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ON THE DISTRIBUTION OF CYCLIC CUBIC FIELDS WITH INDEX 2

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In this paper we prove an analogue of Mertens' theorem for primes of each of the forms a^2+27b^2 and $4a^2+2ab+7b^2$ and then use this result to determine an asymptotic formula for the number of positive integers $n \leq x$ which are discriminants of cyclic cubic fields with each such field having field index 2.

Keywords: Discriminant; field index; cyclic cubic field.

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1. Introduction

Let n be a positive integer. It is known that n is the discriminant of a cyclic cubic field if and only if

$$n = 81, \quad (q_1 \cdots q_r)^2 \quad \text{or} \quad 81(q_1 \cdots q_r)^2,$$

where $r \in \mathbb{N}$ and $q_1, \ldots, q_r$ are distinct primes $\equiv 1 \pmod 3$, see for example [3] and [4]. We showed in [8] that the number of $n \leq x$ which are discriminants of cyclic cubic fields is

$$\frac{3^{1/4}}{\pi} \frac{10}{9} \frac{x^{1/2}}{\sqrt{\log x}} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (1 + o(1)),$$

as $x \to +\infty$, where in the product p runs through primes.

In this paper we determine the number $T(x)$ of $n \leq x$ which are discriminants of cyclic cubic fields with each such field having field index equal to 2. In order to do this, we prove in Sec. 2 an analogue of Mertens' theorem for primes of the form a^2+27b^2 and primes of the form $4a^2+2ab+7b^2$, see Theorem 2.12. A prime is represented by at most one of these two forms, and each form represents infinitely many primes. We also make use of results of Wirsing [11] and Odoni [6], see Proposition 3.1. We prove the following theorem in Sec. 3.

Theorem 1.1. *As* $x \to +\infty$

$$T(x) = 2^{2/3}3^{1/12}\pi^{-1/6}\theta^{-1/3}\Gamma\left(\frac{1}{6}\right)^{-1} \prod_{p=a^2+27b^2} \left(1-\frac{1}{p^2}\right)^{5/6} \times \prod_{p=4a^2+2ab+7b^2} \left(1-\frac{1}{p^2}\right)^{-1/6} \frac{x^{1/2}}{(\log x)^{5/6}} \left(1+O\left(\frac{1}{(\log x)^{1-\epsilon}}\right)\right)$$

for any ϵ *with* $0<\epsilon<1$, *where the constant* θ *is defined in* (2.12).

2. Mertens' Theorem for Primes $p = a^2 + 27b^2$ and $p = 4a^2 + 2ab + 7b^2$

For $x \in \mathbb{R}$ with $x \geq 2$ we define

$$\pi_1(x) = \sum_{\substack{p\leq x\\ p=a^2+27b^2}} 1, \qquad \pi_2(x) = \sum_{\substack{p\leq x\\ p=4a^2+2ab+7b^2}} 1, \tag{2.1}$$

$$\theta_1(x) = \sum_{\substack{p\leq x\\ p=a^2+27b^2}} \log p, \qquad \theta_2(x) = \sum_{\substack{p\leq x\\ p=4a^2+2ab+7b^2}} \log p, \tag{2.2}$$

$$\kappa_1(x) = \sum_{\substack{p\leq x\\ p=a^2+27b^2}} \frac{\log p}{p}, \qquad \kappa_2(x) = \sum_{\substack{p\leq x\\ p=4a^2+2ab+7b^2}} \frac{\log p}{p}, \tag{2.3}$$

$$\lambda_1(x) = \sum_{\substack{p\leq x\\ p=a^2+27b^2}} \frac{1}{p}, \qquad \lambda_2(x) = \sum_{\substack{p\leq x\\ p=4a^2+2ab+7b^2}} \frac{1}{p}. \tag{2.4}$$

Lemma 2.1. *As* $x \to +\infty$

$$\pi_1(x) = \frac{1}{6}\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

and

$$\pi_2(x) = \frac{1}{3}\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Proof. Let $f = f(x,y) = ax^2+bxy+cy^2$ be a primitive integral binary quadratic form with a nonsquare discriminant D. Set $f^{-1} = ax^2-bxy+cy^2$. Let $[f]$ denote the class of f under the action of the modular group. Let $h(D)$ denote the number of classes of forms of discriminant D. Let

$$\epsilon(f) = \begin{cases} 2, & \text{if } [f] = [f^{-1}], \\ 1, & \text{if } [f] \neq [f^{-1}]. \end{cases}$$

Landau [5] has shown that

$$\sum_{\substack{p \le x \\ p \text{ rep. by } f}} 1 = \frac{1}{\varepsilon(f)h(D)} \text{li } x + O_{f,\alpha}\left(xe^{-(\log x)^{1/\alpha}}\right),$$

as $x \to +\infty$, for some positive constant α. We choose $D = -108$. Here $h(-108) = 3$ and representatives of the three classes of positive-definite, primitive, integral, binary quadratic forms of discriminant -108 are

$$x^2 + 27y^2, \quad 4x^2 + 2xy + 7y^2, \quad 4x^2 - 2xy + 7y^2.$$

By Landau's theorem, as

$$\text{li } x = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{2.5}$$

and

$$xe^{-(\log x)^{1/\alpha}} = O\left(\frac{x}{\log^2 x}\right),$$

we have

$$\pi_1(x) = \frac{1}{6}\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

and

$$\pi_2(x) = \frac{1}{3}\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

as asserted. □

Lemma 2.2. *As* $x \to +\infty$

$$\theta_1(x) = \frac{1}{6}x + O\left(\frac{x}{\log x}\right),$$
$$\theta_2(x) = \frac{1}{3}x + O\left(\frac{x}{\log x}\right).$$

Proof. By partial summation [2, Theorem 421, p. 346] we have

$$\theta_1(x) = \pi_1(x)\log x - \int_2^x \frac{\pi_1(t)}{t}dt, \quad x \ge 2.$$

Appealing to Lemma 2.1 we obtain

$$\theta_1(x) = \left(\frac{1}{6}\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)\right)\log x + O\left(\int_2^x \frac{dt}{\log t}\right) = \frac{1}{6}x + O\left(\frac{x}{\log x}\right)$$

by (2.5). We can treat $\theta_2(x)$ similarly. □

Lemma 2.3. *As* $x \to +\infty$

$$\kappa_1(x) = \frac{1}{6}\log x + O(\log\log x),$$
$$\kappa_2(x) = \frac{1}{3}\log x + O(\log\log x).$$

Proof. By partial summation we have

$$\kappa_1(x) = \frac{\theta_1(x)}{x} + \int_2^x \frac{\theta_1(t)}{t^2}dt, \quad x \geq 2.$$

By Lemma 2.2 we obtain

$$\int_2^x \frac{\theta_1(t)}{t^2}dt = \int_2^x \frac{\frac{1}{6}t + O\left(\frac{t}{\log t}\right)}{t^2}dt = \frac{1}{6}\log x - \frac{1}{6}\log 2 + O\left(\int_2^x \frac{dt}{t\log t}\right)$$
$$= \frac{1}{6}\log x + O(1) + O(\log\log x),$$

which gives the asserted result. Similarly for $\kappa_2(x)$. □

Lemma 2.4. *As* $x \to +\infty$

$$\lambda_1(x) = \frac{1}{6}\log\log x + c_1 + O\left(\frac{1}{\log\log x}\right),$$
$$\lambda_2(x) = \frac{1}{3}\log\log x + c_2 + O\left(\frac{1}{\log\log x}\right),$$

where

$$c_1 = \frac{1}{6} - \frac{1}{6}\log\log 2 + \int_2^\infty \frac{\kappa_1(x) - \frac{1}{6}\log x}{x\log^2 x}dx,$$
$$c_2 = \frac{1}{3} - \frac{1}{3}\log\log 2 + \int_2^\infty \frac{\kappa_2(x) - \frac{1}{3}\log x}{x\log^2 x}dx.$$

Proof. Define

$$\tau_1(x) = \kappa_1(x) - \frac{1}{6}\log x$$

so that by Lemma 2.3 we have

$$\kappa_1(x) = \frac{1}{6}\log x + \tau_1(x), \quad \tau_1(x) = O(\log\log x). \tag{2.6}$$

By partial summation we have

$$\lambda_1(x) = \frac{\kappa_1(x)}{\log x} + \int_2^x \frac{\kappa_1(t)}{t\log^2 t}dt.$$

By (2.6) we obtain

$$\lambda_1(x) = \frac{1}{6} + O\left(\frac{\log\log x}{\log x}\right) + \frac{1}{6}\log\log x - \frac{1}{6}\log\log 2 + \int_2^x \frac{\tau_1(t)}{t\log^2 t}dt.$$

Now

$$\begin{aligned}\int_x^\infty \frac{\tau_1(t)}{t\log^2 t} &= O\left(\int_x^\infty \frac{\log\log t}{t\log^2 t}dt\right)\\ &= O\left(\int_x^\infty \frac{dt}{t\log t(\log\log t)^2}\right)\\ &= O\left(\frac{1}{\log\log x}\right),\end{aligned}$$

so that

$$\lambda_1(x) = \frac{1}{6}\log\log x + \frac{1}{6} - \frac{1}{6}\log\log 2 + \int_2^\infty \frac{\tau_1(t)}{t\log^2 t}dt + O\left(\frac{1}{\log\log x}\right),$$

as asserted. Similarly for $\lambda_2(x)$. □

Lemma 2.5. *For each prime p set*

$$\chi(p) = \begin{cases} 2, & \text{if } p = a^2 + 27b^2,\\ -1, & \text{if } p = 4a^2 + 2ab + 27b^2,\\ 0, & \text{otherwise.}\end{cases}$$

Then

(i) $$\sum_{p\le x} \frac{\chi(p)}{p} = 2c_1 - c_2 + O\left(\frac{1}{\log\log x}\right), \qquad \text{as } x \to +\infty.$$

(ii) $$\sum_p \frac{\chi(p)}{p} \ (\text{converges}) = 2c_1 - c_2.$$

(iii) $$2c_1 - c_2 = \int_2^\infty \frac{2\kappa_1(x) - \kappa_2(x)}{x\log^2 x}dx.$$

Proof. (i) As $x \to +\infty$ we have

$$\begin{aligned}\sum_{p\le x} \frac{\chi(p)}{p} &= 2\sum_{\substack{p\,\le\, x\\ p\,=\,a^2+27b^2}} \frac{1}{p} - \sum_{\substack{p\,\le\, x\\ p\,=\,4a^2+2ab+27b^2}} \frac{1}{p}\\ &= 2\lambda_1(x) - \lambda_2(x)\\ &= 2\left(\frac{1}{6}\log\log x + c_1 + O\left(\frac{1}{\log\log x}\right)\right)\\ &\quad - \left(\frac{1}{3}\log\log x + c_2 + O\left(\frac{1}{\log\log x}\right)\right)\\ &= 2c_1 - c_2 + O\left(\frac{1}{\log\log x}\right),\end{aligned}$$

by Lemma 2.4.

(ii) Letting $x \to +\infty$ in part (i) we obtain the asserted result.
(iii) This follows immediately from Lemma 2.4. □

Lemma 2.6. *The infinite product*

$$\prod_p \left(1 - \frac{1}{p}\right)^{\chi(p)}$$

converges.

Proof. Set

$$\gamma(p) = \begin{cases} -2 + \frac{1}{p}, & \text{if } p = a^2 + 27b^2, \\ 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots, & \text{if } p = 4a^2 + 2ab + 7b^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$1 + \frac{\gamma(p)}{p} = \begin{cases} \left(1 - \frac{1}{p}\right)^2, & \text{if } p = a^2 + 27b^2, \\ \frac{1}{1-\frac{1}{p}}, & \text{if } p = 4a^2 + 2ab + 7b^2, \\ 1, & \text{otherwise.} \end{cases}$$

Hence

$$1 + \frac{\gamma(p)}{p} = \left(1 - \frac{1}{p}\right)^{\chi(p)}. \tag{2.7}$$

Further

$$\gamma(p) = -\chi(p) + s(p), \tag{2.8}$$

where

$$s(p) = \begin{cases} \frac{1}{p}, & \text{if } p = a^2 + 27b^2, \\ \frac{1}{p} + \frac{1}{p^2} + \cdots, & \text{if } p = 4a^2 + 2ab + 7b^2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$0 \le s(p) \le \frac{1}{p} + \frac{1}{p^2} + \cdots = \frac{1}{p-1} \le \frac{2}{p}$$

so that the infinite series

$$\sum_p \frac{s(p)}{p} \tag{2.9}$$

converges. Hence, by Lemma 2.5(ii), (2.8) and (2.9), we see that

$$\sum_p \frac{\gamma(p)}{p} = -\sum_p \frac{\chi(p)}{p} + \sum_p \frac{s(p)}{p} \tag{2.10}$$

converges. Further

$$|\gamma(p)| \le |\chi(p)| + |s(p)| \le 2 + \frac{2}{p} \le 3$$

so that the infinite series

$$\sum_p \frac{\gamma(p)^2}{p^2} \tag{2.11}$$

converges. From the convergence of the infinite series (2.10) and (2.11), we deduce from [1, Sec. 41, p. 109] that the infinite product

$$\prod_p \left(1 + \frac{\gamma(p)}{p}\right)$$

converges. Then, from (2.7), we see that the infinite product

$$\prod_p \left(1 - \frac{1}{p}\right)^{\chi(p)}$$

converges as asserted. □

We set

$$\theta := \prod_p \left(1 - \frac{1}{p}\right)^{\chi(p)}. \tag{2.12}$$

Lemma 2.7. *The infinite series*

(i)
$$\sum_{p=a^2+27b^2} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$$

and

(ii)
$$\sum_{p=4a^2+2ab+7b^2} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$$

converge.

Proof. We have

$$\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} = -\frac{1}{2p^2} - \frac{1}{3p^3} - \cdots$$

so that

$$\left|\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right| \le \frac{1}{2p^2} + \frac{1}{2p^3} + \cdots = \frac{1}{2p(p-1)} \le \frac{1}{p^2},$$

proving the assertions. □

In view of Lemma 2.7 we may define constants d_1 and d_2 by

$$d_1 = \sum_{p=a^2+27b^2} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right), \quad d_2 = \sum_{p=4a^2+2ab+7b^2} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right).$$

We also set

$$P_1(x) = \prod_{\substack{p \le x \\ p=a^2+27b^2}} \left(1 - \frac{1}{p}\right), \quad P_2(x) = \prod_{\substack{p \le x \\ p = 4a^2+2ab+7b^2}} \left(1 - \frac{1}{p}\right).$$

Lemma 2.8. *As* $x \to +\infty$

$$P_1(x) = f_1(\log x)^{-1/6} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)$$

and

$$P_2(x) = f_2(\log x)^{-1/3} \left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

where

$$f_1 = e^{-c_1+d_1}, \quad f_2 = e^{-c_2+d_2}.$$

Proof. Appealing to (2.4) and Lemma 2.4, we obtain

$$\begin{aligned}
\log P_1(x) &= \sum_{\substack{p \le x \\ p=a^2+27b^2}} \log\left(1 - \frac{1}{p}\right) \\
&= -\sum_{\substack{p \le x \\ p=a^2+27b^2}} \frac{1}{p} + \sum_{\substack{p \le x \\ p=a^2+27b^2}} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) \\
&= -\lambda_1(x) + d_1 - \sum_{\substack{p > x \\ p=a^2+27b^2}} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) \\
&= -\frac{1}{6}\log\log x - c_1 + O\left(\frac{1}{\log\log x}\right) + d_1 + O\left(\sum_{n>x} \frac{1}{n^2}\right) \\
&= -\frac{1}{6}\log\log x - c_1 + d_1 + O\left(\frac{1}{\log\log x}\right) + O\left(\frac{1}{x}\right) \\
&= -\frac{1}{6}\log\log x - c_1 + d_1 + O\left(\frac{1}{\log\log x}\right),
\end{aligned}$$

so

$$\begin{aligned}
P_1(x) &= e^{-\frac{1}{6}\log\log x - c_1 + d_1 + O\left(\frac{1}{\log\log x}\right)} \\
&= (\log x)^{-1/6} e^{-c_1+d_1} \left(1 + O\left(\frac{1}{\log\log x}\right)\right) \\
&= f_1(\log x)^{-1/6} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)
\end{aligned}$$

with $f_1 = e^{-c_1+d_1}$. $P_2(x)$ can be treated similarly. □

Lemma 2.9.

$$f_1 f_2 = e^{-\gamma/2} 2^{1/2} 3^{-1/4} \pi^{1/2} \prod_{p \equiv 1 \, (\mathrm{mod}\ 3)} \left(1 - \frac{1}{p^2}\right)^{1/2}.$$

Proof. Since

$$p = a^2 + 27b^2 \text{ or } p = 4a^2 + 2ab + 7b^2 \Leftrightarrow p \equiv 1 \, (\mathrm{mod}\, 3),$$

we have by Mertens' theorem for arithmetic progressions, see [8] or [10],

$$\begin{aligned} P_1(x)P_2(x) &= \prod_{\substack{p \le x \\ p \equiv 1 \, (\mathrm{mod}\, 3)}} \left(1 - \frac{1}{p}\right) \\ &= e^{-\gamma/2} 2^{1/2} 3^{-1/4} \pi^{1/2} \prod_{p \equiv 1 \, (\mathrm{mod}\, 3)} \left(1 - \frac{1}{p^2}\right)^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}), \end{aligned}$$

as $x \to +\infty$. By Lemma 2.8 we have

$$P_1(x)P_2(x) = f_1 f_2 (\log x)^{-1/2} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

Hence

$$f_1 f_2 = e^{-\gamma/2} 2^{1/2} 3^{-1/4} \pi^{1/2} \prod_{p \equiv 1 \, (\mathrm{mod}\, 3)} \left(1 - \frac{1}{p^2}\right)^{1/2},$$

as asserted. □

Lemma 2.10.

$$f_1^2 f_2^{-1} = \theta.$$

Proof. By Lemma 2.8 and (2.12) we have

$$\begin{aligned} f_1^2 f_2^{-1} &= \lim_{x \to +\infty} P_1^2(x) P_2^{-1}(x) \\ &= \lim_{x \to +\infty} \prod_{\substack{p \le x \\ p = a^2 + 27b^2}} \left(1 - \frac{1}{p}\right)^2 \prod_{\substack{p \le x \\ p = 4a^2 + 2ab + 7b^2}} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \lim_{x \to +\infty} \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{\chi(p)} \\ &= \prod_{p} \left(1 - \frac{1}{p}\right)^{\chi(p)} \\ &= \theta, \end{aligned}$$

as asserted. □

Lemma 2.11.

$$f_1 = e^{-\gamma/6} 2^{1/6} 3^{-1/12} \pi^{1/6} \theta^{1/3} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/6},$$

$$f_2 = e^{-\gamma/3} 2^{1/3} 3^{-1/6} \pi^{1/3} \theta^{-1/3} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/3}.$$

Proof. This follows immediately from Lemmas 2.9 and 2.10. □

Finally, from Lemmas 2.8 and 2.11, we obtain

Theorem 2.12. *As* $x \to +\infty$

$$\prod_{\substack{p \le x \\ p = a^2 + 27b^2}} \left(1 - \frac{1}{p}\right) = e^{-\gamma/6} 2^{1/6} 3^{-1/12} \pi^{1/6} \theta^{1/3} \prod_{p \equiv 1 (\operatorname{mod} 3)} \left(1 - \frac{1}{p^2}\right)^{1/6} \times (\log x)^{-1/6} \left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

$$\prod_{\substack{p \le x \\ p = 4a^2 + 2ab + 7b^2}} \left(1 - \frac{1}{p}\right) = e^{-\gamma/3} 2^{1/3} 3^{-1/6} \pi^{1/3} \theta^{-1/3} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/3} \times (\log x)^{-1/3} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

3. Proof of Theorem 1.1

It follows from [7] that the only positive integers n which are discriminants of cyclic cubic fields with each such field having field index 2 are those of the form $(q_1 q_2 \cdots q_r)^2$, where $r \in \mathbb{N}$ and $q_1, q_2, \ldots, q_r$ are distinct primes of the form $a^2 + 27b^2$ for some integers a and b. Let A denote the set of positive integers each of which is a product (possibly empty) of distinct primes of the form $a^2 + 27b^2$. Then for $x \ge 1$ we have

$$T(x) = Q(x^{1/2}) - 1,$$

where

$$Q(x) = \sum_{\substack{n \le x \\ n \in A}} 1.$$

From the work of Wirsing [11] and Odoni [6], we have (see [9]).

Proposition 3.1. *Let $f : \mathbb{N} \to \mathbb{R}$ be multiplicative with $0 \le f(n) \le 1$ for all $n \in \mathbb{N}$. Suppose that there are constants τ and β with $\tau > 0$ and $0 < \beta < 1$ such that*

$$\sum_{p \le x} f(p) = \tau \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\beta}}\right).$$

Then

$$\lim_{x \to +\infty} \frac{1}{(\log x)^{\tau}} \prod_{p \le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)$$

exists, and

$$\sum_{n \le x} f(n) = Ex(\log x)^{\tau - 1} + O\left(x(\log x)^{\tau - 1 - \beta}\right)$$

with

$$E = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \lim_{x \to +\infty} \frac{1}{(\log x)^{\tau}} \prod_{p \le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right).$$

We choose in Proposition 3.1

$$f(n) = \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

Clearly f is multiplicative and $0 \le f(n) \le 1$ for all $n \in \mathbb{N}$. Further, by Lemma 2.1, we have

$$\sum_{p \le x} f(p) = \sum_{\substack{p \le x \\ p = a^2 + 27b^2}} 1 = \pi_1(x) = \frac{1}{6} \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

so that we can take

$$\tau = \frac{1}{6}, \quad \beta = 1 - \epsilon \quad (0 < \epsilon < 1).$$

By Proposition 3.1 we see that

$$\begin{aligned} &\lim_{x \to +\infty} \frac{1}{(\log x)^{\tau}} \prod_{p \le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right) \\ &= \lim_{x \to +\infty} (\log x)^{-1/6} \prod_{\substack{p \le x \\ p = a^2 + 27b^2}} \left(1 + \frac{1}{p}\right) \end{aligned}$$

exists, and equals F, say. Hence

$$\begin{aligned} Q(x) &= \sum_{\substack{n \le x \\ n \in A}} 1 \\ &= \sum_{n \le x} f(n) \\ &= Ex(\log x)^{\tau-1} + O(x(\log x)^{\tau-1-\beta}) \\ &= \frac{e^{-\gamma/6}}{\Gamma\left(\frac{1}{6}\right)} Fx(\log x)^{-5/6} + O(x(\log x)^{-11/6+\epsilon}), \end{aligned}$$

as $x \to +\infty$. Next, by Theorem 2.12, we obtain

$$\begin{aligned} &(\log x)^{-1/6} \prod_{\substack{p \le x \\ p = a^2+27b^2}} \left(1 + \frac{1}{p}\right) \\ &= \frac{\prod_{\substack{p \le x \\ p=a^2+27b^2}} \left(1 - \frac{1}{p^2}\right)}{(\log x)^{1/6} \prod_{\substack{p \le x \\ p=a^2+27b^2}} \left(1 - \frac{1}{p}\right)} \\ &= \frac{\prod_{p=a^2+27b^2} \left(1 - \frac{1}{p^2}\right)\left(1 + O\left(\frac{1}{x}\right)\right)}{e^{-\gamma/6} 2^{1/6} 3^{-1/12} \pi^{1/6} \theta^{1/3} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/6} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)} \end{aligned}$$

so that

$$F = e^{\gamma/6} 2^{-1/6} 3^{1/12} \pi^{-1/6} \theta^{-1/3} \prod_{p=a^2+27b^2} \left(1 - \frac{1}{p^2}\right)^{5/6} \prod_{p=4a^2+2ab+7b^2} \left(1 - \frac{1}{p^2}\right)^{-1/6}.$$

Hence

$$\begin{aligned} Q(x) = {} & 2^{-1/6} 3^{1/12} \pi^{-1/6} \theta^{-1/3} \Gamma\left(\frac{1}{6}\right)^{-1} \prod_{p=a^2+27b^2} \left(1 - \frac{1}{p^2}\right)^{5/6} \\ & \times \prod_{p=4a^2+2ab+7b^2} \left(1 - \frac{1}{p^2}\right)^{-1/6} x(\log x)^{-5/6} + O\big(x(\log x)^{-11/6+\epsilon}\big). \end{aligned}$$

Finally

$$\begin{aligned} T(x) = {} & 2^{2/3} 3^{1/12} \pi^{-1/6} \theta^{-1/3} \Gamma\left(\frac{1}{6}\right)^{-1} \prod_{p=a^2+27b^2} \left(1 - \frac{1}{p^2}\right)^{5/6} \\ & \times \prod_{p=4a^2+2ab+7b^2} \left(1 - \frac{1}{p^2}\right)^{-1/6} x^{1/2}(\log x)^{-5/6} + O\big(x^{1/2}(\log x)^{-11/6+\epsilon}\big), \end{aligned}$$

which is the assertion of Theorem 1.1. □

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