Arch. Math., Vol. 67, 290–295 (1996)

0003-889X/96/6704-0290 $ 2.70/0


# Discriminant of the normal closure of a dihedral quartic field

By

JAMES G. HUARD, BLAIR K. SPEARMAN and KENNETH S. WILLIAMS*)

It is known [3, eq. (1)] that a dihedral quartic field $L$ can be expressed in the form

$$L = Q(\sqrt{a + b\sqrt{c}})$$

where $a$, $b$, $c$ are integers with

(1) $\quad (a, b)$ squarefree, $c$ squarefree,

and

(2) $\quad a^2 - b^2 c \neq k^2 \quad \text{or} \quad c k^2$ for any integer $k$.

Let $\hat{L}$ denote the normal closure of $L$, that is,

$$\hat{L} = Q(\sqrt{a + b\sqrt{c}}, \sqrt{a - b\sqrt{c}}).$$

The discriminant of $L$ is given by

(3) $$d(L) = 2^e s c^2 \left(\frac{(a, b)}{(a, b, cs)}\right)^2,$$

where $s$ denotes the squarefree part of $a^2 - b^2 c$, and the value of $e (= -2, 0, 2, 4, 6, 8)$ is given in Tables A ($c \equiv 2 \pmod 4$), B ($c \equiv 3 \pmod 4$), C ($c \equiv 5 \pmod 8$), and D ($c \equiv 1 \pmod 8$) of [3]. Tables A, B, C, D comprise 8, 8, 8, 27 cases respectively. Each case is specified by congruence conditions on $a$, $b$, and $c$. For example B4 is the case $a \equiv b \equiv 1 \pmod 2$, $c \equiv 3 \pmod 4$, and in this case $e = 8$.

In this paper we obtain the following formula for the discriminant of $\hat{L}$.

**Theorem 1.** $d(\hat{L}) = 2^\theta \dfrac{c^4 s^4 (a, b)^4}{(c, s)^2 (a, b, cs)^4}$, *where the value of $\theta$ is given in Table* (i).

*) Research supported by Natural Sciences and Engineering Research Council of Canada Grant A-7233.

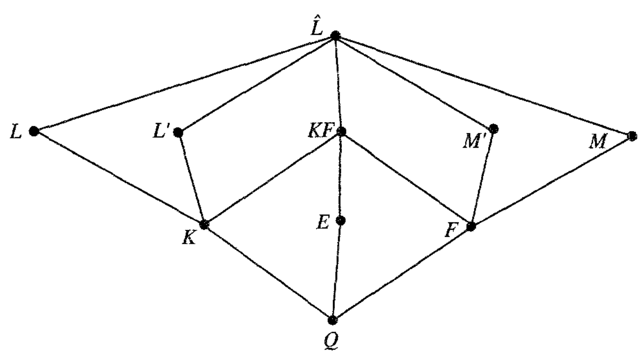
Table (i)

| $\theta$ | $c \equiv 2 \pmod 4$ | $c \equiv 3 \pmod 4$ | $c \equiv 5 \pmod 8$ | $c \equiv 1 \pmod 8$ |
|---|---|---|---|---|
| 18 | A 3, A 7<br>A 4 ($a \equiv 0(4)$)<br>A 8 ($a \equiv 0(8)$) | B 4, B 8 | | |
| 16 | A 4 ($a \equiv 2(4)$)<br>A 8 ($a \equiv 4(8)$) | B 1 | C 4 ($s \equiv 3(4)$) | D 10 ($s \equiv 3(4)$)<br>D 11 ($s \equiv 3(4)$) |
| 12 | A 2, A 6 | B 3, B 7 | C 1<br>C 4 ($s \equiv 1(4)$)<br>C 5 ($s \equiv 3(4)$) | D 6, D 8, D 12<br>D 13, D 26, D 27<br>D 5 ($s \equiv 3(4)$)<br>D 10 ($s \equiv 1(4)$)<br>D 11 ($s \equiv 1(4)$) |
| 8 | A 1, A 5 | B 2, B 6 | C 3, C 8<br>C 5 ($s \equiv 1(4)$)<br>C 6 ($s \equiv 3(4)$) | D 1, D 2, D 4, D 7, D 9<br>D 14, D 15, D 24, D 25<br>D 5 ($s \equiv 1(4)$) |
| 4 | | B 5 | C 6 ($s \equiv 1(4)$) | D 17, D 18, D 19, D 21<br>D 22, D 23 |
| 0 | | | C 2 | D 3 |
| $-4$ | | | C 7 | D 16, D 20 |

Note that in the table we have abbreviated $k \pmod m$ to $k(m)$.

Proof. We set $K = Q(\sqrt{c})$, $E = Q\sqrt{c(a^2 - b^2 c)})$, $F = Q(\sqrt{a^2 - b^2 c})$, $L' = Q(\sqrt{a - b\sqrt{c}})$, $KF = Q(\sqrt{a^2 - b^2 c}, \sqrt{c})$, $M = Q(\sqrt{2a + 2\sqrt{a^2 - b^2 c}})$ and $M' = Q(\sqrt{2a - 2\sqrt{a^2 - b^2 c}})$. The subfield structure of $\hat{L}$ is as shown in the figure. As $L$ and $L'$ are isomorphic fields, we have $d(L) = d(L')$. Similarly $M$ and $M'$ are isomorphic fields so that $d(M) = d(M')$. By a theorem of Artin [1; (1), (2), (3), (4), (20)] (see also Halter-Koch [2; Satz 24, (3)]), we have

$$d(\hat{L}) = d(E)\,d(L)\,d(M). \tag{4}$$



First we determine $d(E)$. As $a^2 - b^2 c = sX^2$ for some positive integer $X$, we have

$$E = Q(\sqrt{c(a^2 - b^2 c)}) = Q(\sqrt{cs}) = Q\left(\sqrt{\frac{cs}{(c,s)^2}}\right).$$

Thus, as $\frac{cs}{(c,s)^2}$ is squarefree, we have

$$d(E) = 2^{\lambda} \frac{cs}{(c,s)^2}, \tag{5}$$

where

$$\lambda = \begin{cases} 0, & \text{if } \frac{cs}{(c,s)^2} \equiv 1 \pmod 4, \\ 2, & \text{if } \frac{cs}{(c,s)^2} \not\equiv 1 \pmod 4. \end{cases}$$

The value of $\lambda$ is given for each case in Table (ii).

Next we treat $d(M)$. It is convenient to set

$$\kappa = \begin{Bmatrix} 1, & \text{if } (a,b) \text{ odd} \\ -1, & \text{if } (a,b) \text{ even} \end{Bmatrix}, \quad \tau = \begin{Bmatrix} 0, & \text{if } cs \text{ odd} \\ \kappa, & \text{if } cs \text{ even} \end{Bmatrix}.$$

As $a^2 - b^2 c = sX^2$ we have

$$M = Q(\sqrt{2a + 2\sqrt{a^2 - b^2 c}}) = Q(\sqrt{2a + 2X\sqrt{s}}) = Q(\sqrt{a' + b'\sqrt{c'}}),$$

where

$$a' = 2^{\kappa} a, \quad b' = 2^{\kappa} X, \quad c' = s.$$

We let $s'$ denote the squarefree part of $a'^2 - b'^2 c'$. We observe that

$$(a', b') = 2^{\kappa}(a, X) = 2^{\kappa}(a, b),$$

so that $(a', b')$ is squarefree;

$$a'^2 - b'^2 c' = 2^{2\kappa}(a^2 - sX^2) = 2^{2\kappa} b^2 c,$$

so that $s' = c$; and

$$(a', b', c's') = (2^{\kappa} a, 2^{\kappa} X, cs) = (2^{\kappa} a, 2^{\kappa} b, cs) = 2^{\tau}(a, b, cs).$$

As $a'$, $b'$, $c'$ satisfy the conditions of (1) and (2), we have by (3)

$$d(M) = 2^{e'} s' c'^2 \left(\frac{(a', b')}{(a', b', c's')}\right)^2 = 2^{e'} cs^2 \left(\frac{2^{\kappa}(a, b)}{2^{\tau}(a, b, cs)}\right)^2$$

that is

$$d(M) = 2^{e' + 2\kappa - 2\tau} cs^2 \left(\frac{(a, b)}{(a, b, cs)}\right)^2, \tag{6}$$

where the values of $e' = e(a', b', c')$ are given in Table (ii).

Hence, as $d(L) = \dfrac{2^e s c^2 (a, b)^2}{(a, b, cs)^2}$ by (3), we obtain from (4), (5) and (6)

$$d(\hat{L}) = 2^{\lambda + e + e' + 2\kappa - 2\tau} \, \frac{c^4 s^4 (a, b)^4}{(c, s)^2 (a, b, cs)^4}\,.$$

The values of $\theta = \lambda + e + e' + 2\kappa - 2\tau$ are given in Table (ii).

Table (ii)

| case of field $L$ | $(a, b)$ (mod 2) | $cs$ (mod 2) | $\frac{cs}{(c, s)^2}$ (mod 4) | type for field $M$ | $\lambda$ | $\kappa$ | $\tau$ | $e$ | $e'$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| A 1 | 1 | 0 | 2 | D 14, D 15, D 24, D 25 | 2 | 1 | 1 | 4 | 2 | 8 |
| A 2 | 1 | 0 | 2 | D 12, D 13, D 26, D 27 | 2 | 1 | 1 | 6 | 4 | 12 |
| A 3 | 1 | 0 | 2 | B 8 | 2 | 1 | 1 | 8 | 8 | 18 |
| A 4, $a \equiv 0(4)$ | 1 | 0 | 3 | A 8, $a' \equiv 0(8)$ | 2 | 1 | 1 | 8 | 8 | 18 |
| A 4, $a \equiv 2(4)$ | 1 | 0 | 1 | A 8, $a' \equiv 4(8)$ | 0 | 1 | 1 | 8 | 8 | 16 |
| A 5 | 0 | 0 | 2 | D 7, D 9 | 2 | −1 | −1 | 4 | 2 | 8 |
| A 6 | 0 | 0 | 2 | D 6, D 8 | 2 | −1 | −1 | 6 | 4 | 12 |
| A 7 | 0 | 0 | 2 | B 4 | 2 | −1 | −1 | 8 | 8 | 18 |
| A 8, $a \equiv 0(8)$ | 0 | 0 | 3 | A 4, $a' \equiv 0(4)$ | 2 | −1 | −1 | 8 | 8 | 18 |
| A 8, $a \equiv 4(8)$ | 0 | 0 | 1 | A 4, $a' \equiv 2(4)$ | 0 | −1 | −1 | 8 | 8 | 16 |
| B 1 | 1 | 1 | 3 | C 5, D 5 | 2 | 1 | 0 | 8 | 4 | 16 |
| B 2 | 1 | 1 | 3 | D 17, D 18, D 21, D 22 | 2 | 1 | 0 | 4 | 0 | 8 |
| B 3 | 1 | 1 | 3 | C 6, C 8 | 2 | 1 | 0 | 6 | 2 | 12 |
| B 4 | 1 | 0 | 2 | A 7 | 2 | 1 | 1 | 8 | 8 | 18 |
| B 5 | 0 | 1 | 3 | D 1, D 2 | 2 | −1 | 0 | 2 | 2 | 4 |
| B 6 | 0 | 1 | 3 | C 1 | 2 | −1 | 0 | 4 | 4 | 8 |
| B 7 | 0 | 1 | 3 | C 4, D 10, D 11 | 2 | −1 | 0 | 6 | 6 | 12 |
| B 8 | 0 | 0 | 2 | A 3 | 2 | −1 | −1 | 8 | 8 | 18 |
| C 1 | 1 | 1 | 3 | B 6 | 2 | 1 | 0 | 4 | 4 | 12 |
| C 2 | 1 | 1 | 1 | C 7, D 16, D 20 | 0 | 1 | 0 | 0 | −2 | 0 |
| C 3 | 1 | 1 | 1 | C 6, D 19, D 23 | 0 | 1 | 0 | 4 | 2 | 8 |
| C 4, $s \equiv 1(4)$ | 1 | 1 | 1 | C 5, D 5 | 0 | 1 | 0 | 6 | 4 | 12 |
| C 4, $s \equiv 3(4)$ | 1 | 1 | 3 | B 7 | 2 | 1 | 0 | 6 | 6 | 16 |
| C 5, $s \equiv 1(4)$ | 0 | 1 | 1 | C 4, D 10, D 11 | 0 | −1 | 0 | 4 | 6 | 8 |
| C 5, $s \equiv 3(4)$ | 0 | 1 | 3 | B 1 | 2 | −1 | 0 | 4 | 8 | 12 |
| C 6, $s \equiv 1(4)$ | 0 | 1 | 1 | C 3, D 4 | 0 | −1 | 0 | 2 | 4 | 4 |
| C 6, $s \equiv 3(4)$ | 0 | 1 | 3 | B 3 | 2 | −1 | 0 | 2 | 6 | 8 |
| C 7 | 0 | 1 | 1 | C 2, D 3 | 0 | −1 | 0 | −2 | 0 | −4 |
| C 8 | 0 | 1 | 3 | B 3 | 2 | −1 | 0 | 2 | 6 | 8 |
| D 1, D 2 | 1 | 1 | 3 | B 5 | 2 | 1 | 0 | 2 | 2 | 8 |
| D 3 | 1 | 1 | 1 | C 7, D 16, D 20 | 0 | 1 | 0 | 0 | −2 | 0 |
| D 4 | 1 | 1 | 1 | C 6, D 19, D 23 | 0 | 1 | 0 | 4 | 2 | 8 |
| D 5, $s \equiv 1(4)$ | 0 | 1 | 1 | C 4, D 10, D 11 | 0 | −1 | 0 | 4 | 6 | 8 |
| D 5, $s \equiv 3(4)$ | 0 | 1 | 3 | B 1 | 2 | −1 | 0 | 4 | 8 | 12 |
| D 6, D 8 | 1 | 0 | 2 | A 6 | 2 | 1 | 1 | 4 | 6 | 12 |
| D 7, D 9 | 1 | 0 | 2 | A 5 | 2 | 1 | 1 | 2 | 4 | 8 |

Table (ii) (Continued)

| case of field $L$ | $(a, b)$ (mod 2) | $cs$ (mod 2) | $\frac{cs}{(c, s)^2}$ (mod 4) | type for field $M$ | $\lambda$ | $\kappa$ | $\tau$ | $e$ | $e'$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| D 10, D 11, $s \equiv 1(4)$ | 1 | 1 | 1 | C 5, D 5 | 0 | 1 | 0 | 6 | 4 | 12 |
| D 10, D 11, $s \equiv 3(4)$ | 1 | 1 | 3 | B 7 | 2 | 1 | 0 | 6 | 6 | 16 |
| D 12, D 13 | 0 | 0 | 2 | A 2 | 2 | −1 | −1 | 4 | 6 | 12 |
| D 14, D 15 | 0 | 0 | 2 | A 1 | 2 | −1 | −1 | 2 | 4 | 8 |
| D 16, D 20 | 0 | 1 | 1 | C 2, D 3 | 0 | −1 | 0 | −2 | 0 | −4 |
| D 17, D 18 | 0 | 1 | 3 | B 2 | 2 | −1 | 0 | 0 | 4 | 4 |
| D 19, D 23 | 0 | 1 | 1 | C 3, D 4 | 0 | −1 | 0 | 2 | 4 | 4 |
| D 21, D 22 | 0 | 1 | 3 | B 2 | 2 | −1 | 0 | 0 | 4 | 4 |
| D 24, D 25 | 0 | 0 | 2 | A 1 | 2 | −1 | −1 | 2 | 4 | 8 |
| D 26, D 27 | 0 | 0 | 2 | A 2 | 2 | −1 | −1 | 4 | 6 | 12 |

This completes the proof of Theorem 1. □

Our second theorem treats the special case of Theorem 1 when $c = -1$ obtaining a result due to Kuroda [4]. We first need a lemma.

**Lemma.** *If $a + bi$ is a squarefree gaussian integer and $a^2 + b^2 = sX^2$, where $s$ is squarefree, then $X$ is squarefree and odd, $(a, b) = X$, $(s, X) = 1$.*

Proof. If $a + bi$ is a unit, then $a^2 + b^2 = 1$, so $s = 1$, $X = 1$, and the Lemma holds. If $a + bi$ is not a unit, since it is squarefree, it can be factored into a product of non-associated gaussian primes. Combining any gaussian prime with its conjugate, if they or their associates both appear in this factorization, we see that we may write

$$a + bi = y \prod_{i=1}^{k} \pi_i , \tag{7}$$

where the integer $y$ is squarefree, odd and relatively prime to each $\pi_i$, and for $i \neq j$ $\pi_i$ is not an associate of either $\pi_j$ or $\bar{\pi}_j$. Thus

$$a - bi = y \prod_{i=1}^{k} \bar{\pi}_i \tag{8}$$

and

$$a^2 + b^2 = y^2 \prod_{i=1}^{k} p_i ,$$

where the $p_i = \pi_i \bar{\pi}_i$ are distinct and coprime to $y$. Hence $s = \prod_{i=1}^{k} p_i$, $X = y$, and $(s, X) = 1$. Finally, adding and subtracting (7) and (8), we see that $(a, b) = y = X$. □

**Theorem 2** (Kuroda [4]). *Let $a$ and $b$ be integers such that $a + bi$ is a squarefree gaussian integer and $a^2 + b^2$ is not a square. Let $s$ denote the squarefree part of $a^2 + b^2$. Then*

$$d(Q(\sqrt{a + bi}, \sqrt{a - bi})) = 2^{\theta} s^2 (a^2 + b^2)^2 ,$$

*where $\theta$ is given by*

$$\theta = \begin{cases} 18, & \text{if } a \equiv b \equiv 1 \pmod 2, \\ 16, & \text{if } a \equiv 0 \pmod 2,\ b \equiv 1 \pmod 2, \\ 12, & \text{if } a \equiv 1 \pmod 2,\ b \equiv 2 \pmod 4, \\ 8, & \text{if } a \equiv 1 \pmod 2,\ b \equiv 0 \pmod 4. \end{cases}$$

Proof. As $a^2 + b^2$ is not a square, by (1) and (2), $Q(\sqrt{a+bi})$ is a dihedral extension of $Q$, and so, by Theorem 1 with $c = -1$, we have

$$d(Q(\sqrt{a+bi}, \sqrt{a-bi})) = \frac{2^\theta s^4 (a, b)^4}{(a, b, s)^4}.$$

Setting $a^2 + b^2 = sX^2$, by the Lemma, we have

$$(a, b) = X, \quad (a, b, s) = (X, s) = 1,$$

so that

$$d(Q(\sqrt{a+bi}, \sqrt{a-bi})) = 2^\theta s^4 X^4 = 2^\theta s^2 (a^2 + b^2)^2.$$

The values of $\theta$ follow from Table (ii) (cases B 4, B 1, B 3, B 2, respectively). □

## References

[1] E. Artin, Die gruppentheoretische Struktur der Diskriminanten algebraischer Zahlkörper. J. Reine Angew. Math. **164**, 1–11 (1931).

[2] F. Halter-Koch, Arithmetische Theorie der Normalkörper von 2-Potenzgrad mit Diedergruppe. J. Number Theory **3**, 412–443 (1971).

[3] J. G. Huard, B. K. Spearman and K. S. Williams, Integral bases for quartic fields with quadratic subfields. J. Number Theory **51**, 87–102 (1995).

[4] S. Kuroda, Über die Zerlegung rationaler Primzahlen in gewissen nicht-abelschen galoisschen Körpern. J. Math. Soc. Japan **3**, 148–156 (1951).

Eingegangen am 15. 11. 1995*)

Anschriften der Autoren:

James G. Huard
Department of Mathematics
Canisius College
Buffalo, New York
USA 14208

Blair K. Spearman
Department of Mathematics
and Statistics
Okanagan University College
Kelowna, B.C.
Canada V1V 1V7

Kenneth S. Williams
Department of Mathematics
and Statistics
Carleton University
Ottawa, Ontario
Canada K1S 5B6

*) Eine Neufassung ging am 10. 4. 1996 ein.