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AN ARITHMETIC PROOF OF JACOBI'S EIGHT SQUARES THEOREM

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Abstract

An elementary proof of Jacobi's eight squares theorem is given.

0. Notation

Let n and s denote positive integers. We let $r_s(n)$ denote the number of representations of n as the sum of s squares. We also let

$$\sigma_s(n) = \sum_{d \mid n} d^s, \qquad \sigma(n) = \sigma_1(n),$$

where d runs through the positive integers dividing n. If x is not a positive integer, we set $\sigma_s(x) = 0$. We also define

$$A_s(n) = \sum_{k < n/s} \sigma(k)\sigma(n - sk),$$

where the summation is over all integers k satisfying $1 \le k < n/s$. Finally, we set

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$$F_s(n) = \begin{cases} 1, & \text{if } s \mid n, \\ 0, & \text{if } s \nmid n. \end{cases}$$

1. Introduction

The formula

$$r_8(n) = 16(-1)^n \sum_{d \mid n} (-1)^d d^3 \tag{1}$$

first appeared implicitly in the work of Jacobi [5], [6, Sections 40-42] and explicitly in the work of Eisenstein [2], [3, p. 501]. The standard arithmetic proof of (1) uses an elementary identity due to Liouville [8], see [10, p. 402], to show that the function on the right hand side of (1) satisfies the same recurrence relation as $r_8(n)$ with the same initial conditions so that the two functions are the same, see, for example, [10, pp. 441-445]. It is the purpose of this note to give a different arithmetic proof of (1). Our starting point is the following elementary identity due to Huard, Ou, Spearman and Williams [4], which is an extension of an identity of Liouville [7, p. 284].

Huard-Ou-Spearman-Williams Identity. Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all integers a, b, x and y. Then

$$\sum_{ax+by=n} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y)$$

$$-f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y))$$

$$= \sum_{d \mid n} \sum_{x<d} (f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x)$$

$$-f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)), \tag{2}$$

where the sum on the left hand side of (2) is over all positive integers a, b, x, y satisfying $ax + by = n$, the inner sum on the right hand side is over all positive integers x satisfying $x < d$, and the outer sum on the right hand side is over all positive integers d dividing n.

The proof in [4, Section 2] of this identity is completely elementary as it only involves the rearrangement of terms in finite sums. The choice $f(a, b, x, y) = xy$ in (2) yields the identity [4, eqn. (16)]

$$A_1(n) = \frac{1}{12}(5\sigma_3(n) + (1 - 6n)\sigma(n)), \tag{3}$$

which originally appeared in a letter from Besge to Liouville [1]. The choice $f(a, b, x, y) = (2a^2 - b^2)F_4(x)$ yields the identity [4, Theorem 4]

$$\begin{aligned} A_4(n) = \frac{1}{48}(&\sigma_3(n) + 3\sigma_3(n/2) + 16\sigma_3(n/4) \\ &+ (2 - 3n)\sigma(n) + (2 - 12n)\sigma(n/4)), \end{aligned} \tag{4}$$

which is an extension of a result of Melfi [9, eqn. (11)]. The choice $f(a, b, x, y) = \left(\frac{-4}{ab}\right)$ (Legendre-Jacobi-Kronecker symbol) gives Jacobi's four squares formula [4, Section 7]

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4). \tag{5}$$

Another arithmetic proof of (5) has been given by Spearman and Williams [11]. Thus formulae (3), (4), (5) can all be proved by entirely elementary means. We now use these three results to give an arithmetic proof of (1).

2. Arithmetic Proof of Jacobi's Eight Squares Theorem

We have

$$r_8(n) = \sum_{k=0}^{n} r_4(k)r_4(n-k) = 2r_4(n) + \sum_{k=1}^{n-1} r_4(k)r_4(n-k), \tag{6}$$

as $r_4(0) = 1$. Appealing to (5), we obtain

$$\sum_{k=1}^{n-1} r_4(k)r_4(n-k) = 64S_1 - 256S_2 - 256S_3 + 1024S_4, \tag{7}$$

where

$$S_1 = \sum_{k=1}^{n-1} \sigma(k)\,\sigma(n-k), \tag{8}$$

$$S_2 = \sum_{k=1}^{n-1} \sigma(k/4)\,\sigma(n-k), \tag{9}$$

$$S_3 = \sum_{k=1}^{n-1} \sigma(k)\,\sigma((n-k)/4), \tag{10}$$

$$S_4 = \sum_{k=1}^{n-1} \sigma(k/4)\,\sigma((n-k)/4). \tag{11}$$

Clearly $S_1 = A_1(n)$ and changing the summation variable in (10) from k to $n-k$ shows that $S_3 = S_2$. Since the only terms in S_2 and S_4 which do not vanish are those for which $4 \mid k$, replacing k by $4k$ in (9) and (11), we find that $S_2 = A_4(n)$ and $S_4 = A_1(n/4)$. Appealing to (3) and (4) for the values of $A_1(n)$ and $A_4(n)$, and to (5) for the value of $r_4(n)$, we obtain from (6)-(11)

$$r_8(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4). \tag{12}$$

Examining the three possibilities $2 \nmid n$, $2 \| n$ and $4 \mid n$ individually, we find that the right hand side of (12) is the same as the right hand side of (1).

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