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DENSITY OF INTEGERS WHICH ARE DISCRIMINANTS OF CYCLIC CUBIC FIELDS

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Abstract

An asymptotic formula is given for the number of integers $\leq x$ which are discriminants of cyclic cubic fields.

Let n be a positive integer. It is known that n is the discriminant of a cyclic cubic field if and only if

$$n = 81, \quad (q_1 \cdots q_r)^2 \quad \text{or} \quad 81(q_1 \cdots q_r)^2, \tag{1}$$

where r is a positive integer and $q_1, \ldots, q_r$ are distinct primes $\equiv 1 \pmod 3$, see for example [2], [3]. Let A denote the set of positive integers which are the product of distinct primes $\equiv 1 \pmod 3$ including the empty product $= 1$. Then the number $K(x)$ of $n \leq x$ which are discriminants of cyclic cubic fields is for $x > 81 \times 7^2$,

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$$K(x) = 1 + \sum_{\substack{1<n\leq x^{1/2} \\ n\in A}} 1 + \sum_{\substack{1<n\leq x^{1/2}/9 \\ n\in A}} 1$$

so that

$$K(x) = Q(x^{1/2}) + Q(x^{1/2}/9) - 1, \tag{2}$$

where

$$Q(x) = \sum_{\substack{n\leq x \\ n\in A}} 1. \tag{3}$$

Our purpose is to determine the behaviour of $K(x)$ for large x. To do this we make use of a theorem of Wirsing [5], the prime number theorem for the arithmetic progression $\{3k+1 : k \in \mathbb{N}\}$, and Mertens' theorem for the arithmetic progression $\{3k+1 : k \in \mathbb{N}\}$ [4]. Throughout this paper p denotes a prime number.

Wirsing's theorem. *Let $f(n)$ be a multiplicative function such that*

$$f(n) \geq 0, \quad \text{for } n = 1, 2, 3, \ldots, \tag{4}$$

$$f(p^k) \leq c_1 c_2^k, \text{ for constants } c_1 \text{ and } c_2 \text{ with } c_2 < 2 \text{ and } k = 1, 2, 3, \ldots, \tag{5}$$

$$\sum_{p\leq x} f(p) = (\tau + o(1))\frac{x}{\log x}, \quad \text{as } x \to \infty, \tag{6}$$

then

$$\sum_{n\leq x} f(n) = \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o(1)\right)\frac{x}{\log x}\prod_{p\leq x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right), \tag{7}$$

as $x \to \infty$, where γ is Euler's constant.

Prime number theorem for primes $p \equiv 1 \pmod 3$. *As $x \to \infty$,*

$$\sum_{\substack{p\leq x \\ p\equiv 1 \pmod 3}} 1 = \left(\frac{1}{2} + o(1)\right)\frac{x}{\log x}. \tag{8}$$

Mertens' theorem for primes $p \equiv 1 (\mathrm{mod}\ 3)$. *As* $x \to \infty$,

$$\prod_{\substack{p \le x \\ p \equiv 1 (\mathrm{mod}\ 3)}} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma/2} 2^{1/2} \pi^{1/2}}{3^{1/4}} \prod_{\substack{p \\ p \equiv 1 (\mathrm{mod}\ 3)}} \left(1 - \frac{1}{p^2}\right)^{1/2} (\log x)^{-1/2}$$

$$+ O((\log x)^{-3/2}). \tag{9}$$

We are now ready to prove

Theorem 1. *As* $x \to \infty$,

$$Q(x) = \left(\frac{3^{1/4}}{2^{1/2} \pi} \prod_{p \equiv 1 (\mathrm{mod}\ 3)} \left(1 - \frac{1}{p^2}\right)^{1/2} + o(1) \right) \frac{x}{\sqrt{\log x}}.$$

Proof. We let

$$f(n) = \begin{cases} 1, & \text{if } n \in A, \\ 0, & \text{if } n \notin A. \end{cases}$$

Clearly $f(n)$ is a multiplicative function satisfying (4), (5) (with $c_1 = c_2 = 1$) and (6) (with $\tau = \frac{1}{2}$ by (8)). Hence, by Wirsing's theorem, we obtain as $\Gamma(1/2) = \sqrt{\pi}$,

$$Q(x) = \sum_{\substack{n \le x \\ n \in A}} 1 = \frac{e^{-\gamma/2}}{\sqrt{\pi}} (1 + o(1)) \frac{x}{\log x} \prod_{\substack{p \le x \\ p \equiv 1 (\mathrm{mod}\ 3)}} \left(1 + \frac{1}{p}\right), \quad \text{as } x \to \infty.$$

Next for $x \to \infty$, we have

$$\prod_{\substack{p \le x \\ p \equiv 1 (\mathrm{mod}\ 3)}} \left(1 + \frac{1}{p}\right) = \frac{R}{S},$$

where

$$R = \prod_{\substack{p \le x \\ p \equiv 1 (\mathrm{mod}\ 3)}} \left(1 - \frac{1}{p^2}\right) = (1 + o(1)) \prod_{p \equiv 1 (\mathrm{mod}\ 3)} \left(1 - \frac{1}{p^2}\right)$$

and by (9)

$$S = \prod_{\substack{p \le x \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right)$$

$$= \frac{e^{-\gamma/2} 2^{1/2} \pi^{1/2}}{3^{1/4}} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (1 + o(1)) \frac{1}{\sqrt{\log x}}$$

so that

$$\prod_{\substack{p \le x \\ p \equiv 1 \pmod 3}} \left(1 + \frac{1}{p}\right) = \frac{e^{\gamma/2} 3^{1/4}}{2^{1/2} \pi^{1/2}} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (1 + o(1)) \sqrt{\log x}\,.$$

Finally

$$Q(x) = \frac{3^{1/4}}{2^{1/2} \pi} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (1 + o(1)) \frac{x}{\sqrt{\log x}}, \quad \text{as } x \to \infty.$$

From Theorem 1 and (2), we obtain

Theorem 2. *As* $x \to \infty$,

$$K(x) = \frac{3^{1/4}}{\pi} \frac{10}{9} \frac{x^{1/2}}{\sqrt{\log x}} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right)^{1/2} (1 + o(1)).$$

H. Cohn [1] has shown that the number $N(x)$ of cyclic cubic fields of discriminant $\le x$ satisfies

$$N(x) = \frac{3^{1/2}}{2\pi} \frac{11}{18} \prod_{p \equiv 1 \pmod 3} \frac{(p+2)(p-1)}{p(p+1)} x^{1/2} (1 + o(1)),$$

as $x \to \infty$.

References

[1] H. Cohn, The density of abelian cubic fields, Proc. Amer. Math. Soc. 5 (1954), 476-477.

[2] H. Hasse, Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen, kubischen und biquadratischen Zahlkörpern, Berlin, 1950, p. 10.

[3] J. G. Huard, B. K. Spearman and K. S. Williams, A short proof of the formula for the conductor of an abelian cubic field, Norske Vid. Selsk. Skr. (Trondheim) 2 (1994), 8 pp.

[4] K. S. Williams, Mertens' theorem for arithmetic progressions, J. Number Theory 6 (1974), 353-359.

[5] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, Math. Annalen 143 (1961), 75-102.

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