + \sum_{n=1}^{\infty} q^n \sum_{d\mid n}\left(1-d+\frac{2n}{d}\right)\cos d\theta - 2\sum_{n=1}^{\infty} q^n \sum_{d\mid n}\sum_{1\le k\le d}\cos k\theta.$$

Simplifying the left-hand side gives

$$2\sum_{a,b,x,y=1}^{\infty} q^{ax+by}\sin a\theta\sin b\theta = 2\Big(\sum_{a,x=1}^{\infty} q^{ax}\sin a\theta\Big)^2 = 2\Big(\sum_{a=1}^{\infty}\sin a\theta\sum_{x=1}^{\infty} q^{ax}\Big)^2$$
$$= 2\bigg(\sum_{a=1}^{\infty}\frac{q^a}{1-q^a}\sin a\theta\bigg)^2 = 2\Big(\sum_{n=1}^{\infty} u_n\sin n\theta\Big)^2.$$

We consider the three sums on the right-hand side separately.

The first sum is

$$\sum_{n=1}^{\infty} q^n(\sigma(n)-d(n)) = \sum_{n=1}^{\infty} q^n\sum_{d\mid n}(d-1) = \sum_{d=1}^{\infty}(d-1)\sum_{e=1}^{\infty} q^{de} = \sum_{d=1}^{\infty}(d-1)\,\frac{q^d}{1-q^d}$$
$$= \sum_{n=1}^{\infty}(n-1)u_n.$$

As

$$\sum_{n=1}^{\infty} nq^{dn} = \frac{q^d}{(1-q^d)^2},$$

the second sum is

$$\sum_{n=1}^{\infty} q^n\sum_{d\mid n}\left(1-d+2\,\frac{n}{d}\right)\cos d\theta = \sum_{d,e=1}^{\infty} q^{de}(1-d+2e)\cos d\theta$$
$$= \sum_{d=1}^{\infty}(1-d)\cos d\theta\sum_{e=1}^{\infty} q^{de} + 2\sum_{d=1}^{\infty}\cos d\theta\sum_{e=1}^{\infty} eq^{de}$$
$$= \sum_{d=1}^{\infty}\frac{q^d}{1-q^d}(1-d)\cos d\theta + 2\sum_{d=1}^{\infty}\frac{q^d}{(1-q^d)^2}\cos d\theta$$
$$= \sum_{n=1}^{\infty}(1-n)u_n\cos n\theta + 2\sum_{n=1}^{\infty}(1+u_n)u_n\cos n\theta$$
$$= \sum_{n=1}^{\infty}(3u_n - nu_n + 2u_n^2)\cos n\theta.$$

The third sum is

$$-2\sum_{n=1}^{\infty} q^n\sum_{d\mid n}\sum_{k=1}^{d}\cos k\theta = -2\sum_{d,e=1}^{\infty} q^{de}\sum_{k=1}^{d}\cos k\theta = -2\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}\sum_{k=1}^{d}\cos k\theta$$
$$= -2\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n}\cos k\theta = -2\sum_{n=1}^{\infty} u_n\sum_{k=1}^{n-1}\cos k\theta - 2\sum_{n=1}^{\infty} u_n\cos n\theta.$$

Therefore the right-hand side is

$$\sum_{n=1}^{\infty}(n-1)u_n + \sum_{n=1}^{\infty} u_n(1-n+2u_n)\cos n\theta - 2\sum_{n=1}^{\infty} u_n \sum_{k=1}^{n-1}\cos k\theta.$$

Equating the left-hand side and the right-hand side gives Theorem 3.1. ■

Proof that Theorem 3.1 implies Theorem 5.1. By Theorem 3.1 we have

$$\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\sin n\theta\right)^2 = \sum_{n=1}^{\infty}\left(\frac{1}{2}(n-1)\frac{q^n}{1-q^n} - \frac{q^{2n}}{(1-q^n)^2}\right)(1-\cos n\theta) + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sum_{k=1}^{n-1}(1-\cos k\theta).$$

Using

$$\frac{q^n}{1-q^n} = \sum_{m=1}^{\infty} q^{mn}, \qquad \frac{q^{2n}}{(1-q^n)^2} = \sum_{m=1}^{\infty}(m-1)q^{mn},$$

on equating coefficients of q^N $(N \in \mathbb{N})$ we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}} \sin a\theta \sin b\theta = \sum_{d|N}\left(\frac{1}{2}(d-1) - \left(\frac{N}{d}-1\right)\right)(1-\cos d\theta) + \sum_{d|N}\sum_{k=1}^{d-1}(1-\cos k\theta)$$

$$= \sum_{d|N}\left(\frac{1}{2}(d-1) - \frac{N}{d}\right)(1-\cos d\theta) + \sum_{d|N}\sum_{k=1}^{d}(1-\cos k\theta).$$

Next, using the identity

$$\sin a\theta \sin b\theta = \frac{1}{2}\left(\cos(a-b)\theta - \cos(a+b)\theta\right)$$

and the series

$$\cos r\theta = \sum_{M=0}^{\infty}(-1)^M \frac{r^{2M}\theta^{2M}}{2M!} \qquad (r \in \mathbb{Z}),$$

on equating coefficients of $(-1)^M\theta^{2M}/(2M)!$, for $M, N \in \mathbb{N}$ we obtain

$$\frac{1}{2}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}}\left((a-b)^{2M} - (a+b)^{2M}\right) = \sum_{d|N}\left(\left(-\frac{1}{2}(d-1) + \frac{N}{d}\right)d^{2M} - \sum_{k=1}^{d} k^{2M}\right).$$

Let $F : \mathbb{Z} \to \mathbb{C}$ be even. Let F_N be the even polynomial of degree at most $2N$ given by Theorem 11.1, say

$$F_N(x) = \sum_{r=0}^{N} A_r x^{2r}.$$

By Theorem 11.1 we have

$$F_N(k) = F(k) \quad \text{for } k \in \mathbb{Z} \text{ with } |k| \le N.$$

Thus

$$A_0 = F_N(0) = F(0).$$

For $(a,b,x,y) \in \mathbb{N}^4$ with $ax+by=N$, we have

$$|a \pm b| \le a+b \le ax+by = N$$

so that

$$F_N(a \pm b) = F(a \pm b).$$

Thus

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}} (F(a-b)-F(a+b)) \\
&\quad= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}} (F_N(a-b)-F_N(a+b)) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}} \sum_{r=1}^{N} A_r((a-b)^{2r}-(a+b)^{2r}) \\
&\quad= \sum_{r=1}^{N} A_r \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N}} ((a-b)^{2r}-(a+b)^{2r}) \\
&\quad= \sum_{r=1}^{N} A_r \sum_{d\mid N} \left(\left(-(d-1) + 2\,\frac{N}{d} \right) d^{2r} - 2\sum_{k=1}^{d} k^{2r} \right) \\
&\quad= \sum_{d\mid N} \left(1-d+2\,\frac{N}{d}\right) \sum_{r=1}^{N} A_r d^{2r} - 2\sum_{d\mid N}\sum_{k=1}^{d}\sum_{r=1}^{N} A_r k^{2r} \\
&\quad= \sum_{d\mid N} \left(1-d+2\,\frac{N}{d}\right)(F_N(d)-F_N(0)) - 2\sum_{d\mid N}\sum_{k=1}^{d}(F_N(k)-F_N(0)) \\
&\quad= \sum_{d\mid N} \left(1-d+2\,\frac{N}{d}\right)(F(d)-F(0)) - 2\sum_{d\mid N}\sum_{k=1}^{d}(F(k)-F(0)) \\
&\quad= F(0)\left(\sum_{d\mid N}\left(d-1-2\,\frac{N}{d}\right) + 2\sum_{d\mid N} d\right) + \sum_{d\mid N}\left(1-d+2\,\frac{N}{d}\right)F(d) - 2\sum_{d\mid N}\sum_{k=1}^{d} F(k) \\
&\quad= F(0)(\sigma(N)-d(N)) + \sum_{d\mid N}\left(1-d+2\,\frac{N}{d}\right)F(d) - 2\sum_{d\mid N}\sum_{k\le d} F(k),
\end{aligned}$$

which is Theorem 5.1. ∎

13. Equivalence of Theorems 4.1–4.3 with Theorems 6.1–6.3

We just prove that Theorem 4.2 is equivalent to Theorem 6.2, as the other two equivalences can be proved in a similar manner.

Proof that Theorem 6.2 implies Theorem 4.2. Let $n \in \mathbb{N}$. We choose $F(x) = \sin x\theta$ in Theorem 6.2. We obtain

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ x,y,z\text{ odd}}} (\sin(a+b+c)\theta-\sin(a-b+c)\theta-\sin(a+b-c)\theta+\sin(a-b-c)\theta)$$
$$=\frac{1}{4}\sum_{\substack{d\mid n\\ n/d\text{ odd}}}\left(2d^2-6n+3\,\frac{n^2}{d^2}+1\right)\sin d\theta-6\sum_{\substack{(a,x)\in\mathbb{N}^2\\ ax<n\\ a\equiv n\ (\mathrm{mod}\ 2)\\ x\text{ odd}}}\sigma\left(\frac{n-ax}{2}\right)\sin a\theta.$$

As

$$\sin(a+b+c)\theta-\sin(a-b+c)\theta-\sin(a+b-c)\theta+\sin(a-b-c)\theta$$
$$=-4\sin a\theta\sin b\theta\sin c\theta$$

and

$$\sigma\left(\frac{n-ax}{2}\right)=\sum_{\substack{(b,y)\in\mathbb{N}^2\\ (n-ax)/2=by}} b,$$

we have

$$4\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ x,y,z\text{ odd}}}\sin a\theta\sin b\theta\sin c\theta$$
$$=6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n\\ x\text{ odd}}} b\sin a\theta-\frac{1}{4}\sum_{\substack{d\mid n\\ n/d\text{ odd}}}\left(2d^2-6n+3\,\frac{n^2}{d^2}+1\right)\sin d\theta.$$

Multiplying both sides by q^n and summing over $n\in\mathbb{N}$ gives

$$(13.1)\qquad 4\sum_{n=1}^{\infty}\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ x,y,z\text{ odd}}}\sin a\theta\sin b\theta\sin c\theta$$
$$=6\sum_{n=1}^{\infty}q^n\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n\\ x\text{ odd}}} b\sin a\theta-\frac{1}{4}\sum_{n=1}^{\infty}q^n\sum_{\substack{d\mid n\\ n/d\text{ odd}}}\left(2d^2-6n+3\,\frac{n^2}{d^2}+1\right)\sin d\theta.$$

The left-hand side of (13.1) is

$$4\sum_{\substack{a,b,c,c,x,y,z=1\\ x,y,z\text{ odd}}}^{\infty} q^{ax+by+cz}\sin a\theta\ \sin b\theta\ \sin c\theta$$
$$=4\Bigg(\sum_{\substack{a,x=1\\ x\text{ odd}}}^{\infty}q^{ax}\sin a\theta\Bigg)^3=4\Bigg(\sum_{a=1}^{\infty}\sin a\theta\,\frac{q^a}{1-q^{2a}}\Bigg)^3=4\Bigg(\sum_{a=1}^{\infty}v_n\sin n\theta\Bigg)^3.$$

The first sum on the right-hand side of (13.1) is

$$6 \sum_{\substack{a,b,x,y=1 \\ x \text{ odd}}}^{\infty} q^{ax+2by} b \sin a\theta = 6\Big(\sum_{\substack{a,x=1 \\ x \text{ odd}}}^{\infty} q^{ax} \sin a\theta \Big)\Big(\sum_{b,y=1}^{\infty} b q^{2by} \Big)$$

$$= 6\Big(\sum_{a=1}^{\infty} \sin a\theta \, \frac{q^a}{1-q^{2a}} \Big)\Big(\sum_{y=1}^{\infty} \frac{q^{2y}}{(1-q^{2y})^2} \Big) = 6\Big(\sum_{n=1}^{\infty} v_n \sin n\theta \Big)\Big(\sum_{n=1}^{\infty} v_n^2 \Big).$$

The second sum on the right-hand side of (13.1) is

$$-\frac{1}{4} \sum_{n=1}^{\infty} q^n \sum_{\substack{d \mid n \\ n/d \text{ odd}}} \left(2d^2 - 6n + 3\frac{n^2}{d^2} + 1 \right) \sin d\theta$$

$$= -\frac{1}{4} \sum_{\substack{d,e=1 \\ e \text{ odd}}}^{\infty} (2d^2 - 6de + 3e^2 + 1) q^{de} \sin d\theta$$

$$= -\frac{1}{2} \sum_{d=1}^{\infty} d^2 \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} q^{de} + \frac{3}{2} \sum_{d=1}^{\infty} d \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} e q^{de}$$

$$- \frac{3}{4} \sum_{d=1}^{\infty} \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} e^2 q^{de} - \frac{1}{4} \sum_{d=1}^{\infty} \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} q^{de}$$

$$-\frac{1}{4} \sum_{n=1}^{\infty} q^n \sum_{\substack{d \mid n \\ n/d \text{ odd}}} \left(2d^2 - 6n + 3\frac{n^2}{d^2} + 1 \right) \sin d\theta$$

$$= -\frac{1}{4} \sum_{\substack{d,e=1 \\ e \text{ odd}}}^{\infty} (2d^2 - 6de + 3e^2 + 1) q^{de} \sin d\theta$$

$$= -\frac{1}{2} \sum_{d=1}^{\infty} d^2 \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} q^{de} + \frac{3}{2} \sum_{d=1}^{\infty} d \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} e q^{de}$$

$$- \frac{3}{4} \sum_{d=1}^{\infty} \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} e^2 q^{de} - \frac{1}{4} \sum_{d=1}^{\infty} \sin d\theta \sum_{\substack{e=1 \\ e \text{ odd}}}^{\infty} q^{de}$$

$$= -\frac{1}{2} \sum_{d=1}^{\infty} d^2 \frac{q^d}{1-q^{2d}} \sin d\theta + \frac{3}{2} \sum_{d=1}^{\infty} \frac{d q^d (1+q^{2d})}{(1-q^{2d})^2} \sin d\theta$$

$$- \frac{3}{4} \sum_{d=1}^{\infty} \frac{q^d (1 + 6q^{2d} + q^{4d})}{(1-q^{2d})^3} \sin d\theta - \frac{1}{4} \sum_{d=1}^{\infty} \frac{q^d}{1-q^{2d}} \sin d\theta$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{2} n^2 + \frac{3}{2} \frac{n(1+q^{2n})}{1-q^{2n}} - \frac{3}{4} \frac{1+6q^{2n}+q^{4n}}{(1-q^{2n})^2} - \frac{1}{4} \right) v_n \sin n\theta.$$

Thus

$$\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big)^3 = \frac{3}{2}\Big(\sum_{n=1}^{\infty} v_n^2\Big)\Big(\sum_{n=1}^{\infty} v_n \sin n\theta\Big) + \sum_{n=1}^{\infty}\Big(-\frac{1}{8}\,n^2 + \frac{3}{8}\,\frac{n(1+q^{2n})}{1-q^{2n}} - \frac{3}{16}\,\frac{1+6q^{2n}+q^{4n}}{(1-q^{2n})^2} - \frac{1}{16}\Big) v_n \sin n\theta.$$

Theorem 4.2 now follows as

$$\begin{aligned} -\frac{1}{8}\,n^2 &+ \frac{3}{8}\,n\,\frac{1+q^{2n}}{1-q^{2n}} - \frac{3}{16}\,\frac{1+6q^{2n}+q^{4n}}{(1-q^{2n})^2} - \frac{1}{16} \\ &= -\frac{1}{8}\,(n^2-3n+2) + \frac{3n}{4}\,\frac{q^{2n}}{1-q^{2n}} - \frac{3}{16}\,\frac{1+6q^{2n}+q^{4n}}{(1-q^{2n})^2} + \frac{3}{16} \\ &= -\frac{1}{8}\,(n^2-3n+2) + \frac{3n}{4}\,u_{2n} - \frac{3}{2}\,\frac{q^{2n}}{(1-q^{2n})^2} = -\frac{1}{8}\,(n^2-3n+2) + \frac{3n}{4}\,u_{2n} - \frac{3}{2}\,v_n^2. \quad \blacksquare \end{aligned}$$

Proof that Theorem 4.2 implies Theorem 6.2. By Theorem 4.2 we have

$$\Big(\sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin n\theta\Big)^3 = \frac{3}{2}\Big(\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}\Big)\Big(\sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin n\theta\Big)$$

$$-\frac{3}{2}\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{2n})^3} \sin n\theta - \frac{1}{8}\sum_{n=1}^{\infty}(n^2-3n+2)\,\frac{q^n}{1-q^{2n}} \sin n\theta + \frac{3}{4}\sum_{n=1}^{\infty} n\frac{q^{3n}}{(1-q^{2n})^2} \sin n\theta.$$

Using

$$\frac{q^n}{1-q^{2n}} = \sum_{\substack{m=1\\ m\ \mathrm{odd}}}^{\infty} q^{mn}, \qquad \frac{q^{2n}}{(1-q^{2n})^2} = \sum_{m=1}^{\infty} mq^{2mn},$$

$$\frac{q^{3n}}{(1-q^{2n})^3} = \sum_{m=1}^{\infty} \frac{(m+1)m}{2}\,q^{(2m+1)n}, \qquad \frac{q^{3n}}{(1-q^{2n})^2} = \sum_{m=1}^{\infty} mq^{(2m+1)n},$$

on equating coefficients of q^N $(N \in \mathbb{N})$ we obtain

$$\begin{aligned} \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=N\\ x,y,z\ \mathrm{odd}}} & \sin a\theta \sin b\theta \sin c\theta \\ &= \frac{3}{2} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\ \mathrm{odd}}} b\sin a\theta - \frac{3}{16} \sum_{\substack{d\mid N\\ N/d\ \mathrm{odd}}} \Big(\frac{N}{d}-1\Big)\Big(\frac{N}{d}+1\Big)\sin d\theta \\ &\quad - \frac{1}{8} \sum_{\substack{d\mid N\\ N/d\ \mathrm{odd}}} (d^2-3d+2)\sin d\theta + \frac{3}{8} \sum_{\substack{d\mid N\\ N/d\ \mathrm{odd}}} d\Big(\frac{N}{d}-1\Big)\sin d\theta \\ &= \frac{3}{2} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\ \mathrm{odd}}} b\sin a\theta - \frac{1}{16} \sum_{\substack{d\mid N\\ N/d\ \mathrm{odd}}} \Big(3\,\frac{N^2}{d^2} + 2d^2 - 6N + 1\Big)\sin d\theta. \end{aligned}$$

Using the identity

$$\begin{aligned}&\sin a\theta \sin b\theta \sin c\theta \\ &\quad = -\frac{1}{4}(\sin(a+b+c)\theta - \sin(a-b+c)\theta - \sin(a+b-c)\theta + \sin(a-b-c)\theta),\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=N \\ x,y,z \text{ odd}}} (\sin(a+b+c)\theta - \sin(a-b+c)\theta - \sin(a+b-c)\theta + \sin(a-b-c)\theta) \\ &\qquad = \frac{1}{4}\sum_{\substack{d\mid N \\ N/d \text{ odd}}} \left(3\frac{N^2}{d^2} + 2d^2 - 6N + 1\right)\sin d\theta - 6 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^6 \\ ax+2by=N \\ x \text{ odd}}} b \sin a\theta.\end{aligned}$$

Next, using the series

$$\sin r\theta = \sum_{M=0}^{\infty} (-1)^M \frac{r^{2M+1}\theta^{2M+1}}{(2M+1)!}, \qquad r \in \mathbb{Z},$$

and equating coefficients of $(-1)^M \theta^{2M+1}/(2M+1)!$ for $M \in \mathbb{N}_0$ and $N \in \mathbb{N}$, we obtain

$$\begin{aligned}&\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=N \\ x,y,z \text{ odd}}} ((a+b+c)^{2M+1} - (a-b+c)^{2M+1} - (a+b-c)^{2M+1} + (a-b-c)^{2M+1}) \\ &\qquad = \frac{1}{4}\sum_{\substack{d\mid N \\ N/d \text{ odd}}} \left(3\frac{N^2}{d^2} + 2d^2 - 6N + 1\right)d^{2M+1} - 6 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+2by=N \\ x \text{ odd}}} b a^{2M+1}.\end{aligned}$$

Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Let F_N be the odd polynomial of degree at most $2N$ given by Theorem 11.1, say

$$F_N(x) = \sum_{r=0}^{N-1} A_r x^{2r+1}.$$

By Theorem 11.1 we have

$$F_N(k) = F(k) \quad \text{ for } k \in \mathbb{Z} \text{ and } |k| \le N.$$

For $(a,b,c,x,y,z) \in \mathbb{N}^6$ with $ax+by+cz=N$ we have

$$|a \pm b \pm c| \le a+b+c \le ax+by+cz = N$$

so that

$$F_N(a \pm b \pm c) = F(a \pm b \pm c).$$

Thus

$$\begin{aligned}&\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=N \\ x,y,z \text{ odd}}} (F(a+b+c) - F(a-b+c) - F(a+b-c) + F(a-b-c)) \\ &\quad = \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=N \\ x,y,z \text{ odd}}} (F_N(a+b+c) - F_N(a-b+c) - F_N(a+b-c) + F_N(a-b-c))\end{aligned}$$

$$= \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=N\\ x,y,z\,\mathrm{odd}}} \sum_{r=0}^{N-1} A_r((a+b+c)^{2r+1}-(a-b+c)^{2r+1}$$
$$-(a+b-c)^{2r+1}+(a-b-c)^{2r+1})$$
$$= \sum_{r=0}^{N-1} A_r \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=N\\ x,y,z\,\mathrm{odd}}} ((a+b+c)^{2r+1}-(a-b+c)^{2r+1}$$
$$-(a+b-c)^{2r+1}+(a-b-c)^{2r+1})$$
$$= \sum_{r=0}^{N-1} A_r\left(\frac{1}{4}\sum_{\substack{d\mid N\\ N/d\,\mathrm{odd}}}\left(3\,\frac{N^2}{d^2}+2d^2-6N+1\right)d^{2r+1} - 6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\,\mathrm{odd}}} ba^{2r+1}\right)$$
$$= \frac{1}{4}\sum_{\substack{d\mid N\\ N/d\,\mathrm{odd}}}\left(3\,\frac{N^2}{d^2}+2d^2-6N+1\right)\sum_{r=0}^{N-1} A_r d^{2r+1} - 6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\,\mathrm{odd}}} b\sum_{r=0}^{N-1} A_r a^{2r+1}$$
$$= \frac{1}{4}\sum_{\substack{d\mid N\\ N/d\,\mathrm{odd}}}\left(3\,\frac{N^2}{d^2}+2d^2-6N+1\right)F_N(d) - 6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\,\mathrm{odd}}} bF_N(a)$$
$$= \frac{1}{4}\sum_{\substack{d\mid N\\ N/d\,\mathrm{odd}}}\left(3\,\frac{N^2}{d^2}+2d^2-6N+1\right)F(d) - 6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=N\\ x\,\mathrm{odd}}} bF(a)$$
$$= \frac{1}{4}\sum_{\substack{d\mid N\\ N/d\,\mathrm{odd}}}\left(3\,\frac{N^2}{d^2}+2d^2-6N+1\right)F(d) - 6\sum_{\substack{(a,x)\in\mathbb{N}^2\\ ax<n\\ a\equiv N \pmod 2\\ x\,\mathrm{odd}}} \sigma\left(\frac{n-ax}{2}\right)F(a),$$

which is Theorem 6.2. ■

14. Equivalence of Theorems 4.4–4.6 with Theorems 6.4–6.6

We just prove that Theorem 4.6 is equivalent to Theorem 6.6. The equivalence of Theorem 4.4 with Theorem 6.4, and of Theorem 4.5 with Theorem 6.5, can be proved similarly.

Proof that Theorem 6.6 implies Theorem 4.6. Let $n \in \mathbb{N}$. We choose $F(x) = \cos x\theta$ in Theorem 6.6 to obtain

$$\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\,\mathrm{odd}}} (\cos(a+b+c+d)\theta - \cos(a-b+c+d)\theta$$
$$-\cos(a+b-c+d)\theta - \cos(a+b+c-d)\theta$$
$$+\cos(a-b-c+d)\theta + \cos(a-b+c-d)\theta$$
$$+\cos(a+b-c-d)\theta - \cos(a-b-c-d)\theta)$$

$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \,\text{odd}}} (d^3 - d)(\cos 2d\theta - 1) - 4 \sum_{\substack{ax<n \\ x \,\text{odd}}} a\sigma(o(n-ax))(\cos 2a\theta - 1).$$

As

(14.1) $$8 \sin a\theta \sin b\theta \sin c\theta \sin d\theta$$
$$= \cos(a+b+c+d)\theta - \cos(a-b+c+d)\theta - \cos(a+b-c+d)\theta - \cos(a+b+c-d)\theta$$
$$+ \cos(a-b-c+d)\theta - \cos(a-b+c-d)\theta + \cos(a+b-c-d)\theta - \cos(a-b-c-d)\theta$$

and

$$\sigma(o(n-ax)) = \sum_{\substack{(b,y)\in\mathbb{N}^2 \\ n-ax=by \\ b \,\text{odd}}} b,$$

we obtain

$$8 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w \,\text{odd}}} \sin a\theta \sin b\theta \sin c\theta \sin d\theta$$
$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \,\text{odd}}} (d^3 - d)(\cos 2d\theta - 1) - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ x,b \,\text{odd}}} ab(\cos 2a\theta - 1).$$

Multiplying both sides by q^{2n}, and summing over $n \in \mathbb{N}$, we obtain

(14.2) $$8 \sum_{n=1}^{\infty} q^{2n} \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w \,\text{odd}}} \sin a\theta \sin b\theta \sin c\theta \sin d\theta$$
$$= \frac{1}{6} \sum_{n=1}^{\infty} q^{2n} \sum_{\substack{d|n \\ n/d \,\text{odd}}} (d^3 - d)(\cos 2d\theta - 1) - 4 \sum_{n=1}^{\infty} q^{2n} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ x,b \,\text{odd}}} ab(\cos 2a\theta - 1).$$

The left-hand side of (14.2) is

$$8 \sum_{\substack{a,b,c,d,x,y,z,w=1 \\ a,b,c,d,x,y,z,w \,\text{odd}}}^{\infty} q^{ax+by+cz+dw} \sin a\theta \sin b\theta \sin c\theta \sin d\theta$$
$$= 8\Big(\sum_{\substack{a,x=1 \\ a,x \,\text{odd}}}^{\infty} q^{ax} \sin a\theta \Big)^4 = 8\Big(\sum_{\substack{a=1 \\ a,x \,\text{odd}}}^{\infty} \sin a\theta \, \frac{q^a}{1-q^{2a}} \Big)^4 = 8\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin(2n+1)\theta \Big)^4.$$

The first sum on the right-hand side of (14.2) is

$$\frac{1}{6} \sum_{\substack{d,e=1 \\ e \,\text{odd}}}^{\infty} q^{2de} (d^3 - d)(\cos 2d\theta - 1) = \frac{1}{6} \sum_{d=1}^{\infty} (d^3 - d)(\cos 2d\theta - 1) \frac{q^{2d}}{1-q^{4d}}$$
$$= \frac{1}{6} \sum_{n=1}^{\infty} (n^3 - n)(\cos 2n\theta - 1) v_{2n}.$$

The second sum on the right-hand side of (14.2) is

$$
\begin{aligned}
-4 \sum_{\substack{a,b,x,y=1 \\ b,x\,\text{odd}}}^{\infty} & q^{2ax+2by} ab(\cos 2a\theta - 1) \\
&= -4\Big(\sum_{\substack{a,x=1 \\ x\,\text{odd}}}^{\infty} q^{2ax} a(\cos 2a\theta - 1)\Big)\Big(\sum_{\substack{b,y=1 \\ b\,\text{odd}}}^{\infty} bq^{2by}\Big) \\
&= -4\Big(\sum_{a=1}^{\infty} \frac{q^{2a}}{1-q^{4a}}\, a(\cos 2a\theta - 1)\Big)\Big(\sum_{\substack{b=1 \\ b\,\text{odd}}}^{\infty} b\, \frac{q^{2b}}{1-q^{2b}}\Big) \\
&= -4\Big(\sum_{n=0}^{\infty} (2n+1) u_{4n+2}\Big)\Big(\sum_{n=1}^{\infty} n v_{2n} (\cos 2n\theta - 1)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Big(\sum_{n=0}^{\infty} v_{2n+1} \sin{(2n+1)\theta} \Big)^4 &= -\frac{1}{2}\Big(\sum_{n=0}^{\infty} (2n+1) u_{4n+2}\Big)\Big(\sum_{n=1}^{\infty} n v_{2n} (\cos 2n\theta - 1)\Big) \\
&\quad + \frac{1}{48} \sum_{n=1}^{\infty} (n^3 - n) v_{2n} (\cos 2n\theta - 1),
\end{aligned}
$$

which gives Theorem 4.6. ■

Proof that Theorem 4.6 implies Theorem 6.6. Starting with Theorem 4.6 and expanding each of the terms in powers of q, we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} q^{2n} & \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w\,\text{odd}}} \sin a\theta \sin b\theta \sin c\theta \sin d\theta \\
&= \frac{1}{48} \sum_{n=1}^{\infty} q^{2n} \sum_{\substack{d\mid n \\ n/d\,\text{odd}}} (d^3 - d)(\cos 2d\theta - 1) - \frac{1}{2} \sum_{n=1}^{\infty} q^{2n} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x\,\text{odd}}} ab(\cos 2a\theta - 1).
\end{aligned}
$$

Equating coefficients of q^{2n} $(n \in \mathbb{N})$ and using the identity (14.1), we obtain

$$
\begin{aligned}
\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w\,\text{odd}}} & (\cos{(a+b+c+d)\theta} - \cos{(a-b+c+d)\theta} \\
&\qquad - \cos{(a+b-c+d)\theta} - \cos{(a+b+c-d)\theta} \\
&\qquad + \cos{(a-b-c+d)\theta} + \cos{(a-b+c-d)\theta} \\
&\qquad + \cos{(a+b-c-d)\theta} - \cos{(a-b-c-d)\theta}) \\
&= \frac{1}{6} \sum_{\substack{d\mid n \\ n/d\,\text{odd}}} (d^3 - d)(\cos 2d\theta - 1) - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x\,\text{odd}}} ab(\cos 2a\theta - 1).
\end{aligned}
$$

Using

$$\cos k\theta = \sum_{k\theta} = \sum_{M=0}^{\infty} (-1)^M k^{2M} \frac{\theta^{2M}}{(2M)!}$$

and equating coefficients of $(-1)^M \theta^{2M}/(2M)!$ $(M \in \mathbb{N}_0)$, for $M \in \mathbb{N}$ we deduce

$$\begin{aligned}\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ \mathrm{odd}}} &((a+b+c+d)^{2M} - (a-b+c+d)^{2M} - (a+b-c+d)^{2M}\\ &- (a+b+c-d)^{2M} + (a-b-c+d)^{2M} + (a-b+c-d)^{2M}\\ &+ (a+b-c-d)^{2M} - (a-b-c-d)^{2M})\\ &= \frac{1}{6} \sum_{\substack{d\mid n\\ n/d\ \mathrm{odd}}} (d^3-d)2^M d^M - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ b,x\ \mathrm{odd}}} ab2^M a^M.\end{aligned}$$

Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Let F_{2n} be the even polynomial of degree at most $4n$ given by Theorem 11.1, say

$$F_{2n}(x) = \sum_{r=0}^{2n} A_r x^{2r}.$$

By Theorem 11.1 we have

$$F_{2n}(k) = F(k) \quad \text{for } k \in \mathbb{Z} \text{ and } |k| \le 2n.$$

In particular $F(0) = F_{2n}(0) = A_0$. For $(a,b,c,d,x,y,z,w) \in \mathbb{N}^8$ with $ax+by+cz+dw = 2n$ we have

$$|a \pm b \pm c \pm d| \le a+b+c+d \le ax+by+cz+dw = 2n$$

so that

$$F_{2n}(a \pm b \pm c \pm d) = F(a \pm b \pm c \pm d).$$

Thus

$$\begin{aligned}\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ \mathrm{odd}}} &(F(a+b+c+d) - F(a-b+c+d)\\ &- F(a+b-c+d) - F(a+b+c-d) + F(a-b-c+d)\\ &+ F(a-b+c-d) + F(a+b-c-d) - F(a-b-c-d))\\ = \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ \mathrm{odd}}} &(F_{2n}(a+b+c+d) - F_{2n}(a-b+c+d)\\ &- F_{2n}(a+b-c+d) - F_{2n}(a+b+c-d) + F_{2n}(a-b-c+d)\\ &+ F_{2n}(a-b+c-d) + F_{2n}(a+b-c-d) - F_{2n}(a-b-c-d))\\ = \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ \mathrm{odd}}} &\sum_{r=0}^{2n} A_r((a+b+c+d)^{2r} - (a-b+c+d)^{2r}\\ &- (a+b-c+d)^{2r} - (a+b+c-d)^{2r} + (a-b-c+d)^{2r}\\ &+ (a-b+c-d)^{2r} + (a+b-c-d)^{2r} - (a-b-c-d)^{2r})\end{aligned}$$

$$= \sum_{r=1}^{2n} A_r \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=2n \\ a,b,c,d,x,y,z,w \text{ odd}}} ((a+b+c+d)^{2r} - (a-b+c+d)^{2r} - (a+b-c+d)^{2r} - (a+b+c-d)^{2r} + (a-b-c+d)^{2r} + (a-b+c-d)^{2r} + (a+b-c-d)^{2r} - (a-b-c-d)^{2r})$$

$$= \sum_{r=1}^{2n} A_r \left(\frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d)2^r d^r - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x \text{ odd}}} ab2^r a^r \right)$$

$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d) \sum_{r=1}^{2n} A_r 2^r d^r - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x \text{ odd}}} ab \sum_{r=1}^{2n} A_r 2^r d^r$$

$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d)(F_{2n}(2d) - A_0) - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x \text{ odd}}} ab(F_{2n}(2a) - A_0)$$

$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d)(F(2d) - F(0)) - 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ b,x \text{ odd}}} ab(F(2a) - F(0))$$

$$= \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d)(F(2d) - F(0)) - 4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x \text{ odd}}} a\sigma(o(n-ax))(F(2a) - F(0))$$

$$= \left(4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x \text{ odd}}} a\sigma(o(n-ax)) - \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d) \right) F(0) + \frac{1}{6} \sum_{\substack{d|n \\ n/d \text{ odd}}} (d^3-d)F(2d) - 4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x \text{ odd}}} a\sigma(o(n-ax))F(2a),$$

which is Theorem 6.6. ∎

15. Other identities

The following result is an identity of Ramanujan's [23], [24, pp. 136–162] (see also [5, pp. 312–313]).

THEOREM 15.1. *If* $\theta \neq 2k\pi$ $(k \in \mathbb{Z})$ *then*

$$\left(\frac{1}{4} \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} u_n \sin n\theta \right)^2 = \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 + \sum_{n=1}^{\infty} u_n(u_n+1) \cos n\theta + \frac{1}{2} \sum_{n=1}^{\infty} nu_n(1-\cos n\theta).$$

Proof. This follows from Theorem 3.1(b). ∎

Replacing θ by $\pi-\theta$ in Theorem 13.1, for $\theta \neq (2k+1)\pi$ $(k \in \mathbb{Z})$ we obtain

$$\left(\frac{1}{4}\tan\frac{\theta}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}u_n\sin n\theta\right)^2$$
$$=\left(\frac{1}{4}\tan\frac{\theta}{2}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}nu_n+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}nu_n\cos n\theta-\sum_{n=1}^{\infty}(-1)^{n-1}nu_n(u_n+1)\cos n\theta,$$

which was given by Ramamani [21, p. 16]. Our next theorem is also due to Ramamani [21, p. 104].

THEOREM 15.2. *If $\theta \neq 2k\pi$ $(k \in \mathbb{Z})$ then*

$$\left(\frac{1}{4}\cot\frac{\theta}{2}+\sum_{n=1}^{\infty}u_n\sin n\theta\right)^3$$
$$=\frac{1}{64}\left(\cot\frac{\theta}{2}\right)^3+\frac{3}{8}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n-\frac{13}{16}\sum_{n=1}^{\infty}u_n\sin n\theta$$
$$+\frac{3}{4}\sum_{n=1}^{\infty}nu_n\sin n\theta-\frac{1}{8}\sum_{n=1}^{\infty}n^2u_n\sin n\theta-\frac{9}{4}\sum_{n=1}^{\infty}u_n^2\sin n\theta$$
$$+\frac{3}{4}\sum_{n=1}^{\infty}nu_n^2\sin n\theta-\frac{3}{2}\sum_{n=1}^{\infty}u_n^3\sin n\theta+\frac{3}{2}\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}u_n\sin n\theta\Big).$$

Proof. This follows from Theorems 3.1(b) and 4.1(b). ∎

THEOREM 15.3. *If $\theta \neq 2k\pi$ $(k \in \mathbb{Z})$ then*

$$\left(\frac{1}{4}\cot\frac{\theta}{2}+\sum_{n=1}^{\infty}u_n\sin n\theta\right)^4$$
$$=\left(\frac{1}{4}\cot\frac{\theta}{2}\right)^4+\frac{1}{8}\cot^2\frac{\theta}{2}\sum_{n=1}^{\infty}nu_n+\frac{1}{16}\sum_{n=1}^{\infty}(3n^3-4n^2)u_n-\frac{1}{4}\sum_{n=1}^{\infty}n^2u_n^2$$
$$+\frac{1}{48}\sum_{n=1}^{\infty}(n^3-12n^2+38n-30)u_n\cos n\theta-\frac{1}{8}\sum_{n=1}^{\infty}(2n^2-18n+29)u_n^2\cos n\theta$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}(3n-12)u_n^3\cos n\theta-3\sum_{n=1}^{\infty}u_n^4\cos n\theta$$
$$+\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}(3u_n^2+3u_n-nu_n)\cos n\theta\Big).$$

Proof. This follows from Theorems 3.1(b), 4.1(b) and 4.4(b). ∎

Our next identity is again due to Ramanujan [23, eqn. (18)], [24, p. 139]. A proof using elliptic functions and Cauchy's residue theorem has recently been given by Liu [9, Theorem 7, p. 146].

THEOREM 15.4. *For $\theta \neq 2k\pi$, where k is an integer, we have*

$$\Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12} + \sum_{n=1}^{\infty} nu_n(1-\cos n\theta)\Big)^2 = \Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\Big)^2 + \frac{1}{12}\sum_{n=1}^{\infty} n^3 u_n(5+\cos n\theta).$$

Proof. Taking $f(n) = g(n) = n$ $(n \in \mathbb{Z})$ in Theorem 2.1(c), we obtain

$$\Big(\sum_{n=1}^{\infty} nu_n \cos n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} n^2 u_n^2 + \frac{1}{12}\sum_{n=1}^{\infty}(n^3-n)u_n\cos n\theta + 2\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} nu_n\cos n\theta\Big) - \sum_{n=1}^{\infty} nu_n\Big(\sum_{k=1}^{n-1}(n-k)\cos k\theta\Big).$$

Taking $\theta = 0$ we deduce

$$\Big(\sum_{n=1}^{\infty} nu_n\Big)^2 = \frac{1}{12}\sum_{n=1}^{\infty}(5n^3-6n^2+n)u_n - \frac{1}{2}\sum_{n=1}^{\infty} n^2u_n^2.$$

From (7.27) and (7.30), for $\theta \neq 2k\pi$ $(k \in \mathbb{Z})$ we obtain

$$\sum_{k=1}^{n-1}(n-k)\cos k\theta = \frac{1}{4}(1-\cos n\theta) + \Big(\frac{1}{4}\cot^2\frac{\theta}{2}\Big)(1-\cos n\theta) - \frac{n}{2}$$

so that

$$\Big(\frac{1}{4}\cot^2\frac{\theta}{2} + \frac{1}{6}\Big)(1-\cos n\theta) = \sum_{k=1}^{n-1}(n-k)\cos k\theta - \frac{1}{12}(1-\cos n\theta) + \frac{n}{2}.$$

Then

$$\begin{aligned}
&\Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12} + \sum_{n=1}^{\infty} nu_n(1-\cos n\theta)\Big)^2 \\
&= \Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\Big)^2 + \Big(\frac{1}{4}\cot^2\frac{\theta}{2} + \frac{1}{6}\Big)\sum_{n=1}^{\infty} nu_n(1-\cos n\theta) + \Big(\sum_{n=1}^{\infty} nu_n(1-\cos n\theta)\Big)^2 \\
&= \Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\Big)^2 + \sum_{n=1}^{\infty} nu_n\Big(\sum_{k=1}^{n-1}(n-k)\cos k\theta - \frac{1}{12}(1-\cos n\theta) + \frac{n}{2}\Big) \\
&\quad + \Big(\sum_{n=1}^{\infty} nu_n\Big)^2 + \Big(\sum_{n=1}^{\infty} nu_n\cos n\theta\Big)^2 - 2\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} nu_n\cos n\theta\Big) \\
&= \Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\Big)^2 + \sum_{n=1}^{\infty} nu_n\sum_{k=1}^{n-1}(n-k)\cos k\theta - \frac{1}{12}\sum_{n=1}^{\infty} nu_n(1-\cos n\theta) \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty} n^2u_n + \frac{1}{12}\sum_{n=1}^{\infty}(5n^3-6n^2+n)u_n - \frac{1}{2}\sum_{n=1}^{\infty} n^2u_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} n^2u_n^2
\end{aligned}$$

$$+\frac{1}{12}\sum_{n=1}^{\infty}(n^3-n)u_n\cos n\theta+2\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}nu_n\cos n\theta\Big)$$

$$-\sum_{n=1}^{\infty}nu_n\sum_{k=1}^{n-1}(n-k)\cos k\theta-2\Big(\sum_{n=1}^{\infty}nu_n\Big)\Big(\sum_{n=1}^{\infty}nu_n\cos n\theta\Big)$$

$$=\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{n=1}^{\infty}n^3u_n(5+\cos n\theta),$$

which is the asserted identity. ∎

The next identity is due to Liu [9, Theorem 8, p. 147]. Notice that the first plus sign in Liu's theorem should be a minus sign. His proof uses elliptic functions and Cauchy's residue theorem. We show that the result is a simple consequence of Theorem 2.2(c).

THEOREM 15.5.

$$\Big(1-24\sum_{n=1}^{\infty}nu_{2n}+24\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)^2=1+240\sum_{n=1}^{\infty}n^3u_{2n}+48\sum_{n=1}^{\infty}n^3v_n\cos n\theta.$$

Proof. Taking $f(n)=g(n)=n$ ($n\in\mathbb{Z}$) in Theorem 2.2(c), we obtain

$$\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)^2=\frac{1}{2}\sum_{n=1}^{\infty}n^2v_n^2+2\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)$$
$$+\frac{1}{12}\Big(\sum_{n=1}^{\infty}(n^3-n)v_n\cos n\theta\Big).$$

Hence

$$\Big(1-24\sum_{n=1}^{\infty}nu_{2n}+24\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)^2$$
$$=\Big(1-24\sum_{n=1}^{\infty}nu_{2n}\Big)^2+48\Big(1-24\sum_{n=1}^{\infty}nu_{2n}\Big)\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)+576\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)^2$$
$$=1-48\sum_{n=1}^{\infty}nu_{2n}+576\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)^2+48\sum_{n=1}^{\infty}nv_n\cos n\theta$$
$$-1152\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)+288\sum_{n=1}^{\infty}n^2v_n^2$$
$$+1152\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)\Big(\sum_{n=1}^{\infty}nv_n\cos n\theta\Big)+48\sum_{n=1}^{\infty}(n^3-n)v_n\cos n\theta$$
$$=1-48\sum_{n=1}^{\infty}nu_{2n}+576\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)^2+288\sum_{n=1}^{\infty}n^2v_n^2+48\sum_{n=1}^{\infty}n^3v_n\cos n\theta.$$

From (10.1) with q replaced by q^2, we have

$$\Big(\sum_{n=1}^{\infty}nu_{2n}\Big)^2=\frac{1}{12}\sum_{n=1}^{\infty}(5n^3-6n^2+n)u_{2n}-\frac{1}{2}\sum_{n=1}^{\infty}n^2u_{2n}^2.$$

Thus

$$\Big(1-24\sum_{n=1}^{\infty} nu_{2n}+24\sum_{n=1}^{\infty} nv_n\cos n\theta\Big)^2 = 1+48\sum_{n=1}^{\infty}(5n^3-6n^2)u_{2n}-288\sum_{n=1}^{\infty} n^2u_{2n}^2 + 288\sum_{n=1}^{\infty} n^2v_n^2+48\sum_{n=1}^{\infty} n^3v_n\cos n\theta.$$

The asserted result now follows as $v_n^2 = u_{2n}^2 + u_{2n}$. ■

16. Arithmetic identities

In this section we just mention a few of the arithmetic formulae that can be obtained from the identities given in Section 6.

Taking $F(x) = x^2$ in Theorem 6.6, we obtain, as

$$(a+b+c+d)^2-(a-b+c+d)^2-(a+b-c+d)^2-(a+b+c-d)^2 + (a+b-c-d)^2+(a-b+c-d)^2+(a-b-c+d)^2-(a-b-c-d)^2 = 0,$$

the following new result.

THEOREM 16.1. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{\substack{(a,x)\in\mathbb{N}^2\\ ax<n\\ x\ odd}} a^3\sigma(o(n-ax)) = \frac{1}{24}\,(\sigma_5^*(n)-\sigma_3^*(n)).$$

Similarly, taking $F(x) = x^2$ in Theorem 6.5 yields

THEOREM 16.2. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{\substack{(a,x)\in\mathbb{N}^2\\ ax<n\\ x\ odd\\ a\equiv n \ (\mathrm{mod}\ 2)}} (2a^3-3a^2x)\sigma\Big(\frac{n-ax}{2}\Big) = \frac{1}{24}\,\sigma_5^*(n)-\frac{1}{12}\,(3n-1)\sigma_3^*(n)+\frac{1}{8}\,(3n^2-n)\sigma^*(n)-\frac{1}{8}\,n^2\sigma(o(n)).$$

Taking $F(x) = x^2$ in Theorem 6.4 yields a linear relationship between the convolution sums

$$\sum_{\substack{(r,s)\in\mathbb{N}^2\\ r+s=n}} \sigma(r)\sigma(s), \qquad \sum_{\substack{(r,s)\in\mathbb{N}^2\\ r+s=n}} \sigma(r)\sigma_3(s), \qquad \sum_{\substack{(r,s)\in\mathbb{N}^2\\ r+s=n}} s\sigma(r)\sigma(s).$$

However the evaluation of each of these sums is known; see for example [6].

Equating coefficients of $F(0)$ in Theorems 6.4, 6.5 and 6.6, we obtain respectively the following three results.

THEOREM 16.3. *Let* $n \in \mathbb{N}$. *Then*

$$3\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ odd\\ a+b=c+d}} 1-4\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w\ odd\\ a=b+c+d}} 1 = -\frac{1}{6}\,(\sigma_3^*(n)-\sigma^*(n))+4\sum_{\substack{(a,x)\in\mathbb{N}^2\\ ax<n\\ x\ odd}} a\sigma(o(n-ax)).$$

Theorem 16.4. *Let $n \in \mathbb{N}$. Then*

$$3 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n \\ x,y,z,w \text{ odd} \\ a+b=c+d}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n \\ x,y,z,w \text{ odd} \\ a=b+c+d}} 1$$
$$= -\frac{1}{6}\left(\sigma_3^*(n) - 3\sigma_3(o(n)) + 2\sigma^*(n) - 6n\sigma^*(n) + 9n\sigma(o(n)) - 3\sigma(o(n))\right)$$
$$+ 4 \sum_{\substack{(a,x)\in\mathbb{N}^2 \\ ax<n \\ x \text{ odd} \\ a\equiv n \ (\mathrm{mod}\ 2)}} (2a-3x)\sigma\left(\frac{n-ax}{2}\right).$$

Theorem 16.5. *Let $n \in \mathbb{N}$. Then*

$$3 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n \\ a+b=c+d}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n \\ a=b+c+d}} 1$$
$$= d(n) - 12nd(n) + \frac{1}{2}\sigma(n) - 2n\sigma(n) + 11\sigma_2(n) + \frac{3}{2}\sigma_3(n) - 12 \sum_{\substack{(r,s)\in\mathbb{N}^2 \\ r+s=n}} d(r)\sigma(s).$$

Theorem 16.5 is particularly interesting as it involves the convolution sum $\sum_{r<n} d(r)\sigma(n-r)$, which, as far as the authors are aware, is untreated in the literature.

Equating the coefficients of $F(k)$ $(k \in \mathbb{N})$ in Theorems 6.1–6.5, we obtain the following five theorems. For $n, k \in \mathbb{N}$ we set

$$\delta(n,k) = \begin{cases} 1 & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Theorem 16.6. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n \\ a+b+c=k}} 1 - 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n \\ a-b+c=k}} 1 + 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n \\ x,y,z,w \text{ odd} \\ a-b-c=k}} 1$$
$$= \left(\frac{1}{2}(k^2-3k+2) + 3\left(\frac{n}{k}-1\right)\left(\frac{n}{k}-k\right)\right)\delta(n,k)$$
$$+ 3 \sum_{\substack{d\mid n \\ d>k}} \left(2d - \frac{2n}{d} - k\right) - 6 \sum_{\substack{x\in\mathbb{N} \\ x<n/k}} \sigma(n-kx).$$

This result was first proved by McAfee [11, Lemma 3.4.4]. (Note that a "2" is missing before the d in the first sum.) McAfee's proof is completely arithmetic.

THEOREM 16.7. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a+b+c=k\\ x,y,z \text{ odd}}} 1 - 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a-b+c=k\\ x,y,z \text{ odd}}} 1 + 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a-b-c=k\\ x,y,z \text{ odd}}} 1$$

$$= \left\{ \begin{array}{ll} \frac{1}{4}\left(2k^2 - 6n + 3\left(\frac{n}{k}\right)^2\right) & \text{if } k \mid n \text{ and } n/k \text{ is odd} \\ 0 & \text{otherwise} \end{array} \right\}$$

$$+ \left\{ \begin{array}{ll} -6 \sum_{\substack{x\in\mathbb{N}\\ x<n/k\\ x \text{ odd}}} \sigma\left(\frac{n-kx}{2}\right) & \text{if } k \equiv n \pmod 2 \\ 0, & \text{if } k \not\equiv n \pmod 2 \end{array} \right\}.$$

This result was not given by McAfee [11] but presumably can be proved in a completely arithmetic way by her methods.

THEOREM 16.8. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a+b+c=k\\ a,b,c,x,y,z \text{ odd}}} 1 - 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a-b+c=k\\ a,b,c,x,y,z \text{ odd}}} 1 + 3 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6\\ ax+by+cz=n\\ a-b-c=k\\ a,b,c,x,y,z \text{ odd}}} 1$$

$$= \frac{(k^2-1)}{8}\,\delta(n,k) - 3 \sum_{\substack{x<n/k\\ x \text{ odd}}} \sigma(o(n-kx)).$$

This theorem was first proved by McAfee [11, Lemma 2.3.4] (see also [13, Lemma 5.4]). Her proof is completely arithmetic.

By the argument given in McAfee [11, Theorem 3.5.1] we see that Theorem 16.1 (resp. 16.2, 16.3) implies Theorem 6.1 (resp. 6.2, 6.3). Thus Theorems 16.1, 16.2, 16.3 are equivalent to Theorems 6.1, 6.2, 6.3 respectively.

THEOREM 16.9. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ a+b+c+d=k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ a-b+c+d=k}} 1 + 6 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ a-b-c+d=k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ a-b-c-d=k}} 1$$

$$= \frac{1}{6}\left(6 + 11k + 6k^2 + k^3 - 24\,\frac{n}{k} + 36\left(\frac{n}{k}\right)^2 - 24\left(\frac{n}{k}\right)^3 - 12nk - 36k + 36\,\frac{n^2}{k}\right)\delta(n,k)$$

$$+ 2 \sum_{\substack{d\mid n\\ d>k}} \left(1 - 12n + 5d^2 + 6\,\frac{n^2}{d^2} - 5kd + 6\,\frac{kn}{d} + k^2\right)$$

$$- 4 \sum_{\substack{x\in\mathbb{N}\\ x<n/k}} (2k - 6x + 3)\sigma(n-kx) - 24 \sum_{\substack{(r,s)\in\mathbb{N}^2\\ r+s=n}} \sigma(r) \sum_{\substack{d\mid s\\ d>k}} 1.$$

THEOREM 16.10. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ x,y,z,w \text{ odd}\\ a+b+c+d=k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ x,y,z,w \text{ odd}\\ a-b+c+d=k}} 1 + 6 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ x,y,z,w \text{ odd}\\ a-b-c+d=k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=n\\ x,y,z,w \text{ odd}\\ a-b-c-d=k}} 1$$

$$= \left\{ \begin{array}{ll} \frac{1}{6}\left(k^3 - 6nk + 2k + 9\frac{n^2}{k} - 3\frac{n}{k} - 3\left(\frac{n}{k}\right)^3\right) & \text{if } k \mid n \text{ and } n/k \text{ is odd} \\ 0 & \text{otherwise} \end{array} \right\}$$

$$+ \left\{ \begin{array}{ll} -4 \sum\limits_{\substack{x\in\mathbb{N}\\ x<n/k\\ x \text{ odd}}} (2k-3x)\sigma\left(\frac{n-ax}{2}\right) & \text{if } k \equiv n \pmod 2 \\ 0 & \text{if } k \not\equiv n \pmod 2 \end{array} \right\}.$$

Equating coefficients of $F(2k)$ ($k \in \mathbb{N}$) in Theorem 6.6 and recalling that F is even, we obtain the following result.

THEOREM 16.11. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w \text{ odd}\\ a+b+c+d=2k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w \text{ odd}\\ a-b+c+d=2k}} 1 + 6 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w \text{ odd}\\ a-b-c+d=2k}} 1 - 4 \sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8\\ ax+by+cz+dw=2n\\ a,b,c,d,x,y,z,w \text{ odd}\\ a-b-c-d=2k}} 1$$

$$= \left\{ \begin{array}{ll} \frac{1}{6}(k^3 - k) & \text{if } k \mid n \text{ and } n/k \text{ is odd} \\ no30 & \text{otherwise} \end{array} \right\} - 4k \sum_{\substack{x\in\mathbb{N}\\ x<n/k\\ x \text{ odd}}} \sigma(o(n-kx)).$$

17. Further identities

In this section we give a few more identities involving u_n and v_n, which can be deduced from our results.

THEOREM 17.1.

(a) $$\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} nv_n\Big) = \frac{1}{24}\sum_{n=1}^{\infty}(8n^3+n)v_n - \frac{3}{8}\sum_{n=1}^{\infty} n^2(u_n^2+u_n) + \frac{1}{2}\sum_{n=1}^{\infty} n^2 v_n^2.$$

(b) $$\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^3v_n\Big) + \Big(\sum_{n=1}^{\infty} n^3u_n\Big)\Big(\sum_{n=1}^{\infty} nv_n\Big)$$
$$= \frac{1}{240}\sum_{n=1}^{\infty}(36n^5+10n^3-n)v_n - \frac{3}{16}\sum_{n=1}^{\infty} n^4(u_n^2+u_n) + \frac{1}{4}\sum_{n=1}^{\infty} n^4 v_n^2.$$

(c) $$3\Big(\sum_{n=1}^{\infty} n^5u_n\Big)\Big(\sum_{n=1}^{\infty} nv_n\Big)+3\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^5v_n\Big)+10\Big(\sum_{n=1}^{\infty} n^3u_n\Big)\Big(\sum_{n=1}^{\infty} n^3v_n\Big)$$
$$=-\frac{63}{168}\sum_{n=1}^{\infty} n^6(u_n^2+u_n)+\frac{1}{168}\sum_{n=1}^{\infty}(48n^7+21n^5-7n^3+n)v_n+\frac{1}{2}\sum_{n=1}^{\infty} n^6v_n^2.$$

(d) $$\Big(\sum_{n=1}^{\infty} nu_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^3v_{2n}\Big)+\Big(\sum_{n=1}^{\infty} n^3u_{2n}\Big)\Big(\sum_{n=1}^{\infty} nv_{2n}\Big)$$
$$=\frac{1}{240}\sum_{n=1}^{\infty}(36n^5+10n^3-n)v_{2n}-\frac{3}{16}\sum_{n=1}^{\infty} n^4v_n^2+\frac{1}{4}\sum_{n=1}^{\infty} n^4v_{2n}^2.$$

Proof. Identities (a), (b) and (c) follow by equating the coefficients of θ^2, θ^4 and θ^6, respectively, in Theorem 3.4. Identity (d) follows by replacing q with q^2 in identity (b) and using $v_n^2 = u_{2n}^2 + u_{2n}$. ∎

THEOREM 17.2.

(a) $$\Big(\sum_{n=1}^{\infty} nu_n\sin n\theta\Big)^2=\frac{1}{2}\sum_{n=1}^{\infty} n^2u_n^2+\frac{1}{12}\sum_{n=1}^{\infty}(n-n^3)u_n\cos n\theta$$
$$+2\sum_{n=1}^{\infty} u_n\cos n\theta\sum_{k=1}^{n} k^2u_k-\sum_{n=1}^{\infty}\cos n\theta\sum_{k=n+1}^{\infty} k^2u_k$$
$$+\sum_{n=1}^{\infty}(1+2u_n)n\cos n\theta\sum_{k=n+1}^{\infty} ku_k.$$

(b) $$\Big(\sum_{n=1}^{\infty} nv_n\sin n\theta\Big)^2=\frac{1}{2}\sum_{n=1}^{\infty} n^2v_n^2+\frac{1}{12}\sum_{n=1}^{\infty}(n-n^3)v_n\cos n\theta$$
$$+2\sum_{n=1}^{\infty} v_n\cos n\theta\sum_{k=1}^{n} k^2u_{2k}+2\sum_{n=1}^{\infty} nv_n\cos n\theta\sum_{k=n+1}^{\infty} ku_{2k}.$$

Proof. Let $f(n) = g(n) = |n|$. Identities (a) and (b) follow from Theorems 2.1(a) and 2.2(a) respectively. ∎

THEOREM 17.3.

(a) $$\Big(\sum_{n=1}^{\infty} n^2u_n\sin n\theta\Big)^2=\frac{1}{2}\sum_{n=1}^{\infty} n^4u_n^2-\frac{1}{60}\sum_{n=1}^{\infty}(n^5-n)u_n\cos n\theta$$
$$+4\Big(\sum_{n=1}^{\infty} n^3u_n\Big)\Big(\sum_{n=1}^{\infty} nu_n\cos n\theta\Big)-\sum_{n=1}^{\infty} n^4u_n\sum_{k=1}^{n-1}\cos k\theta$$
$$+2\sum_{n=1}^{\infty} n^3u_n\sum_{k=1}^{n-1} k\cos k\theta-\sum_{n=1}^{\infty} n^2u_n\sum_{k=1}^{n-1} k^2\cos k\theta.$$

(b) $$\Big(\sum_{n=1}^{\infty} n^2v_n\sin n\theta\Big)^2=\frac{1}{2}\sum_{n=1}^{\infty} n^4v_n^2-\frac{1}{60}\sum_{n=1}^{\infty}(n^5-n)v_n\cos n\theta$$
$$+4\Big(\sum_{n=1}^{\infty} n^3u_{2n}\Big)\Big(\sum_{n=1}^{\infty} nv_n\cos n\theta\Big).$$

(c) $$\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^3u_n\Big) = \frac{1}{240}\sum_{n=1}^{\infty}(21n^5 - 30n^4 + 10n^3 - n)u_n - \frac{1}{8}\sum_{n=1}^{\infty} n^4u_n^2.$$

(d) $$\Big(\sum_{n=1}^{\infty} nv_n\Big)\Big(\sum_{n=1}^{\infty} n^3u_{2n}\Big) = \frac{1}{240}\sum_{n=1}^{\infty}(n^5 - n)v_n - \frac{1}{8}\sum_{n=1}^{\infty} n^4v_n^2.$$

(e) $$\Big(\sum_{n=1}^{\infty} nu_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^3u_{2n}\Big) = \frac{1}{240}\sum_{n=1}^{\infty}(21n^5 + 10n^3 - n)u_{2n} - \frac{1}{8}\sum_{n=1}^{\infty} n^4v_n^2.$$

(f) $$\Big(\sum_{n=1}^{\infty} n^3v_n\Big)^2 = 2\Big(\sum_{n=1}^{\infty} n^3u_n\Big)\Big(\sum_{n=1}^{\infty} n^3v_n\Big) - \frac{1}{120}\sum_{n=1}^{\infty}(n^7 - n^3)v_n.$$

(g) $$\Big(\sum_{n=1}^{\infty} n^3u_n\Big)\Big(\sum_{n=1}^{\infty} n^5v_n\Big) + \Big(\sum_{n=1}^{\infty} n^3v_n\Big)\Big(\sum_{n=1}^{\infty} n^5v_n\Big) = \frac{1}{240}\sum_{n=1}^{\infty}(n^9 - n^5)v_n.$$

Proof. Choose $f(n) = g(n) = n^2$, $n \in \mathbb{Z}$. Identities (a) and (b) follow from Theorems 2.1(a) and 2.2(a) respectively. Identity (c) follows by taking $\theta = 0$ in identity (a). Identity (d) follows by taking $\theta = 0$ in identity (b). Identity (e) follows by replacing q with q^2 in identity (c) and using (7.7). Identity (f) follows by equating the coefficients of θ^2 in identity (b) and using $u_{2n} = u_n - v_n$. Identity (g) follows by equating the coefficients of θ^4 in identity (b) and using $u_{2n} = u_n - v_n$. ∎

THEOREM 17.4.

(a) $$\begin{aligned}\Big(\sum_{n=0}^{\infty}(2n+1)u_{2n+1}\cos(2n+1)\theta\Big)^2 &= \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)^2u_{2n+1}^2 + \frac{1}{6}\sum_{n=1}^{\infty}(2n^3+n)u_{2n}\cos 2n\theta \\ &\quad + 4\Big(\sum_{n=0}^{\infty}(2n+1)u_{2n+1}\Big)\Big(\sum_{n=1}^{\infty} nu_{2n}\cos 2n\theta\Big) \\ &\quad + 2\sum_{n=1}^{\infty}(2n+1)u_{2n+1}\sum_{k=1}^{n} k\cos 2k\theta - \sum_{n=1}^{\infty}(2n+1)^2u_{2n+1}\sum_{k=1}^{n}\cos 2k\theta.\end{aligned}$$

(b) $$\begin{aligned}\Big(\sum_{n=0}^{\infty}(2n+1)u_{2n+1}\Big)^2 &= \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)^2u_{2n+1}^2 - \sum_{n=1}^{\infty}(2n^3+n^2)u_{2n+1} \\ &\quad + \frac{1}{6}\sum_{n=1}^{\infty}(2n^3+n)u_{2n} + 4\Big(\sum_{n=0}^{\infty}(2n+1)u_{2n+1}\Big)\Big(\sum_{n=1}^{\infty} nu_{2n}\Big).\end{aligned}$$

Proof. Identity (a) follows from Theorem 2.1(c) by choosing

$$f(n) = g(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Identity (b) follows by taking $\theta = 0$ in identity (a). ∎

THEOREM 17.5.

$$\Big(\sum_{n=1}^{\infty} n^2 v_n \cos n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} n^4 v_n^2 + \frac{1}{60}\sum_{n=1}^{\infty}(n^5-n)v_n\cos n\theta + 2\sum_{n=1}^{\infty} v_n \cos n\theta \sum_{k=1}^{n} k^4 u_{2k}$$
$$+ 2\sum_{n=1}^{\infty} n^2 v_n \cos n\theta \sum_{k=1}^{n} k^2 u_{2k} + 4\sum_{n=1}^{\infty} n v_n \cos n\theta \sum_{k=n+1}^{\infty} k^3 u_{2k}.$$

Proof. The result follows from Theorem 2.2(c) by choosing

$$f(n) = g(n) = n^2 \operatorname{sgn}(n). \ \blacksquare$$

THEOREM 17.6.

(a) $$\Big(\sum_{n=1}^{\infty} n^3 v_n \cos n\theta\Big)^2 = \frac{1}{2}\sum_{n=1}^{\infty} n^6 v_n^2 + \frac{1}{840}\sum_{n=1}^{\infty}(3n^7+7n^3-10n)v_n\cos n\theta$$
$$+ 6\Big(\sum_{n=1}^{\infty} n^5 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n v_n \cos n\theta\Big) + 2\Big(\sum_{n=1}^{\infty} n^3 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n \cos n\theta\Big).$$

(b) $$\Big(\sum_{n=1}^{\infty} n^3 v_n\Big)^2 = \frac{1}{6}\sum_{n=1}^{\infty} n^6 v_n^2 + \frac{1}{2520}\sum_{n=1}^{\infty}(3n^7+7n^3-10n)v_n$$
$$+ 2\Big(\sum_{n=1}^{\infty} n^5 u_n\Big)\Big(\sum_{n=1}^{\infty} n v_n\Big) - 2\Big(\sum_{n=1}^{\infty} n v_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big).$$

(c) $$\Big(\sum_{n=1}^{\infty} n^3 u_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) + 3\Big(\sum_{n=1}^{\infty} n^5 u_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) - 5\Big(\sum_{n=1}^{\infty} n^3 v_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big)$$
$$= \frac{1}{1680}\sum_{n=1}^{\infty}(10n^3-7n^5-3n^9)v_n.$$

Proof. Identity (a) follows from Theorem 2.2(c) by choosing $f(n) = g(n) = n^3$, $n \in \mathbb{Z}$. Identity (b) follows by taking $\theta = 0$ in identity (a) and using $u_{2n} = u_n - v_n$. Identity (c) follows by equating the coefficients of θ^2 in identity (a) and using $u_{2n} = u_n - v_n$. $\blacksquare$

THEOREM 17.7.

$$\Big(\sum_{n=0}^{\infty} n v_{2n+1} \cos(2n+1)\theta\Big)^2 = \frac{1}{2}\sum_{n=0}^{\infty} n^2 v_{2n+1}^2 + \frac{1}{12}\sum_{n=1}^{\infty}(n^3-3n^2+2n)v_{2n}\cos 2n\theta$$
$$+ 2\Big(\sum_{n=0}^{\infty} n u_{4n+2}\Big)\Big(\sum_{n=1}^{\infty} n v_{2n} \cos 2n\theta\Big) - \sum_{n=1}^{\infty} v_{2n}\cos 2n\theta \sum_{k=0}^{n-1} k u_{4k+2}.$$

Proof. The result follows from Theorem 2.2(c) by choosing $f(n) = g(n) = n$. $\blacksquare$

THEOREM 17.8.

(a) $$\Big(\sum_{n=0}^{\infty}(2n+1) v_{2n+1} \cos(2n+1)\theta\Big)^2 = \frac{1}{6}\sum_{n=1}^{\infty}(2n^3+n)v_{2n}\cos 2n\theta$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)^2 v_{2n+1}^2 + 4\Big(\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big)\Big(\sum_{n=1}^{\infty} n v_{2n}\cos 2n\theta\Big).$$

(b) $$\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big)^2 = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)^2v_{2n+1}^2 + \frac{1}{6}\sum_{n=1}^{\infty}(2n^3+n)v_{2n} + 4\Big(\sum_{n=1}^{\infty}nv_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big).$$

(c) $$\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big) = \frac{1}{3}\sum_{n=1}^{\infty}(2n^5+n^3)v_{2n} + 8\Big(\sum_{n=1}^{\infty}n^3v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big).$$

Proof. Identity (a) follows from Theorem 2.2(c) by choosing

$$f(n) = g(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Identity (b) follows by taking $\theta = 0$ in identity (a). Identity (c) follows by equating the coefficients of θ^2 in identity (a). ∎

THEOREM 17.9.

(a) $$\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big)^2 = \sum_{n=1}^{\infty}n^3v_{2n}.$$

(b) $$\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big) = \sum_{n=1}^{\infty}n^5v_{2n}.$$

(c) $$\Big(\sum_{n=1}^{\infty}n^3v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big) = \Big(\sum_{n=1}^{\infty}n^5v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big).$$

(d) $$3\Big(\sum_{n=1}^{\infty}n^3v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^5v_{2n+1}\Big) + 5\Big(\sum_{n=1}^{\infty}n^5v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big) = 8\Big(\sum_{n=1}^{\infty}n^7v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big).$$

(e) $$\Big(\sum_{n=1}^{\infty}n^5v_{2n}\Big)^2 = \Big(\sum_{n=1}^{\infty}n^3v_{2n}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big)^2.$$

Proof. Identities (a) and (b) follow by equating the coefficients of θ^2 and θ^4, respectively, in Theorem 3.3. Identity (c) follows by multiplying both sides of identity (b) by $\sum_{n=0}^{\infty}(2n+1)v_{2n+1}$ and appealing to identity (a). Identity (e) follows by multiplying identity (b) by $\sum_{n=1}^{\infty}n^5v_{2n}$ and substituting identity (c). Identity (d): Equating the coefficients of θ^6 in Theorem 3.3, we obtain

$$3\Big(\sum_{n=0}^{\infty}(2n+1)v_{2n+1}\Big)\Big(\sum_{n=0}^{\infty}(2n+1)^5v_{2n+1}\Big) + 5\Big(\sum_{n=0}^{\infty}(2n+1)^3v_{2n+1}\Big)^2 = 8\sum_{n=1}^{\infty}n^7v_{2n}.$$

Then, the result follows by multiplying both sides of the above equation by the sum $\sum_{n=0}^{\infty}(2n+1)v_{2n+1}$ and appealing to identities (a) and (b). ∎

THEOREM 17.10.

(a) $$\Big(\sum_{n=1}^{\infty} nv_n \cos n\theta\Big)\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big) = \frac{1}{24}\sum_{n=1}^{\infty}(n^4-n^2)v_n \sin n\theta + \Big(\sum_{n=1}^{\infty} nu_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big).$$

(b) $$2\Big(\sum_{n=1}^{\infty} nv_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) - \Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) = \frac{1}{24}\sum_{n=1}^{\infty}(n^5-n^3)v_n.$$

(c) $$\Big(\sum_{n=1}^{\infty} n^3 v_n\Big)^2 = \frac{1}{72}\sum_{n=1}^{\infty}(n^7-n^5)v_n + \frac{1}{3}\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) - \frac{2}{3}\Big(\sum_{n=1}^{\infty} nv_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big).$$

Proof. Identity (a) follows from Theorem 2.2(b) by taking $f(n) = n$ and $g(n) = n^2$. Identities (b) and (c) follow by equating the coefficients of θ and θ^3 in identity (a) respectively and using $u_{2n} = u_n - v_n$. ■

THEOREM 17.11.

(a) $$\Big(\sum_{n=1}^{\infty} nu_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) = -\frac{1}{8}\sum_{n=1}^{\infty} n^4(u_n^2+u_n) + \frac{1}{24}\sum_{n=1}^{\infty}(2n^5+n^3)v_n + \frac{1}{4}\sum_{n=1}^{\infty} n^4 v_n^2.$$

(b) $$\Big(\sum_{n=1}^{\infty} n^3 u_n\Big)\Big(\sum_{n=1}^{\infty} nv_n\Big) = -\frac{1}{16}\sum_{n=1}^{\infty} n^4(u_n^2+u_n) + \frac{1}{240}\sum_{n=1}^{\infty}(16n^5-n)v_n.$$

(c) $$\Big(\sum_{n=1}^{\infty} nv_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) = -\frac{1}{16}\sum_{n=1}^{\infty} n^4(u_n^2+u_n) + \frac{1}{16}\sum_{n=1}^{\infty} n^5 v_n + \frac{1}{8}\sum_{n=1}^{\infty} n^4 v_n^2.$$

Proof. The asserted results follow from Theorems 17.1(b), 17.3(d) and 17.10(b). ■

THEOREM 17.12.

(a) $$\Big(\sum_{n=1}^{\infty} n^5 v_n \cos n\theta\Big)\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big) = \frac{1}{336}\sum_{n=1}^{\infty}(6n^2-7n^4+n^8)v_n \sin n\theta - 9\Big(\sum_{n=1}^{\infty} n^5 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big) - 5\Big(\sum_{n=1}^{\infty} n^3 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^4 v_n \sin n\theta\Big).$$

(b) $$5\Big(\sum_{n=1}^{\infty} n^3 u_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) + 9\Big(\sum_{n=1}^{\infty} n^5 u_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) - 13\Big(\sum_{n=1}^{\infty} n^3 v_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) = \frac{1}{336}\sum_{n=1}^{\infty}(6n^3-7n^5+n^9)v_n.$$

Proof. Identity (a) follows from Theorem 2.2(b) by choosing $f(n) = n^5$ and $g(n) = n^2$. Identity (b) follows by equating the coefficients of θ in identity (a) and then by using $u_{2n} = u_n - v_n$. ■

THEOREM 17.13.

$$\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big)^3 = -\frac{1}{20160}\sum_{n=1}^{\infty} n^8 v_n \sin n\theta - \frac{1}{1008}\Big(1-504\sum_{n=1}^{\infty} n^5 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big) + \frac{1}{960}\Big(1+240\sum_{n=1}^{\infty} n^3 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^4 v_n \sin n\theta\Big).$$

Proof. By multiplying both sides of the equation in Theorem 17.3(b) by $\sum_{n=1}^{\infty} n^2 v_n \sin n\theta$ and using the equations in Theorems 17.10(a) and 17.12(a), we obtain

$$\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big)^3 = -\frac{1}{20160}\sum_{n=1}^{\infty} n^8 v_n \sin n\theta - \frac{1}{5040}\Big[5-84\sum_{n=1}^{\infty} n u_{2n} + 840\sum_{n=1}^{\infty} n^3 u_{2n} - 20160\Big(\sum_{n=1}^{\infty} n u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^3 u_{2n}\Big) - 756\sum_{n=1}^{\infty} n^5 u_{2n} - 2520\sum_{n=1}^{\infty} n^4 v_n^2\Big]\Big(\sum_{n=1}^{\infty} n^2 v_n \sin n\theta\Big) + \frac{1}{960}\Big(1+240\sum_{n=1}^{\infty} n^3 u_{2n}\Big)\Big(\sum_{n=1}^{\infty} n^4 v_n \sin n\theta\Big).$$

Then the result follows by substituting the equation into Theorem 17.3(e). ■

THEOREM 17.14.

$$\Big(\sum_{n=1}^{\infty} n u_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) - 2\Big(\sum_{n=1}^{\infty} n v_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) - 6\Big(\sum_{n=1}^{\infty} n^3 u_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) = -\frac{1}{120}\sum_{n=1}^{\infty}(8n^7-5n^5-3n^3)v_n.$$

Proof. The result follows by equating the right-hand sides of the equations in Theorems 17.3(f) and 17.10(c). ■

THEOREM 17.15.

$$\Big(\sum_{n=1}^{\infty} n^5 u_n\Big)\Big(\sum_{n=1}^{\infty} n^3 v_n\Big) - 2\Big(\sum_{n=1}^{\infty} n^3 v_n\Big)\Big(\sum_{n=1}^{\infty} n^5 v_n\Big) = \frac{1}{504}\sum_{n=1}^{\infty}(n^3-n^9)v_n.$$

Proof. The result follows by multiplying the equation in Theorem 17.3(g) by -5 and adding to the equation in Theorem 17.12(b). ■

THEOREM 17.16.

$$\Big(\sum_{n=1}^{\infty} n v_{2n} \cos 2n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big) = \frac{1}{4}\sum_{n=0}^{\infty}(n^2+n)v_{2n+1}\sin(2n+1)\theta + \frac{1}{2}\Big(4\sum_{n=1}^{\infty} n u_{4n} - \sum_{n=1}^{\infty} n u_{2n}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big).$$

Proof. Let

$$f(n)=\begin{cases} n/2 & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases} \qquad g(n)=\begin{cases} 1 & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Then, by Theorem 2.2(b), we obtain

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} nv_{2n}\cos 2n\theta\Big)&\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)\\ &= \frac{1}{4}\sum_{n=0}^{\infty}(n^2+n)v_{2n+1}\sin(2n+1)\theta\\ &\quad+\frac{1}{2}\Big(2\sum_{n=1}^{\infty} nu_{4n}-\sum_{n=0}^{\infty}(2n+1)u_{4n+2}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big).\end{aligned}$$

The result follows by using

$$\sum_{n=0}^{\infty}(2n+1)u_{4n+2}=\sum_{n=1}^{\infty} nu_{2n}-2\sum_{n=1}^{\infty} nu_{4n}. \quad \blacksquare \tag{17.1}$$

THEOREM 17.17.

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)^3 &= -\frac{1}{8}\sum_{n=0}^{\infty}(n^2+n)v_{2n+1}\sin(2n+1)\theta\\ &\quad+\frac{3}{4}\Big(\sum_{n=1}^{\infty} nu_{2n}-2\sum_{n=1}^{\infty} nu_{4n}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big).\end{aligned}$$

Proof. Multiplying both sides of the equation in Theorem 3.3 by $\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta$ and using Theorem 17.16, we obtain

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} v_{2n+1}&\sin(2n+1)\theta\Big)^3\\ &= -\frac{1}{8}\sum_{n=0}^{\infty}(n^2+n)v_{2n+1}\sin(2n+1)\theta\\ &\quad+\frac{1}{4}\Big(\sum_{n=1}^{\infty} nu_{2n}-4\sum_{n=1}^{\infty} nu_{4n}+2\sum_{n=1}^{\infty} nv_{2n}\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big).\end{aligned}$$

The result follows by using $v_{2n}=u_{2n}-u_{4n}$. ■

We remark that using (17.1) in Theorem 17.17 we recover Theorem 4.3.

THEOREM 17.18.

$$\begin{aligned}\sum_{n=0}^{\infty}(2n+1)^2 v_{2n+1}\sin(2n+1)\theta &= 64\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big)^3\\ &\quad+\Big(1-24\sum_{n=1}^{\infty} nu_{2n}+48\sum_{n=1}^{\infty} nv_{2n}\cos 2n\theta\Big)\Big(\sum_{n=0}^{\infty} v_{2n+1}\sin(2n+1)\theta\Big).\end{aligned}$$

Proof. The identity follows from Theorems 17.16 and 17.17. ■

Equating coefficients of θ in Theorem 17.18, we recover identities due to Liu [9, formulae (8.15), (8.16)].

18. Conclusion

In investigating the relationship between Liouville's identities and Lambert series, we have exhibited the connection between the Lambert series for

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^2, \quad \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^3, \quad \Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^4$$

and arithmetic formulae for the sums

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (F(a+b) - F(a-b)),$$

$$\sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n}} (F(a+b+c) - F(a-b+c) - F(a+b-c) + F(a-b-c)),$$

$$\begin{aligned}\sum_{\substack{(a,b,c,d,x,y,z,w)\in\mathbb{N}^8 \\ ax+by+cz+dw=n}} & (F(a+b+c+d) - F(a-b+c+d) - F(a+b-c+d) \\ & - F(a+b+c-d) + F(a-b-c+d) + F(a-b+c-d) \\ & + F(a+b-c-d) - F(a-b-c-d))\end{aligned}$$

respectively. These results suggest that a Lambert series for

$$\Big(\sum_{n=1}^{\infty} u_n \sin n\theta\Big)^k \qquad (k \geq 2)$$

might lead to an arithmetic formula for the sum

$$\begin{aligned}\sum_{\substack{(a_1,\ldots,a_k,x_1,\ldots,x_k)\in\mathbb{N}^{2k} \\ a_1x_1+\cdots+a_kx_k=n}} & (F(a_1+a_2+\cdots+a_k) - F(a_1-a_2+a_3+\cdots+a_k) \\ & - F(a_1+a_2-a_3+\cdots+a_k) - \cdots - F(a_1+a_2+a_3+\cdots-a_k) \\ & + F(a_1-a_2-a_3+a_4+\cdots+a_k) \\ & + \cdots + F(a_1+a_2+\cdots-a_{k-1}-a_k) \\ & - \cdots + (-1)^{k-1}F(a_1-a_2-\cdots-a_k)),\end{aligned}$$

where $n \in \mathbb{N}$ and $F(-x) = (-1)^k F(x)$.

In this paper we have shown the equivalence of Liouville's arithmetic identities and identities of Ramanujan type for Lambert series based on our three theorems for products of Lambert series. Along the way we obtained numerous identities for Lambert series. We should mention that some of these Lambert series identities can be proved using properties of theta functions and elliptic functions (see for example Liu [9] and Venkatachaliengar [27]). We should also mention that Venkatachaliengar [27] has given a beautiful multiplicative identity for the product of two series of Lambert type called Jordan–Kronecker functions (see also Cooper [3]). There are many deep and important connections between Lambert series identities, theta function identities, combinatorial number theory and modular forms. We encourage the reader to explore this exciting area of mathematics. Hopefully in this article we have made clear its connection to the wonderful arithmetic identities of Liouville.

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