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Infinitely Many Insolvable Diophantine Equations

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Let $f(x_1,\dots,x_n)$ be a quadratic form in n variables $x_1,\dots,x_n$ with integral coefficients, let p be a prime, and let k be a positive integer. The congruence $f(x_1,\dots,x_n)\equiv 0 \pmod{p^k}$ is said to be *solvable nontrivially* if there exist integers $x_1,\dots,x_n$ such that $f(x_1,\dots,x_n)\equiv 0 \pmod{p^k}$ with at least one of $x_1,\dots,x_n$ not divisible by p. Thus the congruence $x_1^2+x_2^2\equiv 0 \pmod{3^k}$ is solvable (with $x_1=x_2=0$) but is not solvable nontrivially as any solution x_1, x_2 satisfies $x_1\equiv x_2\equiv 0 \pmod 3$. Let m be a positive integer larger than 1. The congruence $f(x_1,\dots,x_n)\equiv 0 \pmod m$ is said to be solvable nontrivially if $f(x_1,\dots,x_n)\equiv 0 \pmod{p^k}$ is solvable nontrivially for each prime divisor p of m and each positive integer k such that p^k is the largest power of p dividing m (written $p^k \parallel m$). We note that the components x_i of the solution that are not divisible by p are not necessarily the same for different prime divisors p of m.

The Hasse-Minkowski theorem [**1**, p. 61] asserts that, if (a) there exist real numbers $r_1, \ldots, r_n$ not all zero such that

$$f(r_1, \ldots, r_n) = 0$$

and (b) the congruence $f(x_1, \ldots, x_n) \equiv 0 \pmod{m}$ is solvable nontrivially for every positive integer m greater than 1, then the equation $f(x_1, \ldots, x_n) = 0$ is solvable in integers $x_1, \ldots, x_n$ not all zero. However, if $f(x_1, \ldots, x_n)$ is a quadratic polynomial that is not a quadratic form (i.e., is not homogeneous), then (a) and (b) do not ensure that $f(x_1, \ldots, x_n) = 0$ is solvable in integers $x_1, \ldots, x_n$. An example is given in [**5**, p. 195]. We give infinitely many quadratic polynomials f in two variables such that (a) and (b) hold but $f(x_1, x_2) = 0$ is not solvable in integers x_1 and x_2.

We make use of a number of elementary arithmetic facts. In (i)–(vi) to follow, p is an odd prime, a, b, and c are integers, α is a positive integer, and $(\frac{*}{p})$ is the *Legendre symbol* defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ is solvable,} \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } p \mid a. \end{cases}$$

(i) If $a \equiv 1 \pmod{8}$, then the congruence $x^2 \equiv a \pmod{2^\alpha}$ is solvable [**2**, p. 13].

(ii) If $(\frac{a}{p}) = 1$, then the congruence $x^2 \equiv a \pmod{p^\alpha}$ is solvable [**2**, p. 13], [**4**, p. 137].

(iii) If $p \nmid a$, then the number of solutions of the congruence $ax^2 + bx + c \equiv 0 \pmod{p}$ is

$$1 + \left(\frac{b^2 - 4ac}{p}\right)$$

[**3**, pp. 68–69].

(iv) If $p \nmid a$ and $p \nmid b^2 - 4ac$, then

$$\sum_{x=0}^{p-1} \left(\frac{ax^2 + bx + c}{p}\right) = -\left(\frac{a}{p}\right)$$

[**3**, p. 82].

(v) If D is a positive integer that is not a perfect square and E is an integer such that $0 < |E| < \sqrt{D}$, then the equation $x^2 - Dy^2 = E$ is solvable in coprime positive integers x and y if and only if $E = h_n^2 - Dk_n^2$ for some convergent h_n/k_n of the continued fraction expansion of $\sqrt{D}$ with n in $\{0, 1, 2, \ldots, l - 1\}$, where l is the length of the period of the continued fraction expansion of $\sqrt{D}$ [**4**, p. 352].

If $p \nmid a$ and $p \nmid b^2 - 4ac$ we infer from (iii) and (iv) that the number of x in $\{0, 1, 2, \ldots, p - 1\}$ such that

$$\left(\frac{ax^2 + bx + c}{p}\right) = 1$$

is

$$\frac{1}{2}\sum_{\substack{x=0\\ p\nmid ax^2+bx+c}}^{p-1}\left(1+\left(\frac{ax^2+bx+c}{p}\right)\right)=\frac{1}{2}\left(p-1-\left(\frac{b^2-4ac}{p}\right)-\left(\frac{a}{p}\right)\right).$$

When $p \geq 5$ we have

$$\frac{1}{2}\left(p-1-\left(\frac{b^2-4ac}{p}\right)-\left(\frac{a}{p}\right)\right)\geq\frac{1}{2}(p-3)\geq 1.$$

Thus we conclude:

(vi) If $p \geq 5$, $p \nmid a$, and $p \nmid b^2-4ac$, then there exists an integer x such that

$$\left(\frac{ax^2+bx+c}{p}\right)=1.$$

Theorem. *If a is an integer greater than 1 each of whose prime divisors is congruent to either 1 or 3 modulo 8, then the equation*

$$2x^2-(2a^4+a^2)y^2+1=0 \tag{1}$$

is not solvable in integers x and y, but the congruence

$$2x^2-(2a^4+a^2)y^2+1\equiv 0 \pmod{m} \tag{2}$$

is solvable nontrivially for every positive integer m greater than 1.

Proof. As

$$(2a^2)^2 < 4a^4+2a^2 < (2a^2+1)^2,$$

the positive integer $4a^4+2a^2$ is not a perfect square. The continued fraction expansion of $\sqrt{4a^4+2a^2}$ is of period two and is given by

$$\sqrt{4a^4+2a^2}=[2a^2,\overline{2,4a^2}].$$

The convergents [**4**, p. 332] of this continued fraction expansion are

$$\frac{h_0}{k_0}=\frac{2a^2}{1},\quad \frac{h_1}{k_1}=\frac{4a^2+1}{2},\ldots.$$

Let

$$g_n=h_n^2-(4a^4+2a^2)k_n^2\quad (n=0,1,2,\ldots),$$

so that

$$g_0=-2a^2,\quad g_1=1,\ldots.$$

Since $|-2| < \sqrt{4a^4+2a^2}$, $g_0 \neq -2$, and $g_1 \neq -2$, statement (v) implies that the equation $x_1^2 - (4a^4+2a^2)y_1^2 = -2$ is not solvable in integers x_1 and y_1. Thus the equation (1) is not solvable in integers x and y.

Now let m be a positive integer larger than 1. We show that the congruence (2) is solvable nontrivially. Let p be a prime with $p \mid m$, say $p^\alpha \parallel m$, where α is a positive integer. We consider three cases according as (A) $p = 2$, (B) $p \neq 2$ and $p \mid 2a^4 + a^2$, or (C) $p \neq 2$ and $p \nmid 2a^4 + a^2$.

Case A: $p = 2$. As a is odd, $a^2 \equiv 1 \pmod 8$, so $2a^4 + a^2 - 2 \equiv 1 \pmod 8$. Thus, by (i), the congruence

$$z^2 \equiv 2a^4 + a^2 - 2 \pmod{2^\alpha}$$

is solvable. Then the congruence $2x_2^2 - (2a^4+a^2)y_2^2 + 1 \equiv 0 \pmod{2^\alpha}$ is solvable nontrivially with $x_2 = y_2 = t$, where t is the inverse of z modulo 2^α.

Case B: $p \neq 2$, $p \mid 2a^4 + a^2$. As $p \mid 2a^4 + a^2$ we have either $p \mid a$ or $p \mid 2a^2 + 1$. In the former case $p \equiv 1$ or $3 \pmod 8$ by assumption, so $(\frac{-2}{p}) = 1$. In the latter case $(2a)^2 \equiv -2 \pmod p$, so $(\frac{-2}{p}) = 1$. According to (ii) there exists in each case an integer w such that $w^2 \equiv -2 \pmod{p^\alpha}$. Then the congruence $2x_p^2 - (2a^4+a^2)y_p^2 + 1 \equiv 0 \pmod{p^\alpha}$ is solvable nontrivially with $x_p = lw$ and $y_p = 0$, where l is the inverse of 2 modulo p^α.

Case C: $p \neq 2$, $p \nmid 2a^4 + a^2$. Because $3 \mid 2a^4 + a^2$, we have $p \neq 3$, implying that $p \geq 5$. Set $b = 2a^4 + a^2$, so that $p \nmid b$. By (vi) there exists an integer y such that

$$\left(\frac{2by^2 - 2}{p}\right) = 1.$$

In view of (ii) there exists an integer z such that

$$z^2 \equiv 2by^2 - 2 \pmod{p^\alpha}.$$

Then the congruence $2x_p^2 - (2a^4+a^2)y_p^2 + 1 \equiv 0 \pmod{p^\alpha}$ is solvable nontrivially with $x_p = lz$ and $y_p = y$, where l is the inverse of 2 modulo p^α.

Finally, appealing to the Chinese remainder theorem, we choose integers x and y such that

$$x \equiv x_p \pmod{p^\alpha}, \qquad y \equiv y_p \pmod{p^\alpha}$$

for every prime divisor p of m and each positive integer α such that $p^\alpha \parallel m$. Then $2x^2 - (2a^4+a^2)y^2 + 1 \equiv 0 \pmod m$. ■

Let

$$f_m(x_1, x_2) = 2x_1^2 - (2 \cdot 3^{4m} + 3^{2m})x_2^2 + 1 \quad (m = 1, 2, \ldots).$$

Clearly each equation $f_m(x_1, x_2) = 0$ has a nontrivial real solution

$$(x_1, x_2) = \left(0, (1/\sqrt{2 \cdot 3^{4m} + 3^{2m}})\right),$$

so condition (a) of the Hasse-Minkowski theorem is satisfied. By the theorem condition (b) is also satisfied. On the other hand, none of these polynomials has a solution (x_1, x_2) in $\mathbb{Z}^2$.

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Forward Shifts and Backward Shifts in a Rearrangement of a Conditionally Convergent Series

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In [**2**, p. 57] it is proved that a rearrangement of a conditionally convergent series remains convergent (with unaltered sum), provided the series is rearranged in such a way that the forward shifts are bounded. It is remarked that there is a clear difference between forward shifts and backward shifts.

The purpose of this note is to show that in the context under consideration there is, in fact, no difference. The result stated in [**2**] holds as well with the assumption that the backward shifts are bounded.

Let $\sum x_n$ be a series, and let $\sum x_{\pi(n)}$ be a rearrangement determined by a permutation π of the natural numbers. The nth term, $x_n = x_{\pi(\pi^{-1}(n))}$, of the original series is shifted to the kth term of the rearranged series, where $k = \pi^{-1}(n)$. The forward (respectively, backward) shifts are bounded if and only if the differences $\pi^{-1}(n) - n$ (respectively, $n - \pi^{-1}(n)$) are bounded above.

We state the following theorem, where we do not a priori assume convergence of the series:

Theorem. *Let $\sum x_n$ be a series in a normed linear space with $\lim x_n = 0$, and let $\sum x_{\pi(n)}$ be a rearrangement of the series. Assume that either the forward shifts are bounded or the backward shifts are bounded. If s_n (respectively, t_n) denotes the nth partial sum of the series $\sum x_n$ (respectively, $\sum x_{\pi(n)}$), then the following statements hold:*