JP Jour. Algebra, Number Theory & Appl. **1**(3) (2001), 179-194

INTEGERS WHICH ARE DISCRIMINANTS OF BICYCLIC OR CYCLIC QUARTIC FIELDS

BLAIR K. SPEARMAN

Department of Mathematics and Statistics, Okanagan University College, Kelowna, British Columbia V1V 1V7, *Canada*

e-mail: bkspearm@okuc02.okanagan.bc.ca

KENNETH S. WILLIAMS

Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario K1S 5B6, *Canada*

e-mail: williams@math.carleton.ca

Abstract

Asymptotic formulae are obtained for the number $B(x)$ of positive integers $n \leq x$ which are discriminants of bicyclic quartic fields and the number $C(x)$ of those which are discriminants of cyclic quartic fields.

1. Notation

The fields of real numbers and rational numbers are denoted by $\mathbb{R}$ and $\mathbb{Q}$ respectively, and the sets of integers, positive integers and nonzero integers by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}^*$ respectively. If $a \in \mathbb{Z}^*$ and $b \in \mathbb{Z}^*$, we

2000 Mathematics Subject Classification: Primary 11R16, 11R29.

Key words and phrases: bicyclic quartic field, cyclic quartic field, discriminant.

Both authors were supported by research grants from the Natural Sciences and Engineering Research Council of Canada.

Communicated by Istvan Gaal

Received December 14, 2001

denote their greatest common divisor by (a, b) and their least common multiple by $[a, b]$ so that $(a, b)[a, b] = ab$. For $x \in \mathbb{R}$, $m \in \mathbb{Z}^*$ and p a prime, we set

$$\omega(m) = \text{number of distinct prime factors of } m,$$

$$d(m) = \text{number of positive divisors of } m,$$

$$v_p(m) = \text{largest integer } t \text{ such that } p^t \mid m,$$

$$\tilde{m} = (-1)^{(m-1)/2} m, \text{ if } m \text{ is odd, so that } \tilde{m} \equiv 1 (\text{mod } 4),$$

$$\pi(x) = \text{number of primes } p \leq x.$$

2. Introduction

Let K be a bicyclic or cyclic quartic extension field of $\mathbb{Q}$. It is well known that the discriminant $d(K)$ of K is a positive integer. For $n \in \mathbb{N}$, we let

$$b(n) = \begin{cases} 1, & \text{if } n = d(K) \text{ for some bicyclic quartic field } K, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c(n) = \begin{cases} 1, & \text{if } n = d(K) \text{ for some cyclic quartic field } K, \\ 0, & \text{otherwise.} \end{cases}$$

We determine asymptotic formulae for

$$B(x) = \sum_{1 \leq n \leq x} b(n) \tag{2.1}$$

and

$$C(x) = \sum_{1 \leq n \leq x} c(n) \tag{2.2}$$

valid as $x \to +\infty$. $B(x)$ counts the number of positive integers $n \leq x$ such that $n = d(K)$ for some bicyclic quartic field K and $C(x)$ the number of positive integers $n \leq x$ such that $n = d(K)$ for some cyclic quartic field K.

3. Some Asymptotic Formulae

As usual, for $n \in \mathbb{N}$, $\mu(n)$ is the Möbius function and $\phi(n)$ is Euler's totient function.

Lemma 1. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{1 \le d \le x \\ (d,k)=1}} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1} + O\left(\frac{1}{x}\right),$$

as $x \to +\infty$, where the constant implied by the O-symbol is absolute.

Proof. We have

$$\sum_{\substack{1 \le d \le x \\ (d,k)=1}} \frac{\mu(d)}{d^2} = \sum_{\substack{d=1 \\ (d,k)=1}}^{\infty} \frac{\mu(d)}{d^2} - \sum_{\substack{d > x \\ (d,k)=1}} \frac{\mu(d)}{d^2}.$$

Now

$$\sum_{\substack{d=1 \\ (d,k)=1}}^{\infty} \frac{\mu(d)}{d^2} = \prod_{p \nmid k} \left(1 - \frac{1}{p^2}\right) = \prod_{p} \left(1 - \frac{1}{p^2}\right) \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1}$$

$$= \frac{6}{\pi^2} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1}$$

and

$$\sum_{\substack{d > x \\ (d,k)=1}} \frac{\mu(d)}{d^2} = O\left(\sum_{d > x} \frac{1}{d^2}\right) = O\left(\frac{1}{x}\right),$$

so the asserted formula follows.

Lemma 2. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{1 \le e \le x \\ (e,k)=1}} 1 = x \frac{\phi(k)}{k} + O(d(k)),$$

as $x \to +\infty$, where the constant implied by the O-symbol is absolute.

Proof. We have

$$\sum_{\substack{1\le e\le x\\(e,k)=1}} 1 = \sum_{1\le e\le x}\ \sum_{d\mid(e,k)} \mu(d)$$

$$= \sum_{d\mid k}\mu(d) \sum_{\substack{1\le e\le x\\ d\mid e}} 1$$

$$= \sum_{d\mid k}\mu(d) \sum_{1\le f\le x/d} 1$$

$$= \sum_{d\mid k}\mu(d)\left[\frac{x}{d}\right]$$

$$= \sum_{d\mid k}\mu(d)\left\{\frac{x}{d} + O(1)\right\}$$

$$= x\sum_{d\mid k}\frac{\mu(d)}{d} + O(d(k))$$

$$= x\,\frac{\phi(k)}{k} + O(d(k)),$$

completing the proof of the lemma.

Lemma 3. *Let* $k \in \mathbb{N}$. *Then*

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\(e,k)=1}} 1 = x\,\frac{6}{\pi^2}\,\frac{\phi(k)}{k}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1} + O(x^{1/2}d(k)),$$

as $x \to +\infty$, *where the constant implied by the O-symbol is absolute.*

Proof. Appealing to Lemmas 1 and 2, we obtain

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\(e,k)=1}} 1 = \sum_{\substack{1\le e\le x\\(e,k)=1}}\ \sum_{d^2\mid e}\mu(d)$$

$$= \sum_{\substack{1\le d^2e\le x\\(d^2e,k)=1}} \mu(d)$$

$$= \sum_{\substack{1 \le d \le x^{1/2} \\ (d,k)=1}} \mu(d) \sum_{\substack{1 \le e \le x/d^2 \\ (e,k)=1}} 1$$

$$= \sum_{\substack{1 \le d \le x^{1/2} \\ (d,k)=1}} \mu(d) \left\{ \frac{x}{d^2} \frac{\phi(k)}{k} + O(d(k)) \right\}$$

$$= x \frac{\phi(k)}{k} \sum_{\substack{1 \le d \le x^{1/2} \\ (d,k)=1}} \frac{\mu(d)}{d^2} + O(x^{1/2} d(k))$$

$$= x \frac{\phi(k)}{k} \left\{ \frac{6}{\pi^2} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1} + O\left(\frac{1}{x^{1/2}}\right) \right\} + O(x^{1/2} d(k))$$

$$= x \frac{6}{\pi^2} \frac{\phi(k)}{k} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{1/2} d(k)),$$

as asserted.

Lemma 4.

$$\sum_{\substack{1 \le e \le x \\ e \text{ squarefree} \\ e \text{ odd}}} 1 = \frac{4}{\pi^2} x + O(x^{1/2}),$$

as $x \to +\infty$, *where the constant implied by the O-symbol is absolute.*

Proof. By Lemma 3 with $k = 2$, we obtain

$$\sum_{\substack{1 \le e \le x \\ e \text{ squarefree} \\ e \text{ odd}}} 1 = x \frac{6}{\pi^2} \frac{\phi(2)}{2} \prod_{p \mid 2} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{1/2}) = \frac{4}{\pi^2} x + O(x^{1/2}),$$

as asserted.

In the proof of the next lemma, we make use of the following weak form of the prime number theorem

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \quad \text{as } x \to +\infty. \tag{3.1}$$

Lemma 5.

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)\ge 2}} 1 = \frac{4}{\pi^2}x - \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \quad \textit{as } x \to +\infty.$$

Proof. We have

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)\ge 2}} 1 = \sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}}} 1 - \sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)=0}} 1 - \sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)=1}} 1.$$

By Lemma 4 the first sum on the right hand side is

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}}} 1 = \frac{4}{\pi^2}x + O(x^{1/2}).$$

Clearly the second sum is

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)=0}} 1 = 1,$$

and the third sum is

$$\sum_{\substack{1\le e\le x\\ e \text{ squarefree}\\ e \text{ odd}\\ \omega(e)=1}} 1 = \sum_{\substack{3\le p\le x\\ p \text{ prime}}} 1 = \pi(x) - 1 = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

by (3.1). The asserted asymptotic formula now follows.

Lemma 6.

$$\sum_{1\le n\le x} \frac{d(n)}{n^{3/4}} = O(x^{1/4}\log x).$$

Proof. By partial summation, see for example [1, Theorem 421], we have

$$\sum_{1\le n\le x} \frac{d(n)}{n^{3/4}} = \frac{1}{x^{3/4}} \sum_{1\le n\le x} d(n) + \frac{3}{4}\int_1^x \frac{\sum_{1\le n\le t} d(n)}{t^{7/4}}\, dt.$$

A very weak form of Dirichlet's divisor problem estimate is [1, Theorem 320]

$$\sum_{1\le n\le x} d(n) = O(x \log x).$$

Using this estimate and the evaluation

$$\int_1^x \frac{\log t}{t^{3/4}}\, dt = 4x^{1/4}\log x - 16x^{1/4} + 16,$$

we obtain the asserted estimate.

4. Bicyclic Quartic Fields

Every bicyclic quartic field K can be expressed in the form

$$K = \mathbb{Q}(\sqrt{m}, \sqrt{n}),$$

where m and n are distinct squarefree integers $\neq 1$. We let

$$l = (m, n),\ m_1 = m/l,\ n_1 = n/l,$$

so that $(m_1, n_1) = 1$. Since

$$\mathbb{Q}(\sqrt{m}, \sqrt{n}) = \mathbb{Q}(\sqrt{n}, \sqrt{m}) = \mathbb{Q}(\sqrt{m}, \sqrt{m_1 n_1}) = \mathbb{Q}(\sqrt{m_1 n_1}, \sqrt{m})$$
$$= \mathbb{Q}(\sqrt{n}, \sqrt{m_1 n_1}) = \mathbb{Q}(\sqrt{m_1 n_1}, \sqrt{n}),$$

without loss of generality we may suppose throughout this section that

$$(m, n) \equiv (1, 1), (1, 2), (2, 3) \text{ or } (3, 3) \pmod 4.$$

Williams [4, Theorem 3, p. 525] has determined the discriminant of K.

Theorem 1. *The discriminant* $d(K)$ *of* $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ *is given by*

$$d(K) = 2^{\alpha}[m, n]^2,$$

where

$$\alpha = \begin{cases} 0, & \textit{if } (m, n) \equiv (1, 1) \pmod 4, \\ 4, & \textit{if } (m, n) \equiv (1, 2) \textit{ or } (3, 3) \pmod 4, \\ 6, & \textit{if } (m, n) \equiv (2, 3) \pmod 4. \end{cases}$$

As an immediate consequence of Theorem 1, we have

Corollary 1. *Let K be a bicyclic quartic field. Then $d(K)$ is a perfect square and $v_2(d(K)) = 0, 4, 6$ or 8.*

From Theorem 1 and Corollary 1 we deduce a necessary and sufficient condition for a positive integer N to be the discriminant of a bicyclic quartic field, see Theorem 2. It is convenient to define

$$A = \{a \in \mathbb{N} \mid a \text{ odd}, a \text{ squarefree}, \omega(a) \geq 2\} \tag{4.1}$$

and for $x \in \mathbb{R}$,

$$A(x) = \sum_{\substack{a \leq x \\ a \in A}} 1. \tag{4.2}$$

By Lemma 5 we know that

$$A(x) = \frac{4}{\pi^2} x - \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \quad \text{as } x \to +\infty. \tag{4.3}$$

We prove

Theorem 2. *Let $N \in \mathbb{N}$. Then*

$$N = d(K) \textit{ for some bicyclic quartic field } K, \tag{4.4}$$

if and only if

$$N = 256, \tag{4.5}$$

or

$$N = 16p^2, 64p^2 \textit{ or } 256p^2, \textit{ where } p \textit{ is an odd prime}, \tag{4.6}$$

or

$$N = a^2, 16a^2, 64a^2 \textit{ or } 256a^2 \textit{ for some } a \in A. \tag{4.7}$$

Proof. If (4.5) holds, then, by Theorem 1, $N = 256 = d(K)$ for $K = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$.

If the first equality holds in (4.6), then, by Theorem 1, for some odd prime p, $N = 16p^2 = d(K)$ for $K = \mathbb{Q}(\sqrt{-\tilde{p}}, \sqrt{-1})$.

If the second equality holds in (4.6), then, by Theorem 1, for some odd prime p, $N = 64p^2 = d(K)$ for $K = \mathbb{Q}(\sqrt{\tilde{p}}, \sqrt{2})$.

If the third equality holds in (4.6), then, by Theorem 1, for some odd prime p, $N = 256p^2 = d(K)$ for $K = \mathbb{Q}(\sqrt{2p}, \sqrt{-1})$.

If (4.7) holds, then $N = 2^k a^2$ for some $a \in A$ and $k = 0, 4, 6$ or 8. As $a \in A$ we have $a = bc$, where

$$b, c \in \mathbb{N},\ b, c \text{ odd, squarefree, } \omega(b) \geq 1,\ \omega(c) \geq 1,\ (b, c) = 1.$$

Set

$$K = \mathbb{Q}\left(\sqrt{b^*}, \sqrt{c^*}\right),$$

where

$$\begin{array}{lll} b^* = \tilde{b}, & c^* = \tilde{c}, & \text{if } k = 0, \\ b^* = -\tilde{b}, & c^* = -\tilde{c}, & \text{if } k = 4, \\ b^* = \tilde{b}, & c^* = 2c, & \text{if } k = 6, \\ b^* = 2b, & c^* = -\tilde{c}, & \text{if } k = 8. \end{array}$$

Clearly, in all four cases b^* and c^* are squarefree, coprime integers $\neq 1$. Moreover

$$\begin{array}{lll} b^* \equiv 1 \pmod 4, & c^* \equiv 1 \pmod 4, & \text{if } k = 0, \\ b^* \equiv 3 \pmod 4, & c^* \equiv 3 \pmod 4, & \text{if } k = 4, \\ b^* \equiv 1 \pmod 4, & c^* \equiv 2 \pmod 4, & \text{if } k = 6, \\ b^* \equiv 2 \pmod 4, & c^* \equiv 3 \pmod 4, & \text{if } k = 8. \end{array}$$

Let

$$g = \begin{cases} 0, & \text{if } k = 0, \\ 4, & \text{if } k = 4, 6, \\ 6, & \text{if } k = 8. \end{cases}$$

Hence, by Theorem 1, we have

$$N = 2^k a^2 = 2^k (bc)^2 = 2^g (b^* c^*)^2 = 2^g [b^*, c^*]^2 = d(K).$$

Conversely, now suppose that $N = d(K)$ for some bicyclic quartic field K. By Corollary 1, we have

$$d(K) = a^2,\ 2^4 a^2,\ 2^6 a^2 \text{ or } 2^8 a^2,$$

for some odd positive integer a.

If $d(K) = a^2$, then, by Theorem 1, $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ for distinct squarefree integers $m \neq 1$ and $n \neq 1$ with $m \equiv n \equiv 1 \pmod 4$ and $a = [m, n] \in \mathbb{N}$. As m and n are squarefree and odd so is a. If $\omega(a) = 0$, then $a = 1$ and $(m, n) = (1, 1), (1, -1), (-1, 1)$ or $(-1, -1)$, none of which satisfy the conditions on m and n. If $\omega(a) = 1$, then $a = p$, where p is an odd prime, so that $(m, n) = (p, 1), (p, -1), (-p, 1), (-p, -1), (1, p), (1, -p), (-1, p), (-1, -p), (p, p), (p, -p), (-p, p)$ or $(-p, -p)$ and again none of these satisfy the conditions on m and n. Hence $\omega(a) \geq 2$ and $a \in A$.

If $d(K) = 16a^2$, then, by Theorem 1, $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ for distinct squarefree integers m and n with $m \equiv n \equiv 3 \pmod 4$ and $a = [m, n] \in \mathbb{N}$. As m and n are squarefree and odd so is a. If $\omega(a) = 0$, then $a = 1$ and $(m, n) = (1, 1), (1, -1), (-1, 1)$ or $(-1, -1)$, all of which cannot occur. If $\omega(a) = 1$, then $a = p$, where p is an odd prime, so that $(m, n) = (p, 1), (p, -1), (-p, 1), (-p, -1), (1, p), (1, -p), (-1, p), (-1, -p), (p, p), (p, -p), (-p, p)$ or $(-p, -p)$. All of these cannot occur except $(m, n) = (-\tilde{p}, -1)$ and $(-1, \tilde{p})$. These give $K = \mathbb{Q}(\sqrt{-1}, \sqrt{-\tilde{p}})$ and $N = d(K) = 16p^2$, which is the first possibility in (4.6). Otherwise $\omega(a) \geq 2$ and $a \in A$.

If $d(K) = 64a^2$, then, by Theorem 1, $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ for distinct squarefree integers $m \neq 1$ and n with $m \equiv 1 \pmod 4$, $n = 2n_1 \equiv 2 \pmod 4$ and $a = [m, n_1] \in \mathbb{N}$. As m and n_1 are squarefree and odd so is a. If $\omega(a) = 0$, then $a = 1$ and $(m, n_1) = (1, 1), (1, -1), (-1, 1)$ or $(-1, -1)$, all of which cannot occur. If $\omega(a) = 1$, then $a = p$, where p is an odd prime, so that $(m, n_1) = (p, 1), (p, -1), (-p, 1), (-p, -1), (1, p), (1, -p), (-1, p), (-1, -p), (p, p), (p, -p), (-p, p)$ or $(-p, -p)$. All of these cannot occur except $(m, n_1) = (\tilde{p}, \pm 1)$ and $(\tilde{p}, \pm p)$. These give $K = \mathbb{Q}(\sqrt{\tilde{p}}, \sqrt{\pm 2})$ and $N = d(K) = 64p^2$, which is the second possibility in (4.6). Otherwise $\omega(a) \geq 2$ and $a \in A$.

If $d(K) = 256a^2$, then, by Theorem 1, $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ for distinct squarefree integers m and n with $m = 2m_1 \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$ and $a = [m_1, n] \in \mathbb{N}$. As m_1 and n are squarefree and odd so is a. If $\omega(a) = 0$, then $a = 1$ and $(m_1, n) = (\pm 1, -1)$, $K = \mathbb{Q}(\sqrt{\pm 2}, \sqrt{-1})$ and $N = d(K) = 256$, which is (4.5). If $\omega(a) = 1$, then $a = p$, where p is an odd prime, so that $(m_1, n) = (\pm p, -1), (\pm 1, -\tilde{p})$ or $(\pm p, -\tilde{p})$. These give $K = \mathbb{Q}(\sqrt{2p}, \sqrt{-1})$ or $K = \mathbb{Q}(\sqrt{\pm 2}, \sqrt{-\tilde{p}})$ and both possibilities give $N = d(K) = 256p^2$, which is the third possibility in (4.6). Otherwise $\omega(a) \geq 2$ and $a \in A$.

We are now ready to determine an asymptotic formula for $B(x)$, which was defined in (2.1).

Theorem 3.

$$B(x) = \frac{23}{4\pi^2} x^{1/2} - \frac{2x^{1/2}}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x}\right),$$

as $x \to +\infty$.

Proof. By Theorem 2, for x sufficiently large, we have

$$B(x) = A(x^{1/2}) + A(x^{1/2}/4) + A(x^{1/2}/8) + A(x^{1/2}/16)$$
$$+ (\pi(x^{1/2}/4) - 1) + (\pi(x^{1/2}/8) - 1) + (\pi(x^{1/2}/16) - 1) + 1,$$

so that

$$B(x) = A(x^{1/2}) + A(x^{1/2}/4) + A(x^{1/2}/8) + A(x^{1/2}/16) + \pi(x^{1/2}/4) + \pi(x^{1/2}/8) + \pi(x^{1/2}/16) - 2.$$

Appealing to (3.1) and (4.3), we obtain the asserted result.

5. Cyclic Quartic Fields

Every cyclic quartic field K can be expressed uniquely in the form

$$K = \mathbb{Q}(\sqrt{A(D + B\sqrt{D})}), \tag{5.1}$$

where A, B, D are integers such that

$$A \text{ is squarefree and odd,} \tag{5.2}$$

$$B \geq 1, \ D \geq 2, \tag{5.3}$$

$$D \text{ is squarefree and } D - B^2 \text{ is a square,} \tag{5.4}$$

$$(A, D) = 1, \tag{5.5}$$

and the discriminant $d(K)$ of K is given by

$$d(K) = \begin{cases} 2^8 A^2 D^3, & \text{if } D \equiv 0 \pmod 2, \\ 2^6 A^2 D^3, & \text{if } D \equiv 1 \pmod 2, \ B \equiv 1 \pmod 2, \\ 2^4 A^2 D^3, & \text{if } D \equiv 1 \pmod 2, \ B \equiv 0 \pmod 2, \ A + B \equiv 3 \pmod 4, \\ A^2 D^3, & \text{if } D \equiv 1 \pmod 2, \ B \equiv 0 \pmod 2, \ A + B \equiv 1 \pmod 4, \end{cases}$$

see for example [2]. Let

$$P(1, 4) = \{n \in \mathbb{N} \mid n = p_1 \cdots p_m, \ m \geq 1, \ p_1, \ldots, p_m \text{ distinct primes} \equiv 1 \pmod 4\}.$$

We note that $1 \notin P(1, 4)$. For $D \in P(1, 4)$ we define

$$S(D) = \{n \in \mathbb{N} \mid n \text{ squarefree}, \ (n, 2D) = 1\}.$$

We note that $1 \in S(D)$. If D is odd and satisfies (5.3) and (5.4), then $D \in P(1, 4)$ and conversely. If D is even and satisfies (5.3) and (5.4), then

$D/2 \in P(1, 4) \cup \{1\}$ and conversely. From (5.1)-(5.5) and the formula for $d(K)$ we obtain the following result.

Theorem 4. *Let $n \in \mathbb{N}$. Then*

$$n = d(K) \text{ for some cyclic quartic field } K,$$

if and only if

$$n = 2^{11}A^2 \text{ for some odd positive squarefree integer } A,$$

or

$$n = A^2D^3,\ 2^4A^2D^3,\ 2^6A^2D^3 \text{ or } 2^{11}A^2D^3$$

$$\text{for some } D \in P(1, 4) \text{ and some } A \in S(D).$$

It is convenient to define

$$T(x) = \sum_{\substack{1 \le A \le x \\ A \in S(D)}} 1.$$

The following estimate for $T(x)$ follows from Lemma 3 (with $k = 2D$),

$$T(x) = x\frac{3}{\pi^2}\frac{\phi(D)}{D}\prod_{p \mid 2D}\left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{1/2}d(D)). \tag{5.6}$$

We are now ready to determine an asymptotic formula for $C(x)$, which was defined in (2.2).

Theorem 5.

$$C(x) = \frac{11}{2\pi^2}\left\{\frac{88 + \sqrt{2}}{88}\prod_{p \equiv 1 (\text{mod } 4)}\left(1 + \frac{1}{(p+1)\sqrt{p}}\right) - 1\right\}x^{1/2} + O(x^{1/3}\log x),$$

as $x \to +\infty$.

Proof. By Theorem 4, we have

$$C(x) = \sum_{\alpha \in \{0, 2, 3, \frac{11}{2}\}}\ \sum_{\substack{D \le x^{1/3} \\ D \in P(1,4)}} T(x^{1/2}D^{-3/2}2^{-\alpha}) + E(x^{1/2}2^{-11/2}), \tag{5.7}$$

where

$$E(x) = \sum_{\substack{1 \le A \le x \\ A \text{ squarefree} \\ A \text{ odd}}} 1.$$

By Lemma 4 we have

$$E(x^{1/2} 2^{-11/2}) = \frac{1}{2^{7/2} \pi^2} x^{1/2} + O(x^{1/4}). \tag{5.8}$$

For $\alpha \in \{0, 2, 3, \frac{11}{2}\}$ we have by (5.6) and Lemma 5

$$\sum_{\substack{D \le x^{1/3} \\ D \in P(1,4)}} T(x^{1/2} D^{-3/2} 2^{-\alpha})$$

$$= \sum_{\substack{D \le x^{1/3} \\ D \in P(1,4)}} \left\{ x^{1/2} D^{-3/2} 2^{-\alpha} \frac{3}{\pi^2} \frac{\phi(D)}{D} \prod_{p \mid 2D} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{1/4} D^{-3/4} d(D)) \right\}$$

$$= \frac{4x^{1/2}}{2^\alpha \pi^2} \sum_{\substack{D \le x^{1/3} \\ D \in P(1,4)}} \frac{\phi(D)}{D^{5/2}} \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} + O\left(x^{1/4} \sum_{1 \le D \le x^{1/3}} \frac{d(D)}{D^{3/4}} \right)$$

$$= \frac{4x^{1/2}}{2^\alpha \pi^2} \sum_{\substack{D \le x^{1/3} \\ D \in P(1,4)}} \frac{\phi(D)}{D^{5/2}} \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{1/3} \log x).$$

Now, as

$$\phi(D) = D \prod_{p \mid D} \left(1 - \frac{1}{p}\right),$$

we have

$$0 < \frac{\phi(D)}{D^{5/2}} \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} \le \frac{1}{D^{3/2}} \prod_{p \mid D} \left(1 + \frac{1}{p}\right)^{-1} \le \frac{1}{D^{3/2}} \tag{5.9}$$

so that

$$\sum_{\substack{D=1 \\ D\in P(1,4)}}^{\infty} \frac{\phi(D)}{D^{5/2}} \prod_{p\mid D} \left(1 - \frac{1}{p^2}\right)^{-1}$$

converges. Remembering that $1 \notin P(1, 4)$, we see that

$$\sum_{\substack{D=1 \\ D\in P(1,4)}}^{\infty} \frac{\phi(D)}{D^{5/2}} \prod_{p\mid D} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{p\equiv 1 \pmod 4} \left(1 + \frac{1}{p^{1/2}(p+1)}\right) - 1.$$

Also, by (5.9), we have

$$\sum_{\substack{D>x^{1/3} \\ D\in P(1,4)}} \frac{\phi(D)}{D^{5/2}} \prod_{p\mid D} \left(1 - \frac{1}{p^2}\right)^{-1} \le \sum_{D>x^{1/3}} \frac{1}{D^{3/2}} = O\left(\frac{1}{x^{1/6}}\right).$$

Hence

$$\sum_{\substack{D\le x^{1/3} \\ D\in P(1,4)}} T(x^{1/2}D^{-3/2}2^{-\alpha})$$

$$= \frac{4x^{1/2}}{2^{\alpha}\pi^2} \left\{ \prod_{p\equiv 1 \pmod 4} \left(1 + \frac{1}{p^{1/2}(p+1)}\right) - 1 \right\} + O(x^{1/3}\log x).$$

Thus

$$\sum_{\alpha\in\{0,2,3,11/2\}} \sum_{\substack{D\le x^{1/3} \\ D\in P(1,4)}} T(x^{1/2}D^{-3/2}2^{-\alpha})$$

$$= \frac{(88+\sqrt{2})}{16\pi^2} x^{1/2} \left\{ \prod_{p\equiv 1 \pmod 4} \left(1 + \frac{1}{p^{1/2}(p+1)}\right) - 1 \right\} + O(x^{1/3}\log x).$$

The theorem now follows from (5.7) and (5.8).

Ou and Williams [3] have shown that the number of cyclic quartic fields with discriminant $\leq x$ is

$$\frac{3}{\pi^2}\left\{\frac{24+\sqrt{2}}{24}\prod_{p\equiv 1(\text{mod } 4)}\left(1+\frac{2}{(p+1)\sqrt{p}}\right)-1\right\}x^{1/2}+O(x^{1/3}\log^3 x).$$

Thus the "average number of cyclic quartic fields per discriminant" is

$$\frac{\frac{3}{\pi^2}\left\{\frac{24+\sqrt{2}}{24}\prod_{p\equiv 1(\text{mod } 4)}\left(1+\frac{2}{(p+1)\sqrt{p}}\right)-1\right\}}{\frac{11}{2\pi^2}\left\{\frac{88+\sqrt{2}}{88}\prod_{p\equiv 1(\text{mod } 4)}\left(1+\frac{1}{(p+1)\sqrt{p}}\right)-1\right\}},$$

which is approximately 1.27.

References

[1] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford Univ. Press, 1960.

[2] R. H. Hudson and K. S. Williams, The integers of a cyclic quartic field, Rocky Mountain J. Math. 29 (1990), 145-150.

[3] Z. M. Ou and K. S. Williams, On the density of cyclic quartic fields, Canad. Math. Bull. 44 (2001), 97-104.

[4] K. S. Williams, Integers of biquadratic fields, Canad. Math. Bull. 13 (1970), 519-526.

■