probability p is. This result not only conforms to our intuition, but our explicit formula for p_{n+1} allows us to check how quickly p_{n+1} converges to 1 for various values of p.

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Uniquely Determined Unknowns in Systems of Linear Equations

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Perhaps the reader has noticed that when solving a consistent system of linear equations (linear system) it can happen that some unknowns are uniquely determined, while others are not?

EXAMPLE. Consider the linear system

$$\begin{cases} 6x_1 + 12x_2 + x_3 + 6x_4 + x_5 = 7, \\ 5x_1 + 10x_2 + x_3 + 5x_4 + x_5 = 6, \\ 13x_1 + 26x_2 + 2x_3 + 13x_4 + 3x_5 = 18, \end{cases}$$

over the field $\mathbb{R}$ of real numbers. The solution set is

$$x_1 = 1 - 2s - t, \quad x_2 = s, \quad x_3 = -2, \quad x_4 = t, \quad x_5 = 3, \quad \text{where } s, t \in \mathbb{R},$$

and in this case x_3, x_5 are uniquely determined while x_1, x_2, x_4 can take infinitely many values.

This example suggests the following three questions.

QUESTION 1. *What is a necessary and sufficient condition for an unknown to be uniquely determined by a consistent linear system?*

QUESTION 2. *How many of the unknowns are uniquely determined by a linear system?*

QUESTION 3. *If an unknown is uniquely determined by a linear system, is there an explicit formula for it?*

In this paper we answer these questions for linear systems defined over an arbitrary field $\mathbb{F}$.

We will write a linear system in its matrix form

$$AX = B, \tag{1}$$

where the coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m,n}(\mathbb{F}),$$

the column vectors of unknowns and constant terms are respectively

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in M_{n,1}(\mathbb{F}) \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in M_{m,1}(\mathbb{F}),$$

and the augmented matrix is

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \in M_{m,n+1}(\mathbb{F}).$$

The linear system defined by (1) is consistent if and only if

$$\operatorname{rank} A = \operatorname{rank}[A \mid B]. \tag{2}$$

From this point on, we assume that (2) holds and so (1) has at least one solution $X \in M_{n,1}(\mathbb{F})$.

Let $A^{(j)}$ denote the $m \times (n-1)$ matrix obtained by removing the jth column of A. Clearly, removing the jth column of A from A decreases the rank of A by at most 1. We can therefore classify the columns of the matrix A as either "rank-preserving" or "rank-decreasing."

DEFINITION 1. *The jth column of A is said to be "rank-preserving" if* $\operatorname{rank} A^{(j)} = \operatorname{rank} A$ *and to be "rank-decreasing" if* $\operatorname{rank} A^{(j)} = \operatorname{rank} A - 1$.

For example the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & i \\ 1 & 2+i & 4+i & 2i \\ 1+i & 3 & 5+2i & 1-3i \end{bmatrix} \in M_{3,4}(\mathbb{C}),$$

where $\mathbb{C}$ denotes the field of complex numbers, has three rank-preserving columns and one rank-decreasing column because

$$\operatorname{rank} A = \operatorname{rank} A^{(1)} = \operatorname{rank} A^{(2)} = \operatorname{rank} A^{(3)} = 3, \quad \operatorname{rank} A^{(4)} = 2.$$

We are now ready to answer Question 1.

THEOREM 1. *Suppose that the linear system defined by* (1) *is consistent. Then the unknown x_j ($j = 1, 2, \ldots, n$) is uniquely determined if and only if the jth column of A is rank-decreasing.*

Proof. We denote the columns of A by $C_1, C_2, \ldots, C_n$. Suppose that the jth column of A is rank-decreasing. Then $\operatorname{rank} A^{(j)} = \operatorname{rank} A - 1$. Hence C_j is not a linear combination of the other columns. Thus every solution of

$$AX = x_1C_1 + \cdots + x_jC_j + \cdots + x_nC_n = 0$$

has $x_j = 0$, and so every solution of $AX = B$ has the same value for x_j. Hence x_j is uniquely determined.

Now suppose that the jth column of A is rank-preserving. Then $\operatorname{rank} A^{(j)} = \operatorname{rank} A$. Hence C_j is a linear combination of the other columns in A and so there are solutions of

$$AX = x_1C_1 + \cdots + x_jC_j + \cdots + x_nC_n = 0$$

with $x_j \neq 0$. Hence $AX = B$ has solutions with different values of x_j. Thus x_j is not uniquely determined. ■

The linear system in the example has coefficient matrix

$$A = \begin{bmatrix} 6 & 12 & 1 & 6 & 1 \\ 5 & 10 & 1 & 5 & 1 \\ 13 & 26 & 2 & 13 & 3 \end{bmatrix},$$

and it is routine to check that $\operatorname{rank} A = \operatorname{rank} A^{(1)} = \operatorname{rank} A^{(2)} = \operatorname{rank} A^{(4)} = 3$ and that $\operatorname{rank} A^{(3)} = \operatorname{rank} A^{(5)} = 2$ so that only the third and fifth columns of A are rank-decreasing. Theorem 1 confirms that only x_3 and x_5 are uniquely determined.

Theorem 1 can now be applied to answer Question 2.

THEOREM 2. *Suppose that the linear system defined by* (1) *is consistent and that* $r = \operatorname{rank} A$. *Then the number of unknowns that are uniquely determined by the system is*

$$nr - \sum_{j=1}^{n} \operatorname{rank} A^{(j)}.$$

Proof. By Theorem 1, the number N of the x_j uniquely determined by (1) is precisely the number of rank-decreasing columns of A, and because

$$r - \operatorname{rank} A^{(j)} = \begin{cases} 1, & \text{if the } j\text{th column of } A \text{ is rank-decreasing,} \\ 0, & \text{if the } j\text{th column of } A \text{ is rank-preserving,} \end{cases}$$

we have

$$N = \sum_{j=1}^{n} (r - \operatorname{rank} A^{(j)}) = nr - \sum_{j=1}^{n} \operatorname{rank} A^{(j)}.$$ ■

Applying Theorem 2 to the system in the example, we have

$$N = 5 \times 3 - (3 + 3 + 2 + 3 + 2) = 15 - 13 = 2.$$

DEFINITION 2. *For* $i = 1, 2, \ldots, n$ *the matrix* $E_i \in M_{1,n}(\mathbb{F})$ *is defined by*

$$E_i = [\, 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0 \,],$$

where 1 *occurs in the* i*th place and* 0 *elsewhere.*

We are now ready to answer Question 3. By eliminating any equations from the system (1) that are linear combinations of other equations, we may suppose without loss of generality that $m = r = \operatorname{rank} A$.

THEOREM 3. *Suppose that the linear system defined by* (1) *is consistent, and that* $m = r$ $(= \operatorname{rank} A)$. *Let* A_i $(i = 1, 2, \ldots, m)$ *denote the* i*th row of* A. *Integers* $k_1, \ldots, k_{n-r}$ *with* $1 \le k_1 < k_2 < \cdots < k_{n-r} \le n$ *may be chosen so that*

$$\operatorname{span}(A_1, \ldots, A_r, E_{k_1}, \ldots, E_{k_{n-r}}) = \mathbb{F}^n.$$

Let $A^{(k_1,\ldots,k_{n-r})} \in M_{r,r}(\mathbb{F})$ *be formed from* A *by deleting columns* $k_1, \ldots, k_{n-r}$. *Let* $j \in \{1, 2, \ldots, n\}$ *be such that* x_j *is a uniquely determined unknown in* (1). *Let* $A^{(k_1,\ldots,k_{n-r})}(j, B) \in M_{r,r}(\mathbb{F})$ *be formed from* A *by replacing the* j*th column by* B *and deleting columns* $k_1, \ldots, k_{n-r}$. *Then*

$$x_j = \frac{\det(A^{(k_1,\ldots,k_{n-r})}(j, B))}{\det(A^{(k_1,\ldots,k_{n-r})})}.$$

Proof. As $\{E_1, \ldots, E_n\}$ span $\mathbb{F}^n$ and $\{A_1, \ldots, A_r\}$ are linearly independent over $\mathbb{F}$, by the Steinitz Exchange Theorem [**2**, p. 276], r of $\{E_1, \ldots, E_n\}$ can be replaced by $\{A_1, \ldots, A_r\}$ so that

$$\operatorname{span}(A_1, \ldots, A_r, E_{k_1}, \ldots, E_{k_{n-r}}) = \mathbb{F}^n,$$

where $1 \le k_1 < k_2 < \cdots < k_{n-r} \le n$.

We note that as x_j is uniquely determined, E_j belongs to the row space of A so that $j \ne k_1, \ldots, k_{n-r}$. Let $A^* \in M_{n,n}(\mathbb{F})$ be formed from A by adjoining $E_{k_1}, \ldots, E_{k_{n-r}}$ as rows $r+1, \ldots, n$. Clearly the set $\{A_1, \ldots, A_r, E_{k_1}, \ldots, E_{k_{n-r}}\}$ is a basis for $\mathbb{F}^n$ and so $\det A^* \ne 0$. Moreover, using the Laplace expansion theorem (see, for example, [**1**, p. 21]) to expand $\det A^*$ by its last $n - r$ rows, we obtain

$$\det A^* = (-1)^{(r+1)+\cdots+n+k_1+\cdots+k_{n-r}} \det A^{(k_1,\ldots,k_{n-r})}.$$

Hence

$$\det A^{(k_1,\ldots,k_{n-r})} \ne 0. \tag{3}$$

Let $X^* \in M_{r,1}(\mathbb{F})$ be the column matrix formed from X by removing $x_{k_1}, \ldots, x_{k_{n-r}}$. Set

$$B^* = B - x_{k_1} A^{(k_1)} - \cdots - x_{k_{n-r}} A^{(k_{n-r})}.$$

Then the linear system defined by (1) can be rewritten as

$$A^{(k_1,\ldots,k_{n-r})} X^* = B^*. \tag{4}$$

From (3) and (4) we see that all the x_v with $v \ne k_1, \ldots, k_{n-r}$ are uniquely determined in terms of the $n - r$ free variables $x_{k_1}, \ldots, x_{k_{n-r}}$. Thus x_j is independent of the choice of $x_{k_1}, \ldots, x_{k_{n-r}}$ and so we may choose $x_{k_1} = \cdots = x_{k_{n-r}} = 0$ in (4) to determine x_j. The matrix form of the linear system becomes

$$A^{(k_1,\ldots,k_{n-r})} X^* = B$$

and Cramer's rule gives

$$x_j = \frac{\det(A^{(k_1,\ldots,k_{n-r})}(j, B))}{\det(A^{(k_1,\ldots,k_{n-r})})}. \qquad \blacksquare$$

We close by revisiting the example to compute the uniquely determined unknowns x_3 and x_5. We have $m = 3$, $n = 5$, $r = 3 = \operatorname{rank} A$ and $n - r = 2$ in this case. It is easy to check that

$$\operatorname{span}(A_1, A_2, A_3, E_1, E_2) = \mathbb{R}^5$$

so that we can take $k_1 = 1, k_2 = 2$ here. By Theorem 3, we obtain

$$x_3 = \frac{\det(A^{(1,2)}(3, B))}{\det(A^{(1,2)})} = \frac{\begin{vmatrix} 7 & 6 & 1 \\ 6 & 5 & 1 \\ 18 & 13 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 6 & 1 \\ 1 & 5 & 1 \\ 2 & 13 & 3 \end{vmatrix}} = \frac{2}{-1} = -2$$

and

$$x_5 = \frac{\det(A^{(1,2)}(5, B))}{\det(A^{(1,2)})} = \frac{\begin{vmatrix} 1 & 6 & 7 \\ 1 & 5 & 6 \\ 2 & 13 & 18 \end{vmatrix}}{\begin{vmatrix} 1 & 6 & 1 \\ 1 & 5 & 1 \\ 2 & 13 & 3 \end{vmatrix}} = \frac{-3}{-1} = 3.$$

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Counterintuitive Aspects of Plane Curvature

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A study of the curvature of a plane curve of the form $y = f(x)$ leads to some counterintuitive results. For instance, the curvature of a function whose graph is concave up may not approach 0 as x approaches ∞, and the curvature of a function with a vertical asymptote at $x = c$ may not approach 0 as x approaches c. In addition, scaling a function affects its curvature qualitatively as well as quantitatively. A discussion of the limit properties of curvature involves ideas from elementary real analysis, while the impact of scaling can be used to create some exploratory exercises for calculus students using a computer algebra system.

Let f be a real-valued twice differentiable function defined on an interval I. The curvature κ of f, which is a measure of the rate at which the graph of $y = f(x)$ is turning, is given by

$$\kappa(x) = \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}$$