Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions

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1 Introduction

Let $\sigma_m(n)$ denote the sum of the m th powers of the positive divisors of the positive integer n. We set $\tau(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$. If l is not a positive integer we set $\sigma_m(l) = 0$. In Section 2 we prove an elementary arithmetic identity (Theorem 1), which generalizes a classical formula of Liouville given in [21]. In Section 3 we use this identity to evaluate in an elementary manner thirty-seven convolution sums involving the function σ_m treated by Lahiri in [14] using more advanced techniques. Some of these convolution sums had been considered earlier by Glaisher [9], [10], [11], MacMahon [24], [25, pp. 303–341] and Ramanujan [30], [31, pp. 136–162 and commentary pp. 365–368]. MacMahon used his formulae to deduce theorems about partitions. In Sections 4 and 5 we prove in an elementary manner some extensions of convolution formulae proved by Melfi in [26] using the theory of modular forms. In Section 6 we use Theorem 1 to derive in an elementary manner some formulae for the number of representations of positive integers as sums of triangular numbers proved by Ono, Robins and Wahl in [27] using more sophisticated methods. In Section 7 we give one illustrative example of how Theorem 1 can be used to determine the number of representations of a positive integer by a quaternary form. Finally in Section 8 we consider some further convolution sums of the type considered in Section 4.

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2 Generalization of Liouville's Formula

We prove

Theorem 1. *Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b) \tag{2.1}$$

for all integers a, b, x and y. Then

$$\sum_{ax+by=n} (f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y)$$
$$-f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y))$$

$$= \sum_{d\,|\,n} \sum_{x<d} (f(0,n/d,x,d) + f(n/d,0,d,x) + f(n/d,n/d,d-x,-x)$$
$$-f(x,x-d,n/d,n/d) - f(x,d,0,n/d) - f(d,x,n/d,0)), \tag{2.2}$$

where the sum on the left hand side of (2.2) is over all positive integers a, b, x, y satisfying $ax+by=n$, the inner sum on the right hand side is over all positive integers x satisfying $x<d$, and the outer sum on the right hand side is over all positive integers d dividing n.

Proof. We set

$$g(a,b,x,y) = f(a,b,x,y) - f(x,y,a,b)$$

so that

$$g(a,-b,x,y) = g(-a,b,x,y)$$

and

$$g(a,b,x,y) = -g(x,y,a,b).$$

Then

$$\begin{aligned}
&\sum_{ax+by=n} (f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y)\\
&\qquad -f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y))\\
&= \sum_{ax+by=n} (f(a,b,x,-y) - f(x,-y,a,b) + f(a,a-b,x+y,y)\\
&\qquad -f(y,x+y,a-b,a) + f(a-b,a,y,x+y) - f(x+y,y,a,a-b))\\
&= \sum_{ax+by=n} (g(a,a-b,x+y,y) + g(a-b,a,y,x+y) + g(a,b,x,-y))
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{d\,|\,n}\sum_{x<d}(f(0,n/d,x,d)+f(n/d,0,d,x)+f(n/d,n/d,d-x,-x)\\
&\qquad -f(x,x-d,n/d,n/d)-f(x,d,0,n/d)-f(d,x,n/d,0))\\
&= \sum_{d\,|\,n}\sum_{t<d}(f(0,n/d,t,d)+f(n/d,0,d,t)+f(n/d,n/d,d-t,-t)\\
&\qquad -f(d-t,-t,n/d,n/d)-f(t,d,0,n/d)-f(d,t,n/d,0))\\
&= \sum_{d\,|\,n}\sum_{t<d}(g(n/d,0,d,t)+g(0,n/d,t,d)+g(n/d,n/d,d-t,-t))
\end{aligned}$$

so we must prove that

$$\begin{aligned}
&\sum_{ax+by=n}(g(a,a-b,x+y,y)+g(a-b,a,y,x+y)+g(a,b,x,-y))\\
&=\sum_{d\,|\,n}\sum_{t<d}(g(n/d,0,d,t)+g(0,n/d,t,d)+g(n/d,n/d,d-t,-t)).
\end{aligned}$$

First we consider the terms with $a=b$ in the left hand sum. We have

$$\begin{aligned}
&\sum_{\substack{ax+by=n\\ a=b}}(g(a,a-b,x+y,y)+g(a-b,a,y,x+y)+g(a,b,x,-y))\\
&= \sum_{a(x+y)=n}(g(a,0,x+y,y)+g(0,a,y,x+y)+g(a,a,x,-y))\\
&= \sum_{d\,|\,n}\sum_{t<d}(g(n/d,0,d,t)+g(0,n/d,t,d)+g(n/d,n/d,d-t,-t)).
\end{aligned}$$

Secondly we consider the terms with $a<b$. We have

$$\begin{aligned}
&\sum_{\substack{ax+by=n\\ a<b}}(g(a,a-b,x+y,y)+g(a-b,a,y,x+y)+g(a,b,x,-y))\\
&= \sum_{\substack{a(x+y)+(b-a)y=n\\ a<b}}(g(a,a-b,x+y,y)+g(a-b,a,y,x+y))\\
&\qquad + \sum_{\substack{ax+by=n\\ a<b}} g(a,b,x,-y)\\
&= \sum_{\substack{ax+by=n\\ x>y}}(g(a,-b,x,y)+g(-b,a,y,x)) \;+ \sum_{\substack{ax+by=n\\ x<y}} g(x,y,a,-b)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{ax+by=n \\ x>y}} g(a,-b,x,y) \ + \sum_{\substack{ax+by=n \\ x>y}} g(b,-a,y,x) \\
&\quad - \sum_{\substack{ax+by=n \\ x<y}} g(a,-b,x,y) \\
&= - \sum_{\substack{ax+by=n \\ x>y}} g(x,y,a,-b) \ + \sum_{\substack{ax+by=n \\ x<y}} g(a,-b,x,y) \\
&\quad - \sum_{\substack{ax+by=n \\ x<y}} g(a,-b,x,y) \\
&= - \sum_{\substack{ax+by=n \\ x>y}} g(x,y,a,-b).
\end{aligned}$$

Thirdly we consider the terms with $a > b$. We have

$$\begin{aligned}
&\sum_{\substack{ax+by=n \\ a>b}} (g(a,a-b,x+y,y) + g(a-b,a,y,x+y) + g(a,b,x,-y)) \\
&= S_1 + S_2,
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{\substack{ax+by=n \\ a>b}} (g(a,a-b,x+y,y) + g(a-b,a,y,x+y)) \\
&= \sum_{ax+b(x+y)=n} (g(a+b,a,x+y,y) + g(a,a+b,y,x+y)) \\
&= \sum_{\substack{ax+by=n \\ y>x}} (g(a+b,a,y,y-x) + g(a,a+b,y-x,y)) \\
&= \sum_{\substack{ax+by=n \\ a>b}} (g(x+y,y,a,a-b) + g(y,x+y,a-b,a)) \\
&= -S_1,
\end{aligned}$$

so that $S_1 = 0$, and

$$\begin{aligned}
S_2 &= \sum_{\substack{ax+by=n \\ a>b}} g(a,b,x,-y) \\
&= \sum_{\substack{ax+by=n \\ x>y}} g(x,y,a,-b).
\end{aligned}$$

This completes the proof of the theorem. □

We remark that if f satisfies

$$f(a,-b,x,y)=f(-a,b,x,y), \quad f(a,b,x,-y)=f(a,b,-x,y), \tag{2.3}$$

for all integers a, b, x and y then f satisfies (2.1).

Finally we observe that if we choose $f(a,b,x,y)=F(a-b,x-y)$ with $F(x,y)=F(-x,y)=F(x,-y)$ then (2.2) becomes Liouville's identity [21, p. 284] (see also [8, p. 331])

$$\begin{aligned}&\sum_{ax+by=n}(F(a-b,x+y)-F(a+b,x-y))\\ &\quad=\sum_{d\,|\,n}(d-1)(F(0,d)-F(d,0))+2\sum_{d\,|\,n}\sum_{e<n/d}(F(d,e)-F(e,d)).\end{aligned} \tag{2.4}$$

We also note that the choice $f(a,b,x,y)=h(a,b)+h(-a,-b)$ with $h(b,b-a)=h(a,b)$ gives Skoruppa's combinatorial identity [33, p. 69]

$$\sum_{ax+by=n}(h(a,b)-h(a,-b))=\sum_{d|n}\left(\frac{n}{d}h(d,0)-\sum_{j=0}^{d-1}h(d,j)\right).$$

Thus our identity is an extension of those of Liouville and Skoruppa. Conversely if one starts with $f(a,b)$ satisfying $f(a,b)=f(-a,-b)$ and defines

$$h(a,b)=f(a,b)+f(b,b-a)+f(b-a,-a)$$

so that $h(b,b-a)=h(a,b)$ then Skoruppa's identity is the special case of Theorem 1 with $f(a,b,x,y)=f(a,b)$.

For convenience in the rest of the paper, we set

$$\begin{aligned}E &= E(a,b,x,y)\\ &= f(a,b,x,-y)-f(a,-b,x,y)+f(a,a-b,x+y,y)\\ &\quad -f(a,a+b,y-x,y)+f(b-a,b,x,x+y)-f(a+b,b,x,x-y),\end{aligned} \tag{2.5}$$

and, for k a positive integer and a an integer, we set

$$F_k(a)=\begin{cases}1, & \text{if } k \mid a,\\ 0, & \text{if } k \nmid a.\end{cases} \tag{2.6}$$

3 Application of Theorem 1 to Lahiri's Identities

Let e, f, g, h be integers such that

$$e \geq 0,\ f \geq 0,\ g \geq 1,\ h \geq 1,\ g \equiv h \equiv 1 \pmod 2, \tag{3.1}$$

and define the sum $S(e\ f\ g\ h)$ by

$$S(e\ f\ g\ h) := \sum_{m=1}^{n-1} m^e (n-m)^f \sigma_g(m) \sigma_h(n-m). \tag{3.2}$$

The change of variable $m \longrightarrow n-m$ shows that

$$S(e\ f\ g\ h) = S(f\ e\ h\ g). \tag{3.3}$$

Thus we may suppose further that

$$g < h \text{ or } g = h,\ e \geq f. \tag{3.4}$$

For nonnegative integers $r,\ s,\ t,\ u$ we set

$$[r\ s\ t\ u] := \sum_{ax+by=n} a^r b^s x^t y^u. \tag{3.5}$$

As the eight permutations (in cycle notation)

$$(a\ b\ x\ y)^i (a\ x)^j \quad (i = 0, 1, 2, 3;\ j = 0, 1)$$

leave $ax + by$ invariant, we have

$$\begin{aligned} & [r\ s\ t\ u] = [r\ u\ t\ s] = [s\ r\ u\ t] = [s\ t\ u\ r] \\ = & [t\ u\ r\ s] = [t\ s\ r\ u] = [u\ r\ s\ t] = [u\ t\ s\ r]. \end{aligned} \tag{3.6}$$

Moreover

$$S(e\ f\ g\ h) = \sum_{m=1}^{n-1} m^e (n-m)^f \sum_{ax=m} x^g \sum_{by=n-m} y^h = \sum_{ax+by=n} a^e b^f x^{e+g} y^{f+h}$$

so that

$$S(e\ f\ g\ h) = [e\ f\ e+g\ f+h] \tag{3.7}$$

and

$$[e\ f\ g\ h] = S(e\ f\ g-e\ h-f), \text{ if } g \geq e,\ h \geq f. \tag{3.8}$$

w	e	f	g	h	w	e	f	g	h	w	e	f	g	h
2	0	0	1	1	10	0	1	1	7	12	0	2	3	5
4	0	0	1	3	10	0	1	3	5	12	0	3	1	5
4	1	0	1	1	10	0	2	1	5	12	0	4	1	3
6	0	0	1	5	10	0	3	1	3	12	1	0	1	9
6	0	0	3	3	10	1	0	1	7	12	1	0	3	7
6	0	1	1	3	10	1	0	3	5	12	1	0	5	5
6	1	0	1	3	10	1	1	1	5	12	1	1	1	7
6	1	1	1	1	10	1	1	3	3	12	1	1	3	5
6	2	0	1	1	10	1	2	1	3	12	1	2	1	5
8	0	0	1	7	10	2	0	1	5	12	1	3	1	3
8	0	0	3	5	10	2	0	3	3	12	2	0	1	7
8	0	1	1	5	10	2	1	1	3	12	2	0	3	5
8	0	2	1	3	10	2	2	1	1	12	2	1	1	5
8	1	0	1	5	10	3	0	1	3	12	2	1	3	3
8	1	0	3	3	10	3	1	1	1	12	2	2	1	3
8	1	1	1	3	10	4	0	1	1	12	3	0	1	5
8	2	0	1	3	12	0	0	1	11	12	3	0	3	3
8	2	1	1	1	12	0	0	3	9	12	3	1	1	3
8	3	0	1	1	12	0	0	5	7	12	3	2	1	1
10	0	0	1	9	12	0	1	1	9	12	4	0	1	3
10	0	0	3	7	12	0	1	3	7	12	4	1	1	1
10	0	0	5	5	12	0	2	1	7	12	5	0	1	1

Table 1. Values of e, f, g, h, corresponding to weights $w \leq 12$

We define the weight of the sum $S(e\ f\ g\ h)$ to be the positive even integer

$$w = w(S) = 2e + 2f + g + h. \tag{3.9}$$

The integers e, f, g, h satisfying (3.1) and (3.4) for which the sum $S(e\ f\ g\ h)$ has weight less than or equal to 12 are given in Table 1. Lahiri [14], making use of ideas of Ramanujan [30], [31, pp. 136–162], has evaluated the single sum of weight 2, the two sums of weight 4, the six sums of weight 6, the ten sums of weight 8, and the first three of the twenty-eight sums of weight 12. We show that the values of these twenty-two sums follow from Theorem 1.

Weight = 2. We take $f(a,b,x,y) = xy$ in Theorem 1. Then $E = 2xy$ and the left hand side of (2.2) is

$$\sum_{ax+by=n} 2xy = 2[0\ 0\ 1\ 1] = 2S(0\ 0\ 1\ 1).$$

The right hand side of (2.2) is

$$\sum_{d \mid n} \sum_{x<d} \left(x^2 + dx - \frac{n^2}{d^2} \right) = \frac{5}{6}\sigma_3(n) + \left(\frac{1}{6} - n \right) \sigma(n).$$

Then Theorem 1 gives

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{1}{12} \left(5\sigma_3(n) + (1-6n)\sigma(n) \right). \tag{3.10}$$

Formula (3.10) is equivalent to formula (3.1) in Lahiri [14] and originally appeared in a letter from Besge to Liouville [4]. Dickson erroneously attributed (3.10) to Lebesgue in [8, p. 338]. Rankin attributed it to Besgue in [32, p. 115]. Lützen [23, p. 81] asserts that Besge/Besgue is a pseudonym for Liouville. The formula (3.10) also appears in the work of Glaisher [9], [10], [11], Lehmer [17, p. 106], [18, p. 678], Ramanujan [30, Table IV], [31, p. 146] and Skoruppa [33].

Weight = 4. We begin by noting that

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) = \sum_{m=1}^{n-1} (n-m)\sigma(n-m)\sigma(m)$$

so that

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) = \frac{n}{2} \sum_{m=1}^{n-1} \sigma(m)\sigma(n-m).$$

Hence, by (3.10), we have

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) = \frac{n}{24} \left(5\sigma_3(n) + (1-6n)\sigma(n) \right), \tag{3.11}$$

which is formula (3.2) of Lahiri [14]. This formula is due to Glaisher [11]. (Note that the multiplier 12 on the left hand side of Glaisher's formula should be replaced by 24.)

Taking $f(a,b,x,y) = xy^3 + x^3y$, we find that

$$E = 8xy^3 + 8x^3y$$

so that the left hand side of (2.2) is

$$\sum_{ax+by=n} (8xy^3 + 8x^3y) = 8[0\ 0\ 1\ 3] + 8[0\ 0\ 3\ 1] = 16[0\ 0\ 1\ 3] = 16S(0\ 0\ 1\ 3).$$

Evaluating the right hand side of (2.2), we obtain

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m) = \frac{1}{240}(21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma(n)), \qquad (3.12)$$

which is a result attributed to Glaisher by MacMahon [24, p. 101], [25, p. 329]. It also appears in Ramanujan [30, Table IV], [31, p. 146] and is also formula (5.1) of Lahiri [14].

Weight = 6. We begin with the sums $S(1\ 0\ 1\ 3)$ and $S(0\ 1\ 1\ 3)$. Clearly

$$\begin{aligned} S(1\ 0\ 1\ 3) + S(0\ 1\ 1\ 3) &= \sum_{m=1}^{n-1} m\sigma(m)\sigma_3(n-m) \\ &\quad + \sum_{m=1}^{n-1} (n-m)\sigma(m)\sigma_3(n-m) \\ &= n\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m) \\ &= \frac{n}{240}(21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma(n)), \end{aligned}$$

by (3.12). Taking

$$f(a,b,x,y) = 3b^2xy^3 + 5abx^3y - 3abx^2y^2 - 2b^2x^3y,$$

we find that

$$E = 12b^2x^3y - 6b^2xy^3 - 4abx^4 + 12a^2xy^3 - 8aby^4 + 6a^2x^3y$$

so that the left hand side of (2.2) is

$$\begin{aligned} &12[0\ 2\ 3\ 1] - 6[0\ 2\ 1\ 3] - 4[1\ 1\ 4\ 0] + 12[2\ 0\ 1\ 3] - 8[1\ 1\ 0\ 4] + 6[2\ 0\ 3\ 1] \\ &\quad = 24[1\ 0\ 2\ 3] - 12[0\ 1\ 1\ 4] = 24S(1\ 0\ 1\ 3) - 12S(0\ 1\ 1\ 3). \end{aligned}$$

The right hand side of (2.2) is found to be after some calculation

$$\frac{1}{10}((6n^2-5n)\sigma_3(n)-n\sigma(n)).$$

Solving for $S(1\ 0\ 1\ 3)$ and $S(0\ 1\ 1\ 3)=S(1\ 0\ 3\ 1)$ we obtain

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma_3(n-m)=\frac{1}{240}(7n\sigma_5(n)-6n^2\sigma_3(n)-n\sigma(n)) \tag{3.13}$$

and

$$\sum_{m=1}^{n-1} m\sigma_3(m)\sigma(n-m)=\frac{1}{120}(7n\sigma_5(n)+(5n-12n^2)\sigma_3(n)). \tag{3.14}$$

Equations (3.13) and (3.14) are given in MacMahon [24, p. 103], [25, p. 331]. They are also formulae (5.4) and (5.3) of Lahiri [14].

To evaluate the remaining four sums of weight 6, namely $S(1\ 1\ 1\ 1)$, $S(2\ 0\ 1\ 1)$, $S(0\ 0\ 3\ 3)$ and $S(0\ 0\ 1\ 5)$, we choose respectively

$$\begin{aligned}
f(a,b,x,y) &= abx^3y-b^2xy^3,\\
f(a,b,x,y) &= b^2y^4-b^2xy^3,\\
f(a,b,x,y) &= xy^5+x^5y-2x^3y^3,\\
f(a,b,x,y) &= xy^5+x^5y-20x^3y^3.
\end{aligned}$$

The corresponding values of E are

$$\begin{aligned}
E &= -4b^2x^3y-2b^2xy^3-2abx^4+4a^2xy^3+2aby^4-6abx^2y^2+2a^2x^3y,\\
E &= 2b^2x^3y+6b^2xy^3-2a^2xy^3,\\
E &= 36x^3y^3,\\
E &= -108x^5y-108xy^5.
\end{aligned}$$

The left hand sides of (2.2) are respectively

$$\begin{aligned}
&-4[0\ 2\ 3\ 1]-2[0\ 2\ 1\ 3]-2[1\ 1\ 4\ 0]+4[2\ 0\ 1\ 3]\\
&\qquad\qquad +2[1\ 1\ 0\ 4]-6[1\ 1\ 2\ 2]+2[2\ 0\ 3\ 1]\\
&\qquad =-6[1\ 1\ 2\ 2]=-6S(1\ 1\ 1\ 1),
\end{aligned}$$

$$2[0\ 2\ 3\ 1]+6[0\ 2\ 1\ 3]-2[2\ 0\ 1\ 3]=6[2\ 0\ 3\ 1]=6S(2\ 0\ 1\ 1),$$

$$36[0\ 0\ 3\ 3]=36S(0\ 0\ 3\ 3),$$

$$-108[0\ 0\ 5\ 1]-108[0\ 0\ 1\ 5]=-216[0\ 0\ 1\ 5]=-216S(0\ 0\ 1\ 5).$$

Evaluating the right hand side of (2.2) for these choices of f, by Theorem 1 we obtain respectively

$$\sum_{m=1}^{n-1} m(n-m)\sigma(m)\sigma(n-m) = \frac{1}{12}(n^2\sigma_3(n) - n^3\sigma(n)), \tag{3.15}$$

$$\sum_{m=1}^{n-1} m^2\sigma(m)\sigma(n-m) = \frac{1}{24}(3n^2\sigma_3(n) + (n^2-4n^3)\sigma(n)), \tag{3.16}$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{1}{120}(\sigma_7(n) - \sigma_3(n)), \tag{3.17}$$

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_5(n-m) = \frac{1}{504}(20\sigma_7(n) + (21-42n)\sigma_5(n) + \sigma(n)). \tag{3.18}$$

Equations (3.15), (3.16), (3.17) and (3.18) are formulae (3.3), (3.4), (7.1) and (7.2) of Lahiri [14] respectively. We remark that formula (3.15) is due to Glaisher [11, p. 35]. Formula (3.16) follows from (3.11) and (3.15), and so is implicit in the work of Glaisher. Formula (3.17) is due to Glaisher [11, p. 35]. It also appears in the work of Ramanujan [30, Table IV], [31, p. 146]. Formula (3.18) is due to Ramanujan [30, Table IV], [31, p. 146]. It is also given in the work of MacMahon [24, p. 103], [25, p. 331]. The recurrence relations of van der Pol [28, eqns. (63), (65)] for $\sigma(n)$ and $\sigma_3(n)$ are simple consequences of (3.15) and (3.16).

Weight = 8. We begin with the sum $S(1\ 0\ 3\ 3)$. We have

$$\begin{aligned} S(1\ 0\ 3\ 3) &= \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_3(n-m) = \sum_{m=1}^{n-1}(n-m)\sigma_3(n-m)\sigma_3(m) \\ &= nS(0\ 0\ 3\ 3) - S(1\ 0\ 3\ 3) \end{aligned}$$

so that

$$S(1\ 0\ 3\ 3) = \frac{n}{2}S(0\ 0\ 3\ 3).$$

Appealing to (3.17) we obtain

$$\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_3(n-m) = \frac{n}{240}(\sigma_7(n) - \sigma_3(n)). \tag{3.19}$$

Equation (3.19) is formula (7.5) of Lahiri [14].

Next we evaluate $S(2\ 1\ 1\ 1) = S(1\ 2\ 1\ 1)$ and $S(3\ 0\ 1\ 1)$. We have

$$\begin{aligned} S(1\ 2\ 1\ 1) - S(3\ 0\ 1\ 1) &= \sum_{m=1}^{n-1} (m(n-m)^2 - m^3)\sigma(m)\sigma(n-m) \\ &= n^2 \sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) \\ &\quad -2n \sum_{m=1}^{n-1} m^2\sigma(m)\sigma(n-m) \\ &= n^2 S(1\ 0\ 1\ 1) - 2nS(2\ 0\ 1\ 1). \end{aligned}$$

Appealing to (3.11) and (3.16), we obtain

$$S(3\ 0\ 1\ 1) - S(1\ 2\ 1\ 1) = \frac{n^3}{24}(\sigma_3(n) + (1-2n)\sigma(n)).$$

Now we choose $f(a,b,x,y) = b^4xy^3$ so that

$$E = 2b^4xy^3 + 12a^2b^2xy^3 - 8ab^3y^4 - 8a^3by^4 + 2a^4xy^3 + 6b^4x^3y.$$

In this case the left hand side of (2.2) is

$$\begin{aligned} 2[0\ 4\ 1\ 3]\ +\ 12[2\ 2\ 1\ 3] \quad &- \quad 8[1\ 3\ 0\ 4]\ -\ 8[3\ 1\ 0\ 4]\ +\ 2[4\ 0\ 1\ 3] \\ &\quad + 6[0\ 4\ 1\ 3] \\ &= \quad -6[3\ 0\ 4\ 1]\ +\ 12[1\ 2\ 2\ 3] \\ &= \quad 12S(1\ 2\ 1\ 1)\ -\ 6S(3\ 0\ 1\ 1). \end{aligned}$$

Evaluating the right hand side of (2.2), we find by Theorem 1 that

$$12S(1\ 2\ 1\ 1) - 6S(3\ 0\ 1\ 1) = \frac{n^3}{4}(n-1)\sigma(n).$$

Solving these two linear equations for $S(1\ 2\ 1\ 1)$ and $S(3\ 0\ 1\ 1)$, we obtain

$$\sum_{m=1}^{n-1} m(n-m)^2\sigma(m)\sigma(n-m) = \frac{n^3}{24}(\sigma_3(n) - n\sigma(n)), \tag{3.20}$$

$$\sum_{m=1}^{n-1} m^3\sigma(m)\sigma(n-m) = \frac{n^3}{24}(2\sigma_3(n) + (1-3n)\sigma(n)). \tag{3.21}$$

Formula (3.20) is due to Glaisher [11, p. 36]. Formulae (3.20) and (3.21) are formulae (3.5) and (3.6) in Lahiri [14].

Next we treat $S(0\ 2\ 1\ 3) = S(2\ 0\ 3\ 1)$ and $S(2\ 0\ 1\ 3)$. First we note that

$$\begin{aligned}
&S(2\ 0\ 3\ 1) - S(2\ 0\ 1\ 3)\\
&\quad = \quad S(0\ 2\ 1\ 3) - S(2\ 0\ 1\ 3)\\
&\quad = \quad \sum_{m=1}^{n-1}((n-m)^2 - m^2)\sigma(m)\sigma_3(n-m)\\
&\quad = \quad n^2\sum_{m=1}^{n-1}\sigma(m)\sigma_3(n-m) - 2n\sum_{m=1}^{n-1}m\sigma(m)\sigma_3(n-m)\\
&\quad = \quad n^2S(0\ 0\ 1\ 3) - 2nS(1\ 0\ 1\ 3)\\
&\quad = \quad \frac{1}{240}(7n^2\sigma_5(n) + (10n^2 - 18n^3)\sigma_3(n) + n^2\sigma(n)),
\end{aligned}$$

by (3.12) and (3.13). Secondly taking

$$f(a,b,x,y) = aby^6 - 5a^2x^3y^3 + 4a^2x^5y,$$

we find that

$$\begin{aligned}
E \quad = \quad & 70a^2x^3y^3 - 2aby^6 - 22a^2x^5y - 10b^2x^5y + 30abx^4y^2\\
& -30abx^2y^4 + 10a^2xy^5 + 30b^2x^3y^3 + 12b^2xy^5 + 2abx^6
\end{aligned}$$

so that the left hand side of (2.2) is

$$70[2\ 0\ 3\ 3] - 2[1\ 1\ 0\ 6] - 22[2\ 0\ 5\ 1] - 10[0\ 2\ 5\ 1] + 30[1\ 1\ 4\ 2]$$

$$-30[1\ 1\ 2\ 4] + 10[2\ 0\ 1\ 5] + 30[0\ 2\ 3\ 3] + 12[0\ 2\ 1\ 5] + 2[1\ 1\ 6\ 0]$$

$$= 100[2\ 0\ 3\ 3] - 10[2\ 0\ 5\ 1] = 100S(2\ 0\ 1\ 3) - 10S(2\ 0\ 3\ 1).$$

Evaluating the right hand side of (2.2), Theorem 1 gives

$$100S(2\ 0\ 1\ 3) - 10S(2\ 0\ 3\ 1) = \frac{5}{12}(2n^2\sigma_5(n) - n^2\sigma_3(n) - n^2\sigma(n)).$$

Solving the two linear equations for $S(2\ 0\ 1\ 3)$ and $S(2\ 0\ 3\ 1)$ we obtain

$$\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma(n-m) = \frac{1}{24}(n^2\sigma_5(n) + (n^2 - 2n^3)\sigma_3(n)) \tag{3.22}$$

$$\sum_{m=1}^{n-1} m^2\sigma(m)\sigma_3(n-m) = \frac{1}{240}(3n^2\sigma_5(n) - 2n^3\sigma_3(n) - n^2\sigma(n)). \tag{3.23}$$

These are formulae (5.7) and (5.8) of Lahiri [14]. Now we turn to the determination of $S(0\ 1\ 1\ 5) = S(1\ 0\ 5\ 1)$ and $S(1\ 0\ 1\ 5)$. First we observe that

$$\begin{aligned} S(1\ 0\ 1\ 5) + S(1\ 0\ 5\ 1) &= S(1\ 0\ 1\ 5) + S(0\ 1\ 1\ 5) \\ &= nS(0\ 0\ 1\ 5) \\ &= \frac{1}{504}(20n\sigma_7(n) + (21n - 42n^2)\sigma_5(n) + n\sigma(n)), \end{aligned}$$

by (3.18). The choice

$$f(a,b,x,y) = -11aby^6 + 30abx^2y^4 + 20a^2x^3y^3 + 6a^2x^5y$$

in Theorem 1 yields

$$540S(1\ 0\ 1\ 5) - 180S(1\ 0\ 5\ 1) = \frac{15}{14}((6n^2 - 7n)\sigma_5(n) + n\sigma(n)).$$

Solving for $S(1\ 0\ 1\ 5)$ and $S(1\ 0\ 5\ 1)$ we obtain

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma_5(n-m) = \frac{1}{504}(5n\sigma_7(n) - 6n^2\sigma_5(n) + n\sigma(n)), \quad (3.24)$$

$$\sum_{m=1}^{n-1} m\sigma_5(m)\sigma(n-m) = \frac{1}{168}(5n\sigma_7(n) + (7n - 12n^2)\sigma_5(n)). \quad (3.25)$$

Formulae (3.24) and (3.25) are formulae (7.7) and (7.6) of Lahiri [14].

Finally we treat $S(1\ 1\ 1\ 3)$, $S(0\ 0\ 3\ 5)$ and $S(0\ 0\ 1\ 7)$. We choose

$$\begin{aligned} f(a,b,x,y) &= -aby^6 - 3abx^2y^4 + a^2x^3y^3 + 3a^2x^5y, \\ f(a,b,x,y) &= xy^7 + x^7y - x^3y^5 - x^5y^3, \\ f(a,b,x,y) &= 11xy^7 + 11x^7y - 56x^3y^5 - 56x^5y^3, \end{aligned}$$

so that

$$\begin{aligned} E &= 12a^2x^5y + 36abx^2y^4 + 62a^2x^3y^3 + 8aby^6 + 24a^2xy^5 \\ &\quad -24b^2x^5y - 62b^2x^3y^3 - 12b^2xy^5 - 8abx^6 + 54abx^4y^2, \\ E &= 90x^3y^5 + 90x^5y^3, \\ E &= -720xy^7 - 720x^7y, \end{aligned}$$

respectively. The left hand sides of (2.2) are

$$\begin{aligned} &12[2\ 0\ 5\ 1] + 36[1\ 1\ 2\ 4] + 62[2\ 0\ 3\ 3] + 8[1\ 1\ 0\ 6] + 24[2\ 0\ 1\ 5] \\ &-24[0\ 2\ 5\ 1] - 62[0\ 2\ 3\ 3] - 12[0\ 2\ 1\ 5] - 8[1\ 1\ 6\ 0] + 54[1\ 1\ 4\ 2] \\ &= 90[1\ 1\ 2\ 4] = 90S(1\ 1\ 1\ 3), \end{aligned}$$

$$90[0\ 0\ 3\ 5] + 90[0\ 0\ 5\ 3] = 180[0\ 0\ 3\ 5] = 180S(0\ 0\ 3\ 5),$$
$$-720[0\ 0\ 1\ 7] - 720[0\ 0\ 7\ 1] = -1440[0\ 0\ 1\ 7] = -1440S(0\ 0\ 1\ 7),$$

respectively. The right hand sides are

$$\frac{3}{2}(n^2\sigma_5(n) - n^3\sigma_3(n)),$$
$$\frac{1}{28}(11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n)),$$
$$-3(11\sigma_9(n) + (20-30n)\sigma_7(n) - \sigma(n)),$$

respectively. Hence, by Theorem 1, we have

$$\sum_{m=1}^{n-1} m(n-m)\sigma(m)\sigma_3(n-m) = \frac{1}{60}(n^2\sigma_5(n) - n^3\sigma_3(n)), \tag{3.26}$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m) = \frac{1}{5040}(11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n)), \tag{3.27}$$

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_7(n-m) = \frac{1}{480}(11\sigma_9(n) + (20-30n)\sigma_7(n) - \sigma(n)). \tag{3.28}$$

These are formulae (5.6), (9.1) and (9.2) of Lahiri [14]. Formula (3.27) is due to Glaisher [11, p. 35]. It also appears in the work of Ramanujan [30, Table IV], [31, p. 146]. Formula (3.28) is due to Ramanujan [30, Table IV], [31, p. 146].

Weight = 12. Of the twenty-eight sums $S(e\ f\ g\ h)$ of weight 12, we know of only three which can be evaluated using Theorem 1, these are $S(0\ 0\ 5\ 7)$, $S(0\ 0\ 3\ 9)$ and $S(0\ 0\ 1\ 11)$. Choosing

$$\begin{aligned}
f(a,b,x,y) &= 8xy^{11} + 8x^{11}y - 35x^3y^9 - 35x^9y^3 + 27x^5y^7 + 27x^7y^5,\\
f(a,b,x,y) &= 2xy^{11} + 2x^{11}y - 11x^3y^9 - 11x^9y^3 + 9x^5y^7 + 9x^7y^5,\\
f(a,b,x,y) &= 271xy^{11} + 271x^{11}y - 1540x^3y^9 - 1540x^9y^3 + 1584x^5y^7\\
&\quad +1584x^7y^5,
\end{aligned}$$

we find that

$$\begin{aligned}
E &= 2520x^5y^7 + 2520x^7y^5,\\
E &= -180x^3y^9 - 180x^9y^3,\\
E &= 7560xy^{11} + 7560x^{11}y,
\end{aligned}$$

respectively. The left hand sides of (2) are $5040S(0\ 0\ 5\ 7)$, $-360S(0\ 0\ 3\ 9)$ and $15120S(0\ 0\ 1\ 11)$ respectively. Evaluating the right hand sides of (2.2) and applying Theorem 1, we obtain

$$\sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m) = \frac{1}{10080}(\sigma_{13}(n) + 20\sigma_7(n) - 21\sigma_5(n)), \tag{3.29}$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_9(n-m) = \frac{1}{2640}(\sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n)), \tag{3.30}$$

$$\begin{aligned}\sum_{m=1}^{n-1} \sigma(m)\sigma_{11}(n-m) &= \frac{1}{65520}(691\sigma_{13}(n) + 2730(1-n)\sigma_{11}(n) \\ &\quad -691\sigma(n)).\end{aligned} \tag{3.31}$$

These are formulae (13.1), (13.2), (13.4) of Lahiri [14]. Formulae (3.29) and (3.30) are due to Glaisher [11, p. 35]. They also appear in Ramanujan [30, Table IV], [31, p. 146]. Formula (3.31) is due to Ramanujan [30, Table IV], [31, p. 146].

Weight = 10. Formulae (3.1)–(3.3), (5.1)–(5.4), (7.1)–(7.5), (9.1)–(9.4), (11.1), (11.3), (11.4) of Lahiri [15] show that the nineteen sums $S(e\ f\ g\ h)$ of weight 10 require the Ramanujan tau function $\tau(n)$ in their evaluation. Thus it is likely that, in order to be able to deduce their evaluations from Theorem 1, representations of Ramanujan's tau function in terms of divisor functions would be needed. We have not explored this.

In addition to the twenty-two identities (3.10)–(3.31) that we have just considered, Lahiri [14] gave fifteen additional identities which give the value of the sum

$$\sum_{m_1+\cdots+m_r=n} {m_1}^{a_1} \cdots {m_r}^{a_r} \sigma_{b_1}(m_1)\cdots\sigma_{b_r}(m_r) \quad (r \geq 3)$$

where the sum is over all positive integers $m_1,\ldots, m_r$ satisfying $m_1 + \cdots + m_r = n$, for the values of $a_1,\ldots, a_r, b_1,\ldots, b_r$ given in Table 2 below. The authors have checked that all of these sums can be evaluated using the sums (3.10)–(3.31) and so can be evaluated in an elementary manner. All of our evaluations agreed with those of Lahiri [14] except for that of

$$\sum_{m_1+m_2+m_3+m_4=n} m_1\sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4),$$

where a typo had crept into Lahiri's evaluation. In Lahiri's equation (7.10) the term $-2^3 3^2(2,5)$ and one of the two terms $3 \cdot 5(1,4)$ should be deleted.

a_1	a_2	a_3	b_1	b_2	b_3	Lahiri's eqn
0	0	0	1	1	1	(5.2)
1	0	0	1	1	1	(5.5)
1	1	0	1	1	1	(5.9)
2	0	0	1	1	1	(5.10)
0	0	0	1	1	3	(7.3)
1	0	0	3	1	1	(7.8)
1	0	0	1	1	3	(7.9)
0	0	0	1	3	3	(9.3)
0	0	0	1	1	5	(9.4)
0	0	0	3	3	5	(13.3)

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	Lahiri's eqn
0	0	0	0	1	1	1	1	(7.4)
1	0	0	0	1	1	1	1	(7.10)
0	0	0	0	1	1	1	3	(9.5)

a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4	b_5	Lahiri's eqn
0	0	0	0	0	1	1	1	1	1	(9.6)
1	0	0	0	0	1	1	1	1	1	(9.7)

Table 2. Lahiri's identities for $r = 3, 4, 5$

The correct evaluation of the above sum is then

$$\frac{1}{13824}(5n\sigma_7(n) + (21n - 42n^2)\sigma_5(n) + (15n - 90n^2 + 108n^3)\sigma_3(n)$$

$$+(n - 18n^2 + 72n^3 - 72n^4)\sigma(n)).$$

We illustrate the derivation of these fifteen identities with just one example. The rest can be treated similarly. We evaluate the sum

$$\sum_{m_1+m_2+m_3=n} m_1 m_2 \sigma(m_1)\sigma(m_2)\sigma(m_3).$$

We have appealing to (3.15), (3.21) and (3.22)

$$288 \sum_{m_1+m_2+m_3=n} m_1 m_2 \sigma(m_1)\sigma(m_2)\sigma(m_3)$$

$$
\begin{aligned}
&= 288 \sum_{m_3=1}^{n-2} \sigma(m_3) \sum_{m_1+m_2=n-m_3} m_1 m_2 \sigma(m_1)\sigma(m_2) \\
&= 24 \sum_{m_3=1}^{n-2} \sigma(m_3)((n-m_3)^2\sigma_3(n-m_3) - (n-m_3)^3\sigma(n-m_3)) \\
&= 24 \sum_{m=1}^{n-2} (n-m)^2\sigma(m)\sigma_3(n-m) - 24 \sum_{m=1}^{n-2} (n-m)^3\sigma(m)\sigma(n-m) \\
&= 24 \sum_{m=2}^{n-1} m^2\sigma_3(m)\sigma(n-m) - 24 \sum_{m=2}^{n-1} m^3\sigma(m)\sigma(n-m) \\
&= 24 \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma(n-m) - 24 \sum_{m=1}^{n-1} m^3\sigma(m)\sigma(n-m) \\
&= (n^2\sigma_5(n) + (n^2-2n^3)\sigma_3(n)) - (2n^3\sigma_3(n) + (n^3-3n^4)\sigma(n)) \\
&= n^2\sigma_5(n) + (n^2-4n^3)\sigma_3(n) - (n^3-3n^4)\sigma(n)).
\end{aligned}
$$

We remark that Lahiri's equations (5.2), (7.4) and (9.6) are implicit in Glaisher [11, p. 33]. Lahiri's identity (5.2) is proved in Bambah and Chowla [1, eqn. (28)], see also Chowla [5], [6, p. 669].

This concludes our proof that all thirty-seven convolution formulae of Lahiri [14] are consequences of Theorem 1.

For further work on convolution sums, see Grosjean [12], [13] and Levitt [19].

4 Application of Theorem 1 to Melfi's Identities

In this section we consider sums of the type

$$\sum_{m<n/k} \sigma(m)\sigma(n-km), \tag{4.1}$$

where k is a given positive integer. Recently Melfi [26] has treated these sums for k= 2, 3, 4, 5, 9 under the restriction that $\gcd(n,k)=1$ using the theory of modular forms. We evaluate these sums using Theorem 1 for k= 2, 3, 4 and all positive integers n thereby extending Melfi's results in these cases. We begin by expressing the sum (4.1) in terms of the quantity

$$\sum_{\substack{ax+by=n \\ x\equiv -y \pmod{k}}} ab \quad + \sum_{\substack{ax+by=n \\ x\equiv y \pmod{k}}} ab, \tag{4.2}$$

which can be evaluated explicitly for $k=1,2,3,4$.

Lemma 1. *Let k be a positive integer. Then*

$$\begin{aligned}\sum_{m<n/k} \sigma(m)\sigma(n-km) &= -\frac{1}{24}\sigma_3(n)+\frac{1}{24}\sigma(n)+\frac{1}{4}\sigma_3(n/k)-\frac{n}{4}\sigma(n/k) \\ &\quad +\frac{1}{4}\sum_{\substack{ax+by=n \\ x\equiv -y \pmod{k}}} ab+\frac{1}{4}\sum_{\substack{ax+by=n \\ x\equiv y \pmod{k}}} ab. \qquad (4.3)\end{aligned}$$

Proof. The identity (4.3) follows from Theorem 1 by taking $f(a,b,x,y) = (2a^2-b^2)F_k(x)$. With this choice, after a long calculation, we find that the left hand side of (2.2) is

$$2\sum_{\substack{ax+by=n \\ x\equiv -y \pmod{k}}} ab+2\sum_{\substack{ax+by=n \\ x\equiv y \pmod{k}}} ab-8\sum_{m<n/k}\sigma(m)\sigma(n-km)$$

and the right hand side is

$$\frac{1}{3}\sigma_3(n)-\frac{1}{3}\sigma(n)-2\sigma_3(n/k)+2n\sigma(n/k).$$

Lemma 1 now follows by Theorem 1. □

When $k=1$ the sum (4.2) is $2\sum_{m=1}^{n-1}\sigma(m)\sigma(n-m)$ and Lemma 1 gives the identity (3.10).

When $k=2$ the sum (4.2) is

$$\begin{aligned}2\sum_{\substack{ax+by=n \\ x\equiv y \pmod{2}}} ab &= 4\sum_{\substack{ax+by=n \\ 2\mid x, 2\mid y}} ab \;+\; 2\sum_{ax+by=n} ab \;-2\sum_{\substack{ax+by=n \\ 2\mid x}} ab \\ &\quad -2\sum_{\substack{ax+by=n \\ 2\mid y}} ab \\ &= 4\sum_{m<n/2}\sigma(m)\sigma(n/2-m)+2\sum_{m=1}^{n-1}\sigma(m)\sigma(n-m) \\ &\quad -4\sum_{m<n/2}\sigma(m)\sigma(n-2m).\end{aligned}$$

Then, from Lemma 1 and (3.10), we obtain

Theorem 2 (see Melfi [26], (8)).

$$\begin{aligned}\sum_{m<n/2}\sigma(m)\sigma(n-2m) &= \frac{1}{24}(2\sigma_3(n)+(1-3n)\sigma(n) \\ &\quad +8\sigma_3(n/2)+(1-6n)\sigma(n/2)). \qquad (4.4)\end{aligned}$$

Replacing n by $n+1$ in (4.4), we obtain the following companion formula

$$\sum_{m\le n/2}\sigma(m)\sigma(n-(2m-1)) = \frac{1}{24}(2\sigma_3(n+1)-(2+3n)\sigma(n+1)$$
$$+8\sigma_3((n+1)/2)-(5+6n)\sigma((n+1)/2)). \quad (4.5)$$

When $k=3$ the sum (4.2) is

$$\begin{aligned}
&3\sum_{\substack{ax+by=n\\ 3\,|\,x,\,3\,|\,y}} ab + \sum_{ax+by=n} ab - \sum_{\substack{ax+by=n\\ 3\,|\,x}} ab - \sum_{\substack{ax+by=n\\ 3\,|\,y}} ab\\
&= 3\sum_{ax+by=n/3} ab + \sum_{ax+by=n} ab - 2\sum_{3ax+by=n} ab\\
&= 3\sum_{m<n/3}\sigma(m)\sigma(n/3-m) + \sum_{m=1}^{n-1}\sigma(m)\sigma(n-m)\\
&\quad -2\sum_{m<n/3}\sigma(m)\sigma(n-3m).
\end{aligned}$$

Appealing to Lemma 1 (with $k=3$) and (3.10), we obtain

Theorem 3 (see Melfi [26], (12)).

$$\sum_{m<n/3}\sigma(m)\sigma(n-3m) = \frac{1}{24}(\sigma_3(n)+(1-2n)\sigma(n)$$
$$+9\sigma_3(n/3)+(1-6n)\sigma(n/3)).$$

When $k=4$ the sum (4.2) is

$$2\sum_{\substack{ax+by=n\\ 4\,|\,x,\,4\,|\,y}} ab + 2\sum_{\substack{ax+by=n\\ 2\,\|\,x,\,2\,\|\,y}} ab + \sum_{\substack{ax+by=n\\ 2\nmid x,\,2\nmid y}} ab.$$

Next we set

$$F(n) := \sum_{ax+by=n} ab, \quad G(n) := \sum_{2ax+by=n} ab,$$

so that the above sum is

$$2F(n/4)+2(F(n/2)-2G(n/2)+F(n/4))+(F(n)-2G(n)+F(n/2))$$
$$=F(n)+3F(n/2)+4F(n/4)-2G(n)-4G(n/2).$$

Since

$$F(n) = \sum_{m=1}^{n-1} \sigma(m)\sigma(n-m), \quad G(n) = \sum_{m<n/2} \sigma(m)\sigma(n-2m),$$

appealing to (3.10), Theorem 2 and Lemma 1 (with $k = 4$), we obtain

Theorem 4 (see Melfi [26], (11)).

$$\sum_{m<n/4} \sigma(m)\sigma(n-4m) = \frac{1}{48}(\sigma_3(n) + (2-3n)\sigma(n) + 3\sigma_3(n/2) + 16\sigma_3(n/4) + (2-12n)\sigma(n/4)).$$

The authors have not been able to use Theorem 1 to evaluate the sum (4.1) for $k \geq 5$.

As a consequence of Theorems 2 and 4 we obtain the following new evaluation.

Theorem 5.

$$\sum_{m<n/2} \sigma(2m)\sigma(n-2m) = \frac{1}{24}(5\sigma_3(n) + 21\sigma_3(n/2) - 16\sigma_3(n/4) + (1-6n)(\sigma(n) + 3\sigma(n/2) - 2\sigma(n/4))).$$

Proof. As $\sigma(2m) = 3\sigma(m) - 2\sigma(m/2)$ we have

$$\begin{aligned} &\sum_{m<n/2} \sigma(2m)\sigma(n-2m) \\ &= 3\sum_{m<n/2} \sigma(m)\sigma(n-2m) - 2\sum_{\substack{m<n/2 \\ 2\,|\,m}} \sigma(m/2)\sigma(n-2m) \\ &= 3\sum_{m<n/2} \sigma(m)\sigma(n-2m) - 2\sum_{m<n/4} \sigma(m)\sigma(n-4m). \end{aligned}$$

The theorem now follows from Theorems 2 and 4. □

As a corollary of Theorem 5 we have the companion result where, instead of running through even integers $2m$ $(m < n/2)$, we run through odd integers $2m-1$ $(m \leq n/2)$.

Corollary 1.

$$\begin{aligned} &\sum_{m\leq n/2} \sigma(2m-1)\sigma(n-(2m-1)) \\ &= \frac{1}{24}(5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\ &\quad +(1-6n)(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4))). \end{aligned}$$

Proof. Corollary 1 follows from Theorem 5 and (3.10). □

Taking $n = 2M$ with M odd in Corollary 1, we obtain

$$\begin{aligned}\sum_{m=1}^{M} \sigma(2m-1)\sigma(2M-2m+1) &= \frac{1}{24}(5\sigma_3(2M) - 21\sigma_3(M) \\ &\quad +(1-12M)(\sigma(2M) - 3\sigma(M)),\end{aligned}$$

that is

$$\sum_{m=1}^{M} \sigma(2m-1)\sigma(2M-2m+1) = \sigma_3(M), \tag{4.6}$$

which is a result of Liouville [20, p. 146], see also [7, p. 287], [8, p. 329].

Our next application of Theorem 1 extends two more of Melfi's identities [26, eqns. (9), (10)] to all positive integers n.

Theorem 6.

$$\begin{aligned}\sum_{m<n/2} \sigma_3(m)\sigma(n-2m) &= \frac{1}{240}(\sigma_5(n) - \sigma(n) + 20\sigma_5(n/2) \\ &\quad +(10-30n)\sigma_3(n/2)), \\ \sum_{m<n/2} \sigma(m)\sigma_3(n-2m) &= \frac{1}{240}(5\sigma_5(n) + (10-15n)\sigma_3(n) \\ &\quad +16\sigma_5(n/2) - \sigma(n/2)).\end{aligned}$$

Proof. We set

$$X := \sum_{m<n/2} \sigma_3(m)\sigma(n-2m), \quad Y := \sum_{m<n/2} \sigma(m)\sigma_3(n-2m).$$

Choosing $f(a,b,x,y) = a^4F_2(x)$ in Theorem 1, the left hand side of (2.2) is

$$\sum_{ax+by=n} ((b-a)^4 - (b+a)^4)F_2(x) = \sum_{2ax+by=n} (-8a^3b - 8ab^3) = -8X - 8Y$$

and the right hand side of (2.2) is (after some calculation)

$$\begin{aligned}&- \frac{1}{5}\sigma_5(n) + \left(\frac{n}{2} - \frac{1}{3}\right)\sigma_3(n) + \frac{1}{30}\sigma(n) - \frac{6}{5}\sigma_5(n/2) \\ &+ \left(n - \frac{1}{3}\right)\sigma_3(n/2) + \frac{1}{30}\sigma(n/2).\end{aligned}$$

Next setting $f(a,b,x,y) = b^4 F_2(a)$ in Theorem 1, the left hand side of (2.2) is

$$\sum_{ax+by=n} ((a-b)^4-(a+b)^4)F_2(a) = \sum_{2ax+by=n} (-64a^3b-16ab^3) = -64X-16Y$$

and the right hand side is (after some calculation)

$$\begin{aligned} - \quad & \frac{3}{5}\sigma_5(n) + \left(n - \frac{2}{3}\right)\sigma_3(n) + \frac{4}{15}\sigma(n) - \frac{32}{5}\sigma_5(n/2) \\ & + \left(8n - \frac{8}{3}\right)\sigma_3(n/2) + \frac{1}{15}\sigma(n/2). \end{aligned}$$

Solving the two linear equations for X and Y resulting from Theorem 1, we obtain the assertions of Theorem 6. □

As a consequence of Theorem 6 we have the following identity.

Corollary 2.

$$\sum_{k=0}^{n-1} \sigma(2k+1)\sigma_3(n-k) = \frac{1}{240}(\sigma_5(2n+1) - \sigma(2n+1)). \tag{4.7}$$

Proof. We have

$$\begin{aligned} \sum_{k=0}^{n-1} \sigma(2k+1)\sigma_3(n-k) &= \sum_{m=1}^{n} \sigma(2n-2m+1)\sigma_3(m) \\ &= \sum_{m<(2n+1)/2} \sigma((2n+1)-2m)\sigma_3(m) \\ &= \frac{1}{240}(\sigma_5(2n+1) - \sigma(2n+1)), \end{aligned}$$

by Theorem 6. This completes the proof of Corollary 2. □

Corollary 2 was explicitly stated but never proved by Ramanujan [30], [31, p. 146]. A result equivalent to (4.7) was first proved by Masser, see Berndt [2, p. 329] and Berndt and Evans [3, p. 136], with later proofs by Atkin (see Berndt [2, p. 329]) and Ramamani [29]. None of these proofs is elementary. The above proof is the first elementary proof of (4.7). Berndt [2, p. 329] has indicated that it would be interesting to have such a proof. As a further consequence of Theorem 6, we obtain the following identity of Liouville [20, p. 147].

Corollary 3. *Let M be an odd positive integer. Then*

$$\sum_{m=0}^{M-1} \sigma(2m+1)\sigma_3(2M-2m-1) = \sigma_5(M).$$

Proof. As

$$\sigma(2k) = 3\sigma(k) - 2\sigma(k/2), \quad \sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3(k/2),$$

we have

$$\begin{aligned}
\sum_{m=1}^{M-1} \sigma(2m)\sigma_3(2M-2m) &= \sum_{m=1}^{M-1} (3\sigma(m) - 2\sigma(m/2))(9\sigma_3(M-m) \\
&\quad -8\sigma_3((M-m)/2)) \\
&= 27S_1 - 18S_2 - 24S_3 + 16S_4,
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{m=1}^{M-1} \sigma(m)\sigma_3(M-m), \\
S_2 &= \sum_{\substack{m=1 \\ 2 \mid m}}^{M-1} \sigma(m/2)\sigma_3(M-m), \\
S_3 &= \sum_{\substack{m=1 \\ 2 \mid M-m}}^{M-1} \sigma(m)\sigma_3((M-m)/2), \\
S_4 &= \sum_{\substack{m=1 \\ 2 \mid m \\ 2 \mid M-m}}^{M-1} \sigma(m/2)\sigma_3((M-m)/2).
\end{aligned}$$

By (3.12) we have

$$S_1 = \frac{21}{240}\sigma_5(M) + \frac{(1-3M)}{24}\sigma_3(M) - \frac{1}{240}\sigma(M).$$

By Theorem 6, we obtain

$$S_2 = \sum_{m<M/2} \sigma(m)\sigma_3(M-2m) = \frac{1}{48}\sigma_5(M) + \frac{(2-3M)}{48}\sigma_3(M).$$

Further, by Corollary 2, we have

$$\begin{aligned}
S_3 &= \sum_{\substack{m=1 \\ 2 \mid M-m}}^{M-1} \sigma(m)\sigma_3((M-m)/2) \\
&= \sum_{k=0}^{(M-3)/2} \sigma(2k+1)\sigma_3\left(\frac{M-1}{2}-k\right) \\
&= \frac{1}{240}\sigma_5(M) - \frac{1}{240}\sigma(M).
\end{aligned}$$

Finally, as m and $N-m$ are of opposite parity, we have $S_4 = 0$. Putting these evaluations together, we obtain

$$\sum_{m=1}^{M-1} \sigma(2m)\sigma_3(2M-2m) = \frac{151}{80}\sigma_5(M) + \frac{(3-18M)}{8}\sigma_3(M) - \frac{1}{80}\sigma(M).$$

Then, appealing to (3.12), we obtain

$$\begin{aligned}
&\sum_{m=0}^{M-1} \sigma(2m+1)\sigma_3(2M-2m-1) \\
&= \sum_{m=1}^{2M-1} \sigma(m)\sigma_3(2M-m) - \sum_{m=1}^{M-1} \sigma(2m)\sigma_3(2M-2m) \\
&= \left(\frac{7}{80}\sigma_5(2M) + \frac{(1-6M)}{24}\sigma_3(2M) - \frac{1}{240}\sigma(2M)\right) \\
&\quad - \left(\frac{151}{80}\sigma_5(M) + \frac{(3-18M)}{8}\sigma_3(M) - \frac{1}{80}\sigma(M)\right) \\
&= \sigma_5(M).
\end{aligned}$$

This completes the proof of Corollary 3. □

We conclude this section with the following result, which is analogous to Theorem 5. We make use of Theorem 3 and a result of Melfi [26, eqn. (14)].

Theorem 7. *If* $n \equiv 0 \pmod 3$ *then*

$$\sum_{m<n/3} \sigma(3m)\sigma(n-3m) = \frac{1}{36}(7\sigma_3(n) + (3-18n)\sigma(n) + 8\sigma_3(n/3)).$$

If $n \equiv 1 \pmod 3$ and there exists a prime $p \equiv 2 \pmod 3$ with $p \,\|\, n$, or if $n \equiv 2 \pmod 3$, then

$$\sum_{m<n/3} \sigma(3m)\sigma(n-3m) = \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)).$$

Proof. As $\sigma(3m) = 4\sigma(m) - 3\sigma(m/3)$ we have

$$\begin{aligned}
&\sum_{m<n/3} \sigma(3m)\sigma(n-3m) \\
&= 4\sum_{m<n/3} \sigma(m)\sigma(n-3m) - 3\sum_{\substack{m<n/3 \\ 3\,|\,m}} \sigma(m/3)\sigma(n-3m) \\
&= 4\sum_{m<n/3} \sigma(m)\sigma(n-3m) - 3\sum_{m<n/9} \sigma(m)\sigma(n-9m) \\
&= \frac{1}{6}(\sigma_3(n) + (1-2n)\sigma(n) + 9\sigma_3(n/3) + (1-6n)\sigma(n/3)) \\
&\quad -3\sum_{m<n/9} \sigma(m)\sigma(n-9m),
\end{aligned}$$

by Theorem 3. We first consider the case $n \equiv 0 \pmod 3$, say $n = 3N$. In this case we have

$$\begin{aligned}
&\sum_{m<n/9} \sigma(m)\sigma(n-9m) \\
&= \sum_{m<N/3} \sigma(m)\sigma(3N-9m) \\
&= 4\sum_{m<N/3} \sigma(m)\sigma(N-3m) - 3\sum_{m<N/3} \sigma(m)\sigma(N/3-m) \\
&= \frac{1}{6}(\sigma_3(N) + (1-2N)\sigma(N) + 9\sigma_3(N/3) + (1-6N)\sigma(N/3)) \\
&\quad -\frac{1}{4}(5\sigma_3(N/3) + (1-2N)\sigma(N/3)) \\
&= \frac{1}{12}(2\sigma_3(N) + (2-4N)\sigma(N) + 3\sigma_3(N/3) - (1+6N)\sigma(N/3)) \\
&= \frac{1}{36}(6\sigma_3(n/3) + (6-4n)\sigma(n/3) + 9\sigma_3(n/9) - (3+6n)\sigma(n/9)),
\end{aligned}$$

by (3.10) and Theorem 3. Hence

$$\begin{aligned}
&\sum_{m<n/3} \sigma(3m)\sigma(n-3m) \\
&= \frac{1}{12}(2\sigma_3(n) + 12\sigma_3(n/3) - 9\sigma_3(n/9) + (2-4n)\sigma(n) \\
&\quad -(4+8n)\sigma(n/3) + (3+6n)\sigma(n/9)).
\end{aligned}$$

Since

$$\sigma_3(n) = 28\sigma_3(n/3) - 27\sigma_3(n/9), \quad \sigma(n) = 4\sigma(n/3) - 3\sigma(n/9),$$

for $n \equiv 0 \pmod 3$, we have

$$\sum_{m<n/3} \sigma(3m)\sigma(n-3m) = \frac{1}{36}(7\sigma_3(n) + 8\sigma_3(n/3) + (3-18n)\sigma(n)).$$

Finally, if $n \equiv 1 \pmod 3$ and there exists a prime $p \equiv 2 \pmod 3$ such that $p \parallel n$, or if $n \equiv 2 \pmod 3$, then, by Melfi [26, eqn. (14)], we have

$$\sum_{m<n/9} \sigma(m)\sigma(n-9m) = \frac{1}{216}(\sigma_3(n) + (9-6n)\sigma(n))$$

so that

$$\sum_{m<n/3} \sigma(3m)\sigma(n-3m) = \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)).$$

This completes the proof of Theorem 7. $\square$

We are unable to give the value of $\sum_{m<n/3} \sigma(3m)\sigma(n-3m)$ for those $n \equiv 1 \pmod 3$ such that $p^2 \mid n$ for every prime $p \equiv 2 \pmod 3$ with $p \mid n$ since the value of $\sum_{m<n/9} \sigma(m)\sigma(n-9m)$ is not known for such n.

5 Sums $\sum_{m=1,\, m\equiv a \pmod b}^{n-1} \sigma(m)\sigma(n-m)$

Let a and b be integers satisfying $b \geq 1$ and $0 \leq a \leq b-1$. We set

$$S(a,b) = \sum_{\substack{m=1 \\ m\equiv a \pmod b}}^{n-1} \sigma(m)\sigma(n-m). \tag{5.1}$$

Clearly, by (3.10), Theorem 5 and Corollary 1, we have

$$S(0,1) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma(n)), \tag{5.2}$$

$$\begin{aligned} S(0,2) &= \frac{1}{24}(5\sigma_3(n) + 21\sigma_3(n/2) - 16\sigma_3(n/4) + (1-6n)(\sigma(n) \\ &\quad +3\sigma(n/2) - 2\sigma(n/4))), \qquad (5.3) \\ S(1,2) &= \frac{1}{24}(5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) + (1-6n)(\sigma(n) \\ &\quad -3\sigma(n/2) + 2\sigma(n/4))). \qquad (5.4) \end{aligned}$$

Melfi [26, eqn. (7)] has shown that

$$S(1,3) = \frac{1}{9}\sigma_3(n), \quad \text{if } n \equiv 2 \pmod 3. \tag{5.5}$$

Appealing to Theorem 7, we obtain the following partial evaluation of $S(i,3)$ $(i = 0,1,2)$.

Theorem 8. *If* $n \equiv 0 \pmod 3$ *then*

$$\begin{aligned} S(0,3) &= \frac{1}{36}(7\sigma_3(n) + (3-18n)\sigma(n) + 8\sigma_3(n/3)), \\ S(1,3) &= \frac{1}{9}(\sigma_3(n) - \sigma_3(n/3)), \\ S(2,3) &= \frac{1}{9}(\sigma_3(n) - \sigma_3(n/3)); \end{aligned}$$

if $n \equiv 1 \pmod 3$ *and there exists a prime* $p \equiv 2 \pmod 3$ *such that* $p \parallel n$ *then*

$$\begin{aligned} S(0,3) &= \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)), \\ S(1,3) &= \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)), \\ S(2,3) &= \frac{1}{9}\sigma_3(n); \end{aligned}$$

and if $n \equiv 2 \pmod 3$ *then*

$$\begin{aligned} S(0,3) &= \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)), \\ S(1,3) &= \frac{1}{9}\sigma_3(n), \\ S(2,3) &= \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)). \end{aligned}$$

Proof. First we note that

$$S(0,3) + S(1,3) + S(2,3) = S(0,1),$$

so that by (5.2) we have

$$S(0,3) + S(1,3) + S(2,3) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma(n)). \tag{5.6}$$

Secondly we note that the change of variable $m \longrightarrow n-m$ yields $S(i,3) = S(n-i,3)$ so that

$$\begin{cases} S(1,3) = S(2,3), & \text{if } n \equiv 0 \pmod 3, \\ S(0,3) = S(1,3), & \text{if } n \equiv 1 \pmod 3, \\ S(0,3) = S(2,3), & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{5.7}$$

Then

$$S(0,3) = \sum_{\substack{m=1 \\ m \equiv 0 \pmod 3}}^{n-1} \sigma(m)\sigma(n-m) = \sum_{m<n/3} \sigma(3m)\sigma(n-3m),$$

and the evaluation of $S(0,3)$ follows from Theorem 7. The values of $S(1,3)$ and $S(2,3)$ then follow from (5.6) and (5.7). This completes the proof of Theorem 8. □

It would be interesting to determine $S(i,3)$ $(i = 0,1,2)$ for all $n \equiv 1 \pmod 3$.

We conclude this section by giving some partial results for the sums $S(i,4)$ $(i = 0,1,2,3)$. We do not know of any evaluations of $S(a,b)$ for $b \geq 5$.

Theorem 9. *If* $n \equiv 0 \pmod 4$ *then*

$$S(1,4) = S(3,4) = \frac{1}{16}(\sigma_3(n) - \sigma_3(n/2))$$

and

$$S(0,4) + S(2,4) = \frac{1}{24}(7\sigma_3(n) + (2-12n)\sigma(n) + 3\sigma_3(n/2)).$$

If $n \equiv 1 \pmod 4$ *then*

$$S(0,4) = S(1,4), \quad S(2,4) = S(3,4)$$

and

$$S(0,4)+S(2,4)=\frac{1}{24}(5\sigma_3(n)+(1-6n)\sigma(n)).$$

If $n \equiv 2 \pmod 4$ *then*

$$S(0,4)=S(2,4)=\frac{1}{72}(11\sigma_3(n)+(3-18n)\sigma(n))$$

and

$$S(1,4)+S(3,4)=\frac{1}{9}\sigma_3(n).$$

If $n \equiv 3 \pmod 4$ *then*

$$S(0,4)=S(3,4), \quad S(1,4)=S(2,4)$$

and

$$S(0,4)+S(1,4)=\frac{1}{24}(5\sigma_3(n)+(1-6n)\sigma(n)).$$

Proof. First we note that by (5.1) and (5.2) we have

$$S(0,4)+S(1,4)+S(2,4)+S(3,4)=\frac{1}{12}(5\sigma_3(n)+(1-6n)\sigma(n)). \quad (5.8)$$

Secondly the change of variable $m \longrightarrow n-m$ yields $S(i,4)=S(n-i,4)$ so that

$$\begin{cases} S(1,4)=S(3,4), & & \text{if } n \equiv 0 \pmod 4, \\ S(0,4)=S(1,4), & S(2,4)=S(3,4), & \text{if } n \equiv 1 \pmod 4, \\ S(0,4)=S(2,4), & & \text{if } n \equiv 2 \pmod 4, \\ S(0,4)=S(3,4), & S(1,4)=S(2,4), & \text{if } n \equiv 3 \pmod 4. \end{cases} \quad (5.9)$$

The asserted results for $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ now follow from (5.8) and (5.9).

For $n \equiv 2 \pmod 4$ we have by (5.9) and (5.3)

$$\begin{aligned} S(0,4) \;=\; S(2,4) &= \frac{1}{2}(S(0,4)+S(2,4)) \\ &= \frac{1}{2}S(0,2) \\ &= \frac{1}{48}(5\sigma_3(n)+21\sigma_3(n/2)+(1-6n)(\sigma(n)+3\sigma(n/2))) \\ &= \frac{1}{72}(11\sigma_3(n)+(3-18n)\sigma(n)). \end{aligned}$$

The value of $S(1,4)+S(3,4)$ then follows from (5.8).

For $n \equiv 0 \pmod 4$ we have by (5.9) and (5.4)

$$\begin{aligned} S(1,4) \ = \ S(3,4) &= \frac{1}{2}(S(1,4)+S(3,4)) \\ &= \frac{1}{2}S(1,2) \\ &= \frac{1}{48}(5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\ &\quad +(1-6n)(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4))) \\ &= \frac{1}{16}(\sigma_3(n) - \sigma_3(n/2)). \end{aligned}$$

The value of $S(0,4)+S(2,4)$ now follows from (5.8). This completes the proof of Theorem 9. □

6 Application of Theorem 1 to Triangular Numbers

The triangular numbers are the nonnegative integers

$$\frac{1}{2}m(m+1), \quad m = 0,1,2,\dots .$$

For k a positive integer, we let $\delta_k(n)$ denote the number of representations of n as the sum of k triangular numbers. It is an easily proved classical result that

$$\delta_2(n) = \sum_{a \mid 4n+1} \left(\frac{-4}{a}\right), \quad n = 0,1,2,\dots . \tag{6.1}$$

We derive the corresponding formulae for $\delta_4(n), \delta_6(n)$ and $\delta_8(n)$ from (6.1) using Theorem 1.

Theorem 10 ([27], Theorem 3).

$$\delta_4(n) = \sigma(2n+1).$$

Proof. First we choose $f(a,b,x,y) = F_4(b)$ in Theorem 1 with n replaced by $4n+2$. We obtain

$$\begin{aligned}
&\sum_{ax+by=4n+2} (F_4(a-b) - F_4(a+b)) \\
&= \sum_{d \mid 4n+2} \sum_{x<d} \left(F_4\left(\frac{4n+2}{d}\right) + F_4(0) + F_4\left(\frac{4n+2}{d}\right) - F_4(x-d) \right. \\
&\qquad \left. -F_4(d) - F_4(x) \right) \\
&= \sum_{d \mid 4n+2} \sum_{x<d} (1 - F_4(x-d) - F_4(x)) \\
&= \sum_{d \mid 4n+2} \left\{ d - 1 - 2\left[\frac{d}{4}\right] \right\} \\
&= \sigma(4n+2) - \tau(4n+2) - 2 \sum_{d \mid 4n+2} \left[\frac{d}{4}\right] \\
&= 3\sigma(2n+1) - 2\tau(2n+1) - 2 \sum_{d \mid 2n+1} \left[\frac{d}{4}\right] - 2 \sum_{d \mid 2n+1} \left[\frac{d}{2}\right] \\
&= 2\sigma(2n+1) - \tau(2n+1) - 2 \sum_{d \mid 2n+1} \left[\frac{d}{4}\right].
\end{aligned}$$

Secondly we choose $f(a,b,x,y) = F_4(a)F_2(x)$ in Theorem 1 with n replaced by $4n+2$. We obtain

$$\begin{aligned}
&\sum_{2ax+by=4n+2} (F_4(a-b) - F_4(a+b)) \\
&= \sum_{d \mid 4n+2} \sum_{x<d} \left(F_4(0)F_2(x) + F_4\left(\frac{4n+2}{d}\right) F_2(d) - F_4(x)F_2\left(\frac{4n+2}{d}\right) \right. \\
&\qquad \left. +F_4\left(\frac{4n+2}{d}\right) F_2(d-x) - F_4(x)F_2(0) - F_4(d)F_2\left(\frac{4n+2}{d}\right) \right) \\
&= \sum_{d \mid 4n+2} \sum_{x<d} \left(F_2(x) - F_4(x)F_2\left(\frac{4n+2}{d}\right) - F_4(x) \right) \\
&= \sum_{d \mid 4n+2} \sum_{x<d/2} 1 - \sum_{d \mid 2n+1} \sum_{x<d/4} 1 - \sum_{d \mid 4n+2} \sum_{x<d/4} 1 \\
&= \sum_{d \mid 2n+1} (d-1) + \sum_{d \mid 2n+1} \frac{(d-1)}{2} - \sum_{d \mid 2n+1} \left[\frac{d}{4}\right] - \sum_{d \mid 2n+1} \frac{(d-1)}{2} \\
&\qquad - \sum_{d \mid 2n+1} \left[\frac{d}{4}\right]
\end{aligned}$$

$$= \quad \sigma(2n+1) - \tau(2n+1) - 2 \sum_{d \mid 2n+1} \left[\frac{d}{4}\right].$$

Subtracting these two results, we deduce that

$$\sum_{ax+by=4n+2} (F_4(a-b) - F_4(a+b)) - \sum_{2ax+by=4n+2} (F_4(a-b) - F_4(a+b))$$

$$= \sigma(2n+1),$$

that is

$$\sum_{\substack{ax+by=4n+2 \\ x \equiv 1 \pmod 2}} (F_4(a-b) - F_4(a+b)) = \sigma(2n+1).$$

A simple consideration of the residues of a and b modulo 4 shows that

$$F_4(a-b) - F_4(a+b) = \left(\frac{-4}{ab}\right)$$

so that

$$\sum_{\substack{ax+by=4n+2 \\ x \equiv 1 \pmod 2}} \left(\frac{-4}{ab}\right) = \sigma(2n+1).$$

As $\left(\frac{-4}{ab}\right) = 0,$ for $a \equiv 0 \pmod 2$, we have

$$\sum_{\substack{ax+by=4n+2 \\ ax \equiv 1 \pmod 2}} \left(\frac{-4}{ab}\right) = \sigma(2n+1).$$

Now

$$\sum_{\substack{ax+by=4n+2 \\ ax \equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right) = \sum_{\substack{ax+by=4n+2 \\ ax \equiv 3 \pmod 4}} \left(\frac{-4}{xb}\right) = - \sum_{\substack{ax+by=4n+2 \\ ax \equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right),$$

as $\left(\frac{-4}{x}\right) = -\left(\frac{-4}{a}\right)$, so that

$$\sum_{\substack{ax+by=4n+2 \\ ax \equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right) = 0.$$

Thus

$$\sum_{\substack{ax+by=4n+2\\ ax\equiv 1 \pmod 4}} \left(\frac{-4}{ab}\right) = \sigma(2n+1).$$

Finally we have

$$\begin{aligned}\delta_4(n) &= \sum_{m=0}^{n} \delta_2(m)\delta_2(n-m)\\ &= \sum_{m=0}^{n} \sum_{a \mid 4m+1} \left(\frac{-4}{a}\right) \sum_{b \mid 4(n-m)+1} \left(\frac{-4}{b}\right)\\ &= \sum_{\substack{ax+by=4n+2\\ ax\equiv 1 \pmod 4}} \left(\frac{-4}{ab}\right)\\ &= \sigma(2n+1).\end{aligned}$$

This is the asserted formula for $\delta_4(n)$. □

Theorem 10 was known to Legendre [16].

Theorem 11 ([27], Theorem 4).

$$\delta_6(n) = -\frac{1}{8} \sum_{d \mid 4n+3} \left(\frac{-4}{d}\right) d^2.$$

Proof. We have by (6.1) and Theorem 10

$$\begin{aligned}\delta_6(n) &= \sum_{m=0}^{n} \delta_2(m)\delta_4(n-m)\\ &= \sum_{m=0}^{n} \sum_{a \mid 4m+1} \left(\frac{-4}{a}\right) \sum_{b \mid 2(n-m)+1} b\\ &= \sum_{ax+2by=4n+3} \left(\frac{-4}{a}\right) b,\end{aligned}$$

as

$$\begin{aligned}\sum_{\substack{ax+2by=4n+3\\ b\equiv 0 \pmod 2}} \left(\frac{-4}{a}\right) b &= 2 \sum_{ax+4by=4n+3} \left(\frac{-4}{a}\right) b\\ &= \sum_{ax+4by=4n+3} \left\{\left(\frac{-4}{a}\right) + \left(\frac{-4}{x}\right)\right\} b\\ &= 0\end{aligned}$$

and similarly

$$\sum_{\substack{ax+2by=4n+3 \\ y\equiv 0 \pmod 2 \\ b\equiv 1 \pmod 2}} \left(\frac{-4}{a}\right) b = 0.$$

Next, as

$$\left(\frac{-4}{a+2b}\right) = \left(\frac{-4}{a}\right), \text{ if } b \equiv 0 \pmod 2,$$

and

$$\left(\frac{-4}{a+2b}\right) = \left(\frac{-4}{a+2}\right) = -\left(\frac{-4}{a}\right), \text{ if } b \equiv 1 \pmod 2,$$

we have

$$\begin{aligned}
&\sum_{ax+2by=4n+3} \left\{ \left(\frac{-4}{a}\right) b + \left(\frac{-4}{a+2b}\right) b \right\} \\
&= 2 \sum_{\substack{ax+2by=4n+3 \\ b\equiv 0 \pmod 2}} \left(\frac{-4}{a}\right) b \\
&= 0,
\end{aligned}$$

so that

$$\delta_6(n) = - \sum_{ax+2by=4n+3} \left(\frac{-4}{a+2b}\right) b.$$

Now choose

$$f(a,b,x,y) = \left(\frac{-4}{a}\right) b$$

in Theorem 1. Then

$$\begin{aligned}
E &= \left(\frac{-4}{a}\right) b - \left(\frac{-4}{a}\right)(-b) + \left(\frac{-4}{a}\right)(a-b) - \left(\frac{-4}{a}\right)(a+b) \\
&\quad + \left(\frac{-4}{b-a}\right) b - \left(\frac{-4}{a+b}\right) b \\
&= \left\{ \left(\frac{-4}{b-a}\right) - \left(\frac{-4}{a+b}\right) \right\} b
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{ax+by=4n+3} \left\{ \left(\frac{-4}{b-a} \right) - \left(\frac{-4}{a+b} \right) \right\} b \\
&= \sum_{\substack{ax+by=4n+3 \\ a+1 \equiv b \equiv 0 \pmod 2}} \left\{ \left(\frac{-4}{b-a} \right) - \left(\frac{-4}{a+b} \right) \right\} b \\
&\quad + \sum_{\substack{ax+by=4n+3 \\ a \equiv b+1 \equiv 0 \pmod 2}} \left\{ \left(\frac{-4}{b-a} \right) - \left(\frac{-4}{a+b} \right) \right\} b \\
&= 2 \sum_{ax+2by=4n+3} \left\{ \left(\frac{-4}{2b-a} \right) - \left(\frac{-4}{2b+a} \right) \right\} b \\
&\quad + \sum_{2ax+by=4n+3} \left\{ \left(\frac{-4}{b-2a} \right) - \left(\frac{-4}{b+2a} \right) \right\} b \\
&= -4 \sum_{ax+2by=4n+3} \left(\frac{-4}{a+2b} \right) b,
\end{aligned}$$

as

$$\left(\frac{-4}{2b-a} \right) \left(\frac{-4}{2b+a} \right) = \left(\frac{-4}{4b^2-a^2} \right) = \left(\frac{-4}{-a^2} \right) = \left(\frac{-4}{3} \right) = -1,$$

for a odd, and

$$\left(\frac{-4}{b-2a} \right) \left(\frac{-4}{b+2a} \right) = \left(\frac{-4}{b^2-4a^2} \right) = \left(\frac{-4}{b^2} \right) = 1,$$

for b odd. We have shown that

$$\sum_{ax+by=4n+3} E = 4\delta_6(n).$$

Then, by Theorem 1, we have

$$\begin{aligned}
4\delta_6(n) &= \sum_{d \mid 4n+3} \sum_{x<d} \left\{ \left(\frac{-4}{(4n+3)/d} \right) \frac{4n+3}{d} - \left(\frac{-4}{x} \right) x - \left(\frac{-4}{d} \right) x \right\} \\
&= A_1 - A_2 - A_3,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{d\,|\,4n+3}\ \sum_{x<(4n+3)/d} \left(\frac{-4}{d}\right) d \\
&= \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d \left\{\frac{4n+3}{d} - 1\right\} \\
&= (4n+3) \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) - \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d \\
&= -\sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d; \\
A_2 &= \sum_{d\,|\,4n+3}\ \sum_{x<d} \left(\frac{-4}{x}\right) x \\
&= -\sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) \frac{d-1}{2} \\
&= -\frac{1}{2} \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d; \\
A_3 &= \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) \sum_{x<d} x \\
&= \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) \frac{(d-1)d}{2} \\
&= \frac{1}{2} \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d^2 - \frac{1}{2} \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d,
\end{aligned}$$

so that

$$4\delta_6(n) = -\frac{1}{2} \sum_{d\,|\,4n+3} \left(\frac{-4}{d}\right) d^2,$$

which gives the asserted formula. □

Theorem 12 ([27], Theorem 5).

$$\delta_8(n) = \sigma_3(n+1) - \sigma_3((n+1)/2).$$

Proof. We have appealing to Theorem 10 and Corollary 1

$$\begin{aligned}
\delta_8(n) &= \sum_{m=0}^{n} \delta_4(m)\delta_4(n-m) \\
&= \sum_{m=0}^{n} \sigma(2m+1)\sigma(2n-2m+1) \\
&= \sum_{m=1}^{n+1} \sigma(2m-1)\sigma((2n+2)-(2m-1)) \\
&= \frac{1}{24}(5\sigma_3(2n+2) - 21\sigma_3(n+1) + 16\sigma_3((n+1)/2) \\
&\quad -(11+12n)(\sigma(2n+2) - 3\sigma(n+1) + 2\sigma((n+1)/2))) \\
&= \sigma_3(n+1) - \sigma_3((n+1)/2),
\end{aligned}$$

as

$$\sigma_3(2n+2) = 9\sigma_3(n+1) - 8\sigma_3((n+1)/2)$$

and

$$\sigma(2n+2) = 3\sigma(n+1) - 2\sigma((n+1)/2).$$

This completes the proof of Theorem 12. □

7 Application of Theorem 1 to the Representations of a Positive Integer by Certain Quaternary Forms

Theorem 1 can be used to determine in an elementary manner the classical formulae for the number of representations of a positive integer n by certain quaternary forms such as $x^2+xy+y^2+u^2+uv+v^2$, $x^2+y^2+u^2+v^2$, $x^2+2y^2+u^2+2v^2$, etc. We just give one example to illustrate the ideas.

Theorem 13. *The number of representations of a positive integer n by the quaternary form $x^2+xy+y^2+u^2+uv+v^2$ is $12\sigma(n) - 36\sigma(n/3)$.*

Proof. Let k be a nonnegative integer. We set

$$r(k) = \text{card}\{(x,y) \in \mathbb{Z}^2 \mid k = x^2+xy+y^2\}$$

and

$$R(k) = \text{card}\{(x,y,u,v) \in \mathbb{Z}^4 \mid k = x^2+xy+y^2+u^2+uv+v^2\}.$$

Let n be a positive integer. Clearly

$$R(n) = \sum_{k=0}^{n} r(k)r(n-k)$$

so that as $r(0)=1$ we have

$$R(n)-2r(n)=\sum_{k=1}^{n-1} r(k)r(n-k). \tag{7.1}$$

It is a classical result that

$$r(n)=6\sum_{d\mid n}\left(\frac{-3}{d}\right)=6\tau_{1,3}(n)-6\tau_{2,3}(n), \tag{7.2}$$

where

$$\tau_{i,3}(n)=\sum_{\substack{d\mid n\\ d\equiv i \pmod 3}} 1, \quad i=0,1,2.$$

Since $\tau_{0,3}(n)=\tau(n/3)$ and $\tau_{0,3}(n)+\tau_{1,3}(n)+\tau_{2,3}(n)=\tau(n)$ we have

$$\begin{aligned}
\tau_{0,3}(n) &= \tau(n/3),\\
\tau_{1,3}(n) &= \frac{1}{2}\tau(n)-\frac{1}{2}\tau(n/3)+\frac{1}{12}r(n),\\
\tau_{2,3}(n) &= \frac{1}{2}\tau(n)-\frac{1}{2}\tau(n/3)-\frac{1}{12}r(n).
\end{aligned} \tag{7.3}$$

From (7.1) and (7.2) we obtain

$$R(n)-2r(n)=36\sum_{k=1}^{n-1}\sum_{a\mid k}\left(\frac{-3}{a}\right)\sum_{b\mid n-k}\left(\frac{-3}{b}\right)=36\sum_{ax+by=n}\left(\frac{-3}{ab}\right)$$

so that

$$\sum_{ax+by=n}\left(\frac{-3}{ab}\right)=\frac{1}{36}R(n)-\frac{1}{18}r(n). \tag{7.4}$$

We now choose

$$f(a,b,x,y)=\left(\frac{-3}{ab}\right).$$

Clearly this choice of f satisfies (2.1) so we may apply Theorem 1. We obtain after a little simplification

$$\begin{aligned}
&\sum_{ax+by=n}\left\{2\left(\frac{-3}{ab}\right)+2\left(\frac{-3}{a(a-b)}\right)-2\left(\frac{-3}{a(a+b)}\right)\right\}\\
&\quad=\sum_{d\,\mid\, n}\sum_{x<d}\left\{\left(\frac{-3}{(n/d)^2}\right)-\left(\frac{-3}{x(x-d)}\right)-2\left(\frac{-3}{dx}\right)\right\}.
\end{aligned} \tag{7.5}$$

A simple examination of the possible residues of a and b modulo 3 shows that

$$\left(\frac{-3}{a(a-b)}\right) - \left(\frac{-3}{a(a+b)}\right) = \left(\frac{-3}{ab}\right)$$

so that the left hand side of (7.5) is

$$4 \sum_{ax+by=n} \left(\frac{-3}{ab}\right) = \frac{1}{9}R(n) - \frac{2}{9}r(n)$$

by (7.4). Next we determine the sums on the right hand side of (7.5). First

$$\begin{aligned}
\sum_{d \mid n} \sum_{x<d} \left(\frac{-3}{(n/d)^2}\right) &= \sum_{d \mid n} \sum_{x<n/d} \left(\frac{-3}{d^2}\right) \\
&= \sum_{d \mid n,\, 3\nmid d} \sum_{x<n/d} 1 \\
&= \sum_{d \mid n,\, 3\nmid d} \left(\frac{n}{d} - 1\right) \\
&= (\sigma(n) - \tau(n)) - (\sigma(n/3) - \tau(n/3)).
\end{aligned}$$

Secondly

$$\begin{aligned}
&-\sum_{d \mid n} \sum_{x<d} \left(\frac{-3}{x(x-d)}\right) \\
&= \sum_{d \mid n} \sum_{\substack{x<d \\ 3\nmid x}} \left(\frac{-3}{x(d-x)}\right) \\
&= \sum_{d \mid n} \sum_{\substack{x<d \\ 3\nmid x}} \left(\frac{-3}{xd-1}\right) \\
&= \sum_{d \mid n} \left(\frac{-3}{d-1}\right) \sum_{\substack{x<d \\ x\equiv 1 \pmod 3}} 1 + \sum_{d \mid n} \left(\frac{-3}{2d-1}\right) \sum_{\substack{x<d \\ x\equiv 2 \pmod 3}} 1 \\
&= \sum_{d \mid n} \left(\frac{-3}{d-1}\right) \left[\frac{d+1}{3}\right] + \sum_{d \mid n} \left(\frac{-3}{2d-1}\right) \left[\frac{d}{3}\right] \\
&= \frac{1}{3}\sigma(n) - 3\sigma(n/3) - \frac{1}{18}r(n),
\end{aligned}$$

by (7.2). Thirdly

$$\begin{aligned} -2\sum_{d\mid n}\sum_{x<d}\left(\frac{-3}{dx}\right) &= -2\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\left[\frac{d+1}{3}\right]-\left[\frac{d}{3}\right]\right) \\ &= 2\tau_{2,3}(n) \\ &= \tau(n)-\tau(n/3)-\frac{1}{6}r(n), \end{aligned}$$

by (7.3). Thus the right hand side of (7.5) is

$$\frac{4}{3}\sigma(n)-4\sigma(n/3)-\frac{2}{9}r(n).$$

Hence

$$\frac{1}{9}R(n)-\frac{2}{9}r(n)=\frac{4}{3}\sigma(n)-4\sigma(n/3)-\frac{2}{9}r(n)$$

so that

$$R(n)=12\sigma(n)-36\sigma(n/3)$$

as asserted. □

Theorem 13 can be found for example in [22].

8 Further Convolution Sums

We conclude this paper by considering the sums

$$\begin{aligned} R &= \sum_{m<n/3}\sigma(n-3m)\sigma_3(m), \\ S &= \sum_{m<n/3}\sigma(m)\sigma_3(n-3m), \\ A &= \sum_{m<n/2}\sigma(m)\sigma_5(n-2m), \\ B &= \sum_{m<n/2}\sigma_3(m)\sigma_3(n-2m), \\ C &= \sum_{m<n/2}\sigma_5(m)\sigma(n-2m). \end{aligned}$$

Although we cannot evaluate any of R, S, A, B, C individually, Theorem 1 enables us to determine certain linear combinations of them.

Theorem 14.

$$\begin{aligned} 3R+S &= \frac{1}{240}(3\sigma_5(n)+(10-10n)\sigma_3(n)-3\sigma(n) \\ &\quad +81\sigma_5(n/3)+(30-90n)\sigma_3(n/3)-\sigma(n/3)). \end{aligned}$$

Proof. We set

$$G(x,y)=\begin{cases} 12, & \text{if } x\equiv y\equiv 0 \pmod 3, \\ 9, & \text{if } x\equiv y\not\equiv 0 \pmod 3, \\ 1, & \text{if } x\not\equiv y \pmod 3, \end{cases}$$

and choose

$$f(a,b,x,y)=a^3b\,G(x,y).$$

Clearly (2.3) is satisfied and a straightforward calculation shows that

$$E=\begin{cases} -24a^3b - 72a^3b, & \text{if } x\equiv y\equiv 0 \pmod 3, \\ -18a^3b - 6ab^3, & \text{if } x\equiv 0 \pmod 3,\ y\not\equiv 0 \pmod 3, \\ -18a^3b - 54ab^3, & \text{if } x\not\equiv 0 \pmod 3,\ y\equiv 0 \pmod 3, \\ 6a^3b - 6ab^3, & \text{if } x\not\equiv 0 \pmod 3,\ y\not\equiv 0 \pmod 3. \end{cases}$$

Firstly we have

$$\begin{aligned} \sum_{\substack{ax+by=n \\ x\equiv y\equiv 0 \pmod 3}} (-24a^3b-72ab^3) &= \sum_{ax+by=n/3} (-24a^3b-72ab^3) \\ &= -96 \sum_{ax+by=n/3} ab^3 \\ &= -96 \sum_{m<n/3} \sigma(m)\sigma_3(n/3-m). \end{aligned}$$

Secondly

$$\begin{aligned} &\sum_{\substack{ax+by=n \\ x\equiv 0 \pmod 3 \\ y\not\equiv 0 \pmod 3}} (-18a^3b-6ab^3) \\ &= \sum_{3ax+by=n} (-18a^3b-6ab^3) - \sum_{ax+by=n/3} (-18a^3b-6ab^3) \\ &= -18R-6S+24\sum_{m<n/3} \sigma(m)\sigma_3(n/3-m). \end{aligned}$$

Thirdly

$$\begin{aligned}
&\sum_{\substack{ax+by=n\\ x\not\equiv 0 \pmod 3\\ y\equiv 0 \pmod 3}} (-18a^3b-54ab^3)\\
&\quad = \sum_{\substack{ax+by=n\\ x\equiv 0 \pmod 3\\ y\not\equiv 0 \pmod 3}} (-18ab^3-54a^3b)\\
&\quad = -54R-18S+72\sum_{m<n/3}\sigma(m)\sigma_3(n/3-m)
\end{aligned}$$

as above. Fourthly the change of variables $(a,b,x,y)\longrightarrow(b,a,y,x)$ shows that

$$\sum_{\substack{ax+by=n\\ x,y\not\equiv 0 \pmod 3}} (6a^3b-6ab^3) = \sum_{\substack{ax+by=n\\ x,y\not\equiv 0 \pmod 3}} (6ab^3-6a^3b)$$

so that

$$\sum_{\substack{ax+by=n\\ x,y\not\equiv 0 \pmod 3}} (6a^3b-6ab^3) = 0.$$

Set

$$T := \sum_{m<n/3}\sigma(m)\sigma_3(n/3-m).$$

Hence the left hand side of (2.2) is

$$(-96T-18R-6S)+(24T-54R-18S)+72T = -72R-24S.$$

The right hand side of (2.2) is (after a long straightforward calculation)

$$\begin{aligned}
-\frac{1}{10}(3\sigma_5(n)&+(10-10n)\sigma_3(n)-3\sigma(n)+81\sigma_5(n/3)\\
&+(30-90n)\sigma_3(n/3)-\sigma(n/3)).
\end{aligned}$$

Theorem 14 follows by equating both sides of (2.2). □

Theorem 15.

$$\begin{aligned}
3A+8B &= \frac{1}{840}(28\sigma_7(n)+(105-105n)\sigma_5(n)-28\sigma_3(n)+128\sigma_7(n/2)\\
&\quad -28\sigma_3(n/2)+5\sigma(n/2)),\\
2B+3C &= \frac{1}{840}(2\sigma_7(n)-7\sigma_3(n)+5\sigma(n)+112\sigma_7(n/2)\\
&\quad +(105-210n)\sigma_5(n/2)-7\sigma_3(n/2)).
\end{aligned}$$

Proof. The choice

$$f(a,b,x,y) = (-ab^5 + 10a^3b^3 - 12a^4b^2)F_2(x)$$

in Theorem 1 yields $36A + 96B$ and the choice

$$f(a,b,x,y) = (ab^5 - 10a^3b^3 + 12a^4b^2 - 36a^6)F_2(x)$$

yields $24B + 36C$. The details are left to the reader. □

9 Conclusion

It is very likely that there are other choices of $f(a,b,x,y)$ for which Theorem 1 will yield new arithmetic identities. Moreover Theorem 1 itself may be capable of being generalized. It would also be interesting to know if the sums R, S, A, B, C of Section 8 can be determined individually, perhaps in terms of Ramanujan's tau function, and also whether $\delta_{16}(n)$ can be evaluated using Theorem 1.

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