Bernoulli's Identity Without Calculus

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A very simple way of defining the Bernoulli numbers B_m $(m = 0, 1, 2, \dots)$ is by means of the recurrence relation

$$B_m = -\frac{1}{m+1}\sum_{j=0}^{m-1}\binom{m+1}{j}B_j, \qquad m = 1, 2, 3, \dots, \tag{1}$$

where $B_0 = 1$ ([**2**, p. 229]). The more usual (and indeed equivalent) way is by means of the power series expansion of $x/(e^x - 1)$, namely,

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \qquad |x| < 2\pi. \tag{2}$$

The definition (1) has the advantage over the definition (2) of not requiring an infinite series and its radius of convergence. The first few Bernoulli numbers are easily calculated from (1). Taking $m = 1$, 2, 3, and 4 in (1), we obtain

$$B_1 = -\frac{1}{2}B_0 = -\frac{1}{2},$$

$$B_2 = -\frac{1}{3}(B_0 + 3B_1) = \frac{1}{6},$$

$$B_3 = -\frac{1}{4}(B_0 + 4B_1 + 6B_2) = 0,$$

$$B_4 = -\frac{1}{5}(B_0 + 5B_1 + 10B_2 + 10B_3) = -\frac{1}{30}.$$

The sequence of Bernoulli numbers is one of the most important number sequences in mathematics. Bernoulli numbers arise naturally in the calculus of finite differences, in combinatorics, in connection with Fermat's last theorem, in the study of class numbers of algebraic number fields, in the Euler–Maclaurin summation formula, and in connection with values of the Riemann zeta function. The main properties of Bernoulli numbers can be found for example in Ireland and Rosen [**2**, Chap. 15]. An extensive bibliography of Bernoulli numbers has been compiled by Dilcher, Skula, and Slavutskii [**1**].

In 1987, Nunemacher and Young [**3**] made use of (2) to give an elegant proof of Bernoulli's identity

$$\sum_{r=1}^{n-1} r^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}, \quad n, k = 1, 2, \dots, \tag{3}$$

which expresses the sum of the kth powers of the first $n-1$ natural numbers as a polynomial in n. For example, with $k=2$ we have

$$\begin{aligned}1^2+2^2+\cdots+(n-1)^2 &= \frac{1}{3}\left(\binom{3}{0}B_0n^3+\binom{3}{1}B_1n^2+\binom{3}{2}B_2n\right)\\ &= \frac{1}{3}\left(n^3-\frac{3}{2}n^2+\frac{1}{2}n\right)\\ &= (n-1)n(2n-1)/6\end{aligned}$$

and with $k=3$ we have

$$\begin{aligned}1^3+2^3+\cdots+(n-1)^3 &= \frac{1}{4}\left(\binom{4}{0}B_0n^4+\binom{4}{1}B_1n^3+\binom{4}{2}B_2n^2+\binom{4}{3}B_3n\right)\\ &= \frac{1}{4}(n^4-2n^3+n^2)\\ &= ((n-1)n/2)^2.\end{aligned}$$

The identity (3) for $k=1,2,\ldots,10$ was given by Jakob Bernoulli (1654–1705) in his work "Ars Conjectandi" published posthumously in 1713. This was the first occurrence of Bernoulli numbers in the literature. The classical proofs of (3) make use of the Euler–Maclaurin summation formula or the derivatives and integrals of Bernoulli polynomials. They thus involve knowledge of calculus. This paper was motivated by the desire to find a proof of (3) which proceeds directly from the recurrence relation (1) to obtain (3) without any use of calculus or infinite series. We now present such a proof which is therefore accessible to students who have not studied calculus. It only uses the basic properties of binomial coefficients.

We begin by giving a sketch of the proof and then give the missing details. For a nonnegative integer m we define the Kronecker delta symbol δ_m by

$$\delta_m=\begin{cases}1, & \text{if } m=1,\\ 0, & \text{if } m\neq 1.\end{cases} \tag{4}$$

An easy deduction from the recurrence relation (1) is the following expression for δ_m, namely,

$$\delta_m=\sum_{j=0}^{m}\binom{m}{j}B_j-B_m, \qquad m=0,1,2,\ldots. \tag{5}$$

Let k and n be positive integers and let r and l be integers with $1\le r\le n-1$ and $0\le l\le k+1$. Appealing to (4) we see that

$$(k+1)r^k=\sum_{l=0}^{k+1}\binom{k+1}{l}\delta_{k+1-l}r^l. \tag{6}$$

Using (5) in (6) we obtain

$$(k+1)r^k=\sum_{l=0}^{k+1}\binom{k+1}{l}\left\{\sum_{j=0}^{k+1-l}\binom{k+1-l}{j}B_j-B_{k+1-l}\right\}r^l. \tag{7}$$

Manipulation of the right side of (7) yields

$$(k+1)r^k = \sum_{j=0}^{k+1} \binom{k+1}{j} B_j \{(r+1)^{k+1-j} - r^{k+1-j}\}. \tag{8}$$

Then summing (8) over $r = 1, 2, \ldots, n-1$ the right side telescopes and we obtain (3).

Now we give the details. First we prove (5). For $m = 0$ and $m = 1$ (5) is easily checked. For $m \geq 2$, appealing to (1), we see that

$$\sum_{j=0}^{m} \binom{m}{j} B_j - B_m = \sum_{j=0}^{m-2} \binom{m}{j} B_j + mB_{m-1} = -mB_{m-1} + mB_{m-1} = 0 = \delta_m.$$

Next we deduce (8) from (7). From (7) we see that

$$(k+1)r^k = S_1 - S_2,$$

where

$$S_1 = \sum_{l=0}^{k+1} \sum_{j=0}^{k+1-l} \binom{k+1}{l} \binom{k+1-l}{j} B_j r^l$$

and

$$S_2 = \sum_{l=0}^{k+1} \binom{k+1}{l} B_{k+1-l} r^l.$$

Now

$$\begin{aligned} \binom{k+1}{l}\binom{k+1-l}{j} &= \frac{(k+1)!}{l!(k+1-l)!} \frac{(k+1-l)!}{j!(k+1-l-j)!} \\ &= \frac{(k+1)!}{j!(k+1-j)!} \frac{(k+1-j)!}{l!(k+1-l-j)!} \\ &= \binom{k+1}{j}\binom{k+1-j}{l} \end{aligned}$$

so

$$S_1 = \sum_{l=0}^{k+1} \sum_{j=0}^{k+1-l} \binom{k+1}{j} \binom{k+1-j}{l} B_j r^l.$$

Interchanging the order of summation, we obtain

$$S_1 = \sum_{j=0}^{k+1} \binom{k+1}{j} B_j \sum_{l=0}^{k+1-j} \binom{k+1-j}{l} r^l.$$

By the binomial theorem, the inner sum is $(r+1)^{k+1-j}$, so that

$$S_1 = \sum_{j=0}^{k+1} \binom{k+1}{j} B_j (r+1)^{k+1-j}.$$

Changing the summation variable l in the sum S_2 to $j = k + 1 - l$, we deduce that

$$S_2 = \sum_{j=0}^{k+1} \binom{k+1}{k+1-j} B_j r^{k+1-j}$$
$$= \sum_{j=0}^{k+1} \binom{k+1}{j} B_j r^{k+1-j}$$

Then we have

$$(k+1)r^k = S_1 - S_2 = \sum_{j=0}^{k+1} \binom{k+1}{j} B_j\left((r+1)^{k+1-j} - r^{k+1-j}\right),$$

which is (7).

Finally, we sum (8) over $r = 1, 2, \ldots, n-1$ and obtain

$$(k+1)\sum_{r=1}^{n-1} r^k = \sum_{j=0}^{k+1} \binom{k+1}{j} B_j(n^{k+1-j} - 1)$$
$$= \sum_{j=0}^{k+1} \binom{k+1}{j} B_j n^{k+1-j} - B_{k+1} \quad \text{(by (5))}$$
$$= \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j},$$

completing the proof of Bernoulli's identity (3).

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