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VALUES OF THE RIEMANN ZETA FUNCTION AND INTEGRALS INVOLVING $\log\left(2\sinh\frac{\theta}{2}\right)$ AND $\log\left(2\sin\frac{\theta}{2}\right)$

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Integrals involving the functions $\log\left(2\sinh(\theta/2)\right)$ and $\log\left(2\sin(\theta/2)\right)$ are studied, particularly their relationship to the values of the Riemann zeta function at integral arguments. For example general formulae are proved which contain the known results

$$\int_0^{\frac{\pi}{3}} \log^2\left(2\sin(\theta/2)\right)d\theta = 7\pi^3/108,$$

$$\int_0^{\frac{\pi}{3}} \theta\log^2\left(2\sin(\theta/2)\right)d\theta = 17\pi^4/6480,$$

$$\int_0^{\frac{\pi}{3}} \left(\log^4(2\sin(\theta/2)) - \frac{3}{2}\theta^2\log^2(2\sin(\theta/2))\right)d\theta = 253\pi^5/3240,$$

$$\int_0^{\frac{\pi}{3}} \left(\theta\log^4(2\sin(\theta/2)) - \frac{\theta^3}{2}\log(2\sin(\theta/2))\right)d\theta = 313\pi^6/408240,$$

as special cases.

1. Introduction. Since the discovery of the formulae

$$\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}} = 2\int_0^{\frac{\pi}{3}}\theta\log^2\left(2\sin\frac{\theta}{2}\right)d\theta = \frac{17\pi^4}{2^3\cdot 3^4\cdot 5}, \tag{1.1}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} &= -2\int_0^{2\log\tau}\theta\log\left(2\sinh\frac{\theta}{2}\right)d\theta \\ &= \frac{2}{5}\zeta(3), \text{ where } \tau = \frac{1}{2}(1+\sqrt{5}),\end{aligned} \tag{1.2}$$

the relationship between the values of the Riemann zeta function and integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$ and $\log\left(2\sinh\frac{\theta}{2}\right)$ has been studied by many authors, see for example [**2**], [**4**], [**5**], [**7**], [**9**].

Recently Butzer, Markett and Schmidt [**2**] made use of central and Stirling numbers to obtain a representation of $\zeta(2m+1)$ by integrals involving $\log\left(2\sinh\frac{\theta}{2}\right)$ (see (2.13)). In §2 of this paper, we reprove (2.13) and at the same time prove the analogous formula for $\zeta(2m)$ (see (2.14)). Note that (1.2) is the special case of (2.13) when $m=1$.

In [**7**], van der Poorten proves (1.1), as well as the formula

$$\int_0^{\frac{\pi}{3}} \log^2\left(2\sin\frac{\theta}{2}\right)d\theta = \frac{7\pi^3}{108} \tag{1.3}$$

and remarks that "It appears that (1.1) and (1.3) are not representative of a much larger class of similar formulas". However in [**9**] Zucker establishes the two formulae

(1.4)
$$\int_0^{\frac{\pi}{3}} \left[\log^4\left(2\sin\frac{\theta}{2}\right) - \frac{3\theta^2}{2}\log^2\left(2\sin\frac{\theta}{2}\right)\right]d\theta = \frac{253\pi^5}{2^3\cdot 3^4\cdot 5},$$

(1.5)
$$\int_0^{\frac{\pi}{3}} \left[\theta\log^4\left(2\sin\frac{\theta}{2}\right) - \frac{\theta^3}{2}\log\left(2\sin\frac{\theta}{2}\right)\right]d\theta = \frac{313\pi^6}{2^4\cdot 3^6\cdot 5\cdot 7}.$$

In §3, we prove the general formulae

(1.6)
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-2} \frac{(-1)^k\binom{2m-2}{2k}}{(2k+1)2^{2k}}\theta^{2k+1}\log^{2m-2k-2}\left(2\sin\frac{\theta}{2}\right)d\theta \\ = \frac{(-1)^m\pi^{2m}}{4m(2m-1)}\left[\left(\frac{1}{6}\right)^{2m-1} - 2\left(1-\frac{1}{2^{2m-1}}\right)B_{2m}\right],\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k\binom{2m}{2k}}{2^{2k}}\theta^{2k}\log^{2m-2k}\left(2\sin\frac{\theta}{2}\right)d\theta \\ = \frac{(-1)^m\pi^{2m+1}}{2^{2m+2}}\left[E_{2m} - \frac{1}{(2m+1)3^{2m}}\right],\end{aligned} \tag{1.7}$$

where the B_{2m} and E_{2m} are the Bernoulli and Euler numbers respectively. We remark that (1.1), (1.3), (1.4) and (1.5) are all special cases of (1.6) and (1.7). Formula (1.6) is basically formula (3.10b) of [**2**] and formula (1.7) is essentially Theorem 4.1 of [**1**].

In §4, we establish the relations:

(1.8)
$$\sum_{n=1}^{\infty} \frac{1}{n^{m+2}\binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{\frac{\pi}{3}} \theta \log^m \left(2\sin\frac{\theta}{2}\right) d\theta,$$

(1.9)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{m+2}\binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{2\log\tau} \theta \log^m \left(2\sinh\frac{\theta}{2}\right) d\theta,$$

which generalize the formulae (1.1) and (1.2). The formula (1.8) was given by Zucker [**4**, (2.5)]. Formula (1.9) can be established in an analogous fashion.

2. Representation of $\zeta(n)$ by integrals involving $\log\left(2\sinh\frac{\theta}{2}\right)$. For $k \geq 1$, $x \geq 1$, we have

$$(2.1)\quad \int_{x^2}^{1} \frac{\log^k(t-1)}{t} dt \overset{t=e^\theta}{=} -\int_0^{2\log x} \left[\log\left(2\sinh\frac{\theta}{2}\right) + \frac{\theta}{2}\right]^k d\theta,$$

since

$$e^\theta - 1 = e^{\frac{\theta}{2}}\left(e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}\right) = 2e^{\frac{\theta}{2}}\sinh\frac{\theta}{2},$$
$$\log(e^\theta - 1) = \log\left(2\sinh\frac{\theta}{2}\right) + \frac{\theta}{2}.$$

Similarly, we have

$$(2.2)\quad \int_{\frac{1}{x^2}}^{1} \frac{\log^k(1-t)}{t} dt \overset{t=e^{-\theta}}{=} \int_0^{2\log x} \left[\log\left(2\sinh\frac{\theta}{2}\right) - \frac{\theta}{2}\right]^k d\theta.$$

We set

(2.3)
$$A(\theta) = \log\left(2\sinh\frac{\theta}{2}\right),$$
$$I(k) = \int_0^1 \frac{\log^k(1-t)}{t} dt = \int_0^1 \frac{\log^k t}{1-t} dt = (-1)^k k!\zeta(k+1).$$

Taking $x = \tau = \frac{1}{2}(1+\sqrt{5})$ in (2.2) and (2.1) gives respectively

$$(2.4) \qquad f(k) = \int_{1/\tau^2}^{1} \frac{\log^k(1-t)}{t}dt = \int_0^{2\log\tau} \left[A(\theta) - \frac{\theta}{2}\right]^k d\theta,$$

$$(2.5) \qquad g(k) = \int_{\tau^2}^{1} \frac{\log^k(t-1)}{t}dt = -\int_0^{2\log\tau} \left[A(\theta) + \frac{\theta}{2}\right]^k d\theta.$$

Now we evaluate $f(k)$ and $g(k)$. We have

$$(2.6) \qquad f(k) = \int_{1/\tau^2}^{1} \frac{\log^k(1-t)}{t}dt \overset{u=1-t}{=} \int_0^{1/\tau} \frac{\log^k u}{1-u}du$$

and

$$\begin{aligned}
g(k) &= \int_{\tau^2}^{1} \frac{\log^k(t-1)}{t}dt \overset{u=t-1}{=} -\int_0^{\tau} \frac{\log^k u}{1+u}du \\
&= \int_0^{\tau}\left(\frac{1}{1-u} - \frac{1}{1+u}\right)\log^k u du - \int_0^{\tau} \frac{\log^k u}{1-u}du \\
&= \int_0^{\tau} \frac{\log^k u}{1-u^2}du^2 - \int_0^{\tau} \frac{\log^k u}{1-u}du \\
&= \frac{1}{2^k}\int_0^{\tau^2} \frac{\log^k u}{1-u}du - \int_0^{\tau} \frac{\log^k u}{1-u}du \\
&= \frac{1}{2^k}\left[I(k) - \int_{\tau^2}^{1} \frac{\log^k u}{1-u}du\right] - \left[I(k) - \int_{\tau}^{1} \frac{\log^k u}{1-u}du\right],
\end{aligned}$$

that is

$$g(k) = \left(\frac{1}{2^k} - 1\right)I(k) - \frac{1}{2^k}\int_{\tau^2}^{1} \frac{\log^k u}{1-u}du + \int_{\tau}^{1} \frac{\log^k u}{1-u}du.$$

In the two integrals, using the substitution $t = \frac{1}{n}$ and noting $\frac{1}{(1-t)t} = \frac{1}{1-t} + \frac{1}{t}$, we have

$$\begin{aligned}
-\frac{1}{2^k}\int_{\tau^2}^{1} \frac{\log^k u}{1-u}du &= \frac{(-1)^{k+1}}{2^k}\int_{\frac{1}{\tau^2}}^{1}\left(\frac{1}{1-t} + \frac{1}{t}\right)\log^k t\, dt \\
&= \frac{(-1)^{k+1}}{2^k}I(k) + \frac{(-1)^k}{2^k}\int_0^{\frac{1}{\tau^2}} \frac{\log^k t}{1-t}dt \\
&\quad - \frac{2}{k+1}\log^{k+1}\tau
\end{aligned}$$

and

$$\int_\tau^1 \frac{\log^k u}{1-u}du = (-1)^k \int_{1/\tau}^1 \left(\frac{1}{1-t} + \frac{1}{t}\right) \log^k t dt$$
$$= (-1)^k I(k) + (-1)^{k+1} \int_0^{\frac{1}{\tau}} \frac{\log^k t}{1-t} dt$$
$$+ \frac{1}{k+1} \log^{k+1} t.$$

Hence we have

$$g(k) = \left(\frac{1}{2^k} - 1\right)\left(1 + (-1)^{k+1}\right) I(k) + (-1)^{k+1} \int_0^{\frac{1}{\tau}} \frac{\log^k t}{1-t} dt$$
$$+ \frac{(-1)^k}{2^k} \int_0^{\frac{1}{\tau^2}} \frac{\log^k t}{1-t} dt - \frac{\log^{k+1} \tau}{k+1}.$$

Thus

$$g(2m) = -\int_0^{\frac{1}{\tau}} \frac{\log^{2m} t}{1-t} dt + \frac{1}{2^{2m}} \int_0^{\frac{1}{\tau^2}} \frac{\log^{2m} t}{1-t} dt - \frac{\log^{2m+1} \tau}{2m+1},$$
$$g(2m-1) = 2\left(\frac{1}{2^{2m-1}} - 1\right) I(2m-1) + \int_0^{\frac{1}{\tau}} \frac{\log^{2m-1} t}{1-t} dt$$
$$- \frac{1}{2^{2m-1}} \int_0^{\frac{1}{\tau^2}} \frac{\log^{2m-1} t}{1-t} dt - \frac{\log^{2m} \tau}{2m}.$$

From (2.6), we obtain

$$f(2m) + g(2m) = \frac{1}{2^{2m}} \int_0^{\frac{1}{\tau^2}} \frac{\log^{2m} t}{1-t} dt - \frac{\log^{2m+1} \tau}{2m+1}, \tag{2.7}$$

(2.8)

$$f(2m-1) - g(2m-1) = 2\left(1 - \frac{1}{2^{2m-1}}\right) I(2m-1)$$
$$+ \frac{1}{2^{2m-1}} \int_0^{\frac{1}{\tau^2}} \frac{\log^{2m-1} t}{1-t} dt + \frac{\log^{2m} \tau}{2m}.$$

Next we evaluate the integral $\int_0^{1/\tau^2} \frac{\log^k t}{1-t} dt$. Since

$$\int_0^{1/\tau^2} \frac{\log^k t}{1-t} dt = \int_{1/\tau}^1 \frac{\log^k (1-u)}{u} du = I(k) - \int_0^{1/\tau} \frac{\log^k (1-u)}{u} du,$$

it suffices to evaluate

$$\begin{aligned}
\int_0^{1/\tau} \frac{\log^k(1-u)}{u} du
&= \int_0^{1/\tau} \log^k(1-u) d\log u \\
&= \log^k(1-u)\log u \Big|_0^{1/\tau} + k\int_0^{1/\tau} \frac{\log^{k-1}(1-u)\log u}{1-u} du \\
&= (-1)^{k+1}2^k \log^{k+1}\tau + k(-1)^{k+1}\int_0^{2\log\tau} \theta^{k-1}\Big[A(\theta)-\frac{\theta}{2}\Big]d\theta \\
&= \frac{(-1)^{k+1}2^k}{k+1}\log^{k+1}\tau + k(-1)^{k+1}\int_0^{2\log\tau}\theta^{k-1}A(\theta)d\theta.
\end{aligned}$$

Hence we have

(2.9)
$$\begin{aligned}
\int_0^{1/\tau^2} \frac{\log^{2m} t}{1-t} dt =& I(2m) + \frac{2^{2m}\log^{2m+1}\tau}{2m+1} \\
&+ 2m\int_0^{2\log\tau}\theta^{2m-1}A(\theta)d\theta,
\end{aligned}$$

(2.10)
$$\begin{aligned}
\int_0^{1/\tau^2} \frac{\log^{2m-1} t}{1-t} dt =& I(2m-1) - \frac{2^{2m-1}\log^{2m} t}{2m} \\
&- (2m-1)\int_0^{2\log\tau}\theta^{2m-2}A(\theta)d\theta.
\end{aligned}$$

Substituting (2.9), (2.10) into (2.7), (2.8) respectively, we obtain

$$f(2m)+g(2m) = \frac{I(2m)}{2^{2m}} + \frac{2m}{2^{2m}}\int_0^{2\log\tau}\theta^{2m-1}A(\theta)d\theta, \tag{2.11}$$

(2.12)
$$\begin{aligned}
f(2m-1)-g(2m-1) =& 2\Big(1-\frac{1}{2^{2m}}\Big)I(2m-1) \\
&- \frac{2m-1}{2^{2m-1}}\int_0^{2\log\tau}\theta^{2m-2}A(\theta)d\theta.
\end{aligned}$$

Combining (2.11), (2.12) with (2.3), (2.4), (2.5) gives

THEOREM 1. *For* $m = 1, 2, \ldots$

(2.13)
$$\zeta(2m+1) = -\frac{1}{(2m-1)!}\int_0^{2\log\tau} \theta^{2m-1}\log\left(2\sinh\frac{\theta}{2}\right)d\theta + \frac{2^{2m}}{(2m)!}\int_0^{2\log\tau}\left\{\left[\log\left(2\sinh\frac{\theta}{2}\right)-\frac{\theta}{2}\right]^{2m} - \left[\log\left(2\sinh\frac{\theta}{2}\right)+\frac{\theta}{2}\right]^{2m}\right\}d\theta,$$

and

(2.14)
$$\left(1-\frac{1}{2^{2m}}\right)\zeta(2m) = -\frac{1}{2^{2m}(2m-2)!}\int_0^{2\log\tau}\theta^{2m-2}\log\left(2\sinh\frac{\theta}{2}\right)d\theta - \frac{1}{2(2m-1)!}\int_0^{2\log\tau}\left\{\left[\log\left(2\sinh\frac{\theta}{2}\right)-\frac{\theta}{2}\right]^{2m-1} + \left[\log\left(2\sinh\frac{\theta}{2}\right)+\frac{\theta}{2}\right]^{2m-1}\right\}d\theta.$$

As previously remarked (2.13) is due to Butzer, Markett and Schmidt [**2**], while (2.14) appears to be new. We note that (2.13) can be rewritten as

$$(2.15)\quad \int_0^{2\log\tau}\sum_{k=1}^{m-1}\frac{\binom{2m-1}{2k-2}}{(2k-1)2^{2k-2}}\theta^{2k-1}\log^{2m-2k+1}\left(2\sinh\frac{\theta}{2}\right)d\theta + \frac{5}{2^{2m}}\int_0^{2\log\tau}\theta^{2m-1}\log\left(2\sinh\frac{\theta}{2}\right)d\theta = -\frac{(2m-1)!}{2^{2m}}\zeta(2m+1),$$

and (2.14) as

$$(2.16)\quad \int_0^{2\log\tau}\sum_{k=0}^{m-1}\frac{\binom{2m-1}{2k}}{2^{2k}}\theta^{2k}\log^{2m-2k-1}\left(2\sinh\frac{\theta}{2}\right)d\theta + \frac{2m-1}{2^{2m}}\int_0^{2\log\tau}\theta^{2m-2}\log\left(2\sinh\frac{\theta}{2}\right)d\theta = \left(\frac{1}{2^{2m}}-1\right)(2m-1)!\zeta(2m) = \frac{(-1)^m}{4m}\pi^{2m}(2^{2m}-1)B_{2m},$$

since

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}. \tag{2.17}$$

Taking $m = 1, 2$ in (2.15), we have

$$\int_0^{2\log\tau} \theta \log\left(2\sinh\frac{\theta}{2}\right)d\theta = -\frac{1}{5}\zeta(3), \tag{2.18}$$

(2.19)

$$\int_0^{2\log\tau}\left[\theta\log^3\left(2\sinh\frac{\theta}{2}\right) + \frac{5}{16}\theta^3\log\left(2\sinh\frac{\theta}{2}\right)\right]d\theta = -\frac{3}{8}\zeta(5).$$

Taking $m = 1, 2$ in (2.16) gives

$$\int_0^{2\log\tau} \log\left(2\sinh\frac{\theta}{2}\right)d\theta = -\frac{\pi^2}{10}, \tag{2.20}$$

(2.21)

$$\int_0^{2\log\tau}\left[\log^3\left(2\sinh\frac{\theta}{2}\right) + \frac{15}{16}\theta^2\log\left(2\sinh\frac{\theta}{2}\right)\right]d\theta = -\frac{\pi^4}{16}.$$

3. Representation of $\zeta(n)$ by integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$. We set

$$B = B(\theta) = \log\left(2\sin\frac{\theta}{2}\right),$$

so that

$$\log i(1 - e^{i\theta}) = B(\theta) + \frac{i\theta}{2}.$$

We consider the integral

$$J(k) = \int_1^\omega \frac{[\log i(1-u)]^k}{u}du \overset{u=e^{i\theta}}{=} i\int_0^{\frac{\pi}{3}}\left(B + \frac{i\theta}{2}\right)^k d\theta \tag{3.1}$$

where the first integral is along the arc of the unit circle $|u| = 1$ from $u = 1$ to $u = \omega = e^{\pi i/3}$ in a counter-clockwise direction. Making the substitution $t = i(1-u)$, $u = i(t-i)$, $du = idt$, we obtain

$$J(k) = \int_0^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i}dt = \int_0^1 \frac{\log^k t}{t-i}dt + \int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i}dt. \tag{3.2}$$

We now evaluate the two integrals on the right side of (3.2). We have

$$\int_0^1 \frac{\log^k t}{t-i}dt = \int_0^1 \frac{t+i}{t^2+1}\log^k t\,dt$$
$$= \frac{1}{2}\int_0^1 \frac{\log^k t}{1+t^2}dt^2 + i\int_0^1 \frac{\log^k t}{1+t^2}dt.$$

Since

$$\int_0^1 \frac{\log^k t}{1+t^2}dt = (-1)^k k! S(k+1),$$

where $S(s) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$ ($\mathrm{Re}(s) > 0$) is a particular Dirichlet L-function (see (4.4) in [**8**]) and

$$\int_0^1 \frac{\log^k t}{1+t^2}dt^2 = \frac{1}{2^k}\int_0^1 \frac{\log^k t}{1+t}dt$$
$$= \frac{1}{2^k}\left[\int_0^1 \left(\frac{1}{1+t} - \frac{1}{1-t}\right)\log^k t\,dt + \int_0^1 \frac{\log^k t}{1-t}dt\right]$$
$$= \frac{1}{2^k}\left[I(k) - \int_0^1 \frac{\log^k t}{1-t^2}dt^2\right] = \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)I(k),$$

we have

$$\int_0^1 \frac{\log^k t}{t-i}dt = \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)I(k) + i(-1)^k k! S(k+1). \tag{3.3}$$

For the second integral in (3.2), we have

$$\int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i}dt = \int_1^{\omega^{\frac{1}{2}}} \frac{t+i}{1+t^2}\log^k t\ dt$$
$$\overset{t=e^{\frac{i\theta}{2}}}{=} \int_0^{\frac{\pi}{3}} \frac{\left(e^{\frac{i\theta}{2}}+i\right)}{1+e^{i\theta}}\left(\frac{i\theta}{2}\right)^k \frac{i}{2}e^{\frac{i\theta}{2}}d\theta$$
$$= \frac{i^{k+1}}{2^{k+2}}\int_0^{\frac{\pi}{3}} \frac{\left[\cos\frac{\theta}{2} + i\left(1+\sin\frac{\theta}{2}\right)\right]}{\cos\frac{\theta}{2}}\theta^k d\theta$$
$$= \frac{i^{k+1}}{2^{k+2}}\left[\frac{1}{k+1}\left(\frac{\pi}{3}\right)^{k+1} + i\int_0^{\frac{\pi}{3}} \frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\theta^k d\theta\right]$$
$$= \frac{1}{2(k+1)}\left(\frac{\pi}{6}\right)^{k+1} i^{k+1} - \frac{i^k}{2^{k+2}}\int_0^{\frac{\pi}{3}} \frac{\theta^k\cos\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}d\theta.$$

Since

$$\frac{-1}{2}\int_0^{\frac{\pi}{3}} \frac{\theta^k \cos\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}d\theta = \int_0^{\frac{\pi}{3}} \theta^k d\log\left(1-\sin\frac{\theta}{2}\right)$$

$$= \theta^k \log\left(1-\sin\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{3}}$$

$$- k\int_0^{\frac{\pi}{3}} \theta^{k-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta$$

$$= -\left(\frac{\pi}{3}\right)^k \log 2 - k\int_0^{\frac{\pi}{3}} \theta^{k-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta,$$

we have

(3.4)

$$\int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i}dt = \frac{1}{2(k+1)}\left(\frac{\pi i}{6}\right)^{k+1}$$

$$- \frac{i^k}{2^{k+1}}\left[\left(\frac{\pi}{3}\right)^k \log 2 + k\int_0^{\frac{\pi}{3}} \theta^{k-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta\right].$$

From (3.1)–(3.4), we obtain

(3.5)

$$i\int_0^{\frac{\pi}{3}}\left(B+\frac{i\theta}{2}\right)^k d\theta$$

$$= \frac{1}{2^{k+1}}\left(1-\frac{1}{2^k}\right)(-1)^k k!\zeta(k+1)$$

$$+ i(-1)^k k! S(k+1) + \frac{1}{2(k+1)}\left(\frac{\pi i}{6}\right)^{k+1}$$

$$- \frac{i^k}{2}\left[\left(\frac{\pi}{6}\right)^k \log 2 + \frac{k}{2^k}\int_0^{\frac{\pi}{3}} \theta^{k-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta\right].$$

Taking $k = 2m-1$ and $k = 2m$ in (3.5) gives respectively

(3.6)

$$i\int_0^{\frac{\pi}{3}}\left(B+\frac{i\theta}{2}\right)^{2m-1} d\theta$$

$$= -\frac{1}{2^{2m}}\left(1-\frac{1}{2^{2m-1}}\right)(2m-1)!\zeta(2m) + \frac{(-1)^m}{4m}\left(\frac{\pi}{6}\right)^{2m}$$

$$+ i\left[-(2m-1)!S(2m) + \frac{(-1)^m}{2}\left(\frac{\pi}{6}\right)^{2m-1}\log 2\right.$$

$$\left. + \frac{(-1)^m(2m-1)}{2^{2m}}\int_0^{\frac{\pi}{3}} \theta^{2m-2}\log\left(1-\sin\frac{\theta}{2}\right)d\theta\right],$$

(3.7)

$$
\begin{aligned}
&i\int_0^{\frac{\pi}{3}}\left(B+\frac{i\theta}{2}\right)^{2m}d\theta\\
&\quad=\frac{1}{2^{2m+1}}\left(1-\frac{1}{2^{2m}}\right)(2m)!\zeta(2m+1)\\
&\quad+\frac{(-1)^{m+1}}{2}\left[\left(\frac{\pi}{6}\right)^{2m}\log 2+\frac{2m}{2^{2m}}\int_0^{\frac{\pi}{3}}\theta^{2m-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta\right]\\
&\quad+i\left[(2m)!S(2m+1)+\frac{(-1)^m}{2(2m+1)}\left(\frac{\pi}{6}\right)^{2m+1}\right].
\end{aligned}
$$

Taking the real part of (3.6) yields

$$
\begin{aligned}
(3.8)\qquad &\left(1-\frac{1}{2^{2m-1}}\right)\zeta(2m)\\
&=\frac{(-1)^m}{2(2m)!}\left(\frac{\pi}{3}\right)^{2m}+\frac{2^{2m}}{(2m-1)!}\int_0^{\frac{\pi}{3}}Im\left(B+\frac{i\theta}{2}\right)^{2m-1}d\theta.
\end{aligned}
$$

Since

$$
\begin{aligned}
Im\left(B+\frac{i\theta}{2}\right)^{2m-1}&=\sum_{k=0}^{m-1}(-1)^k\binom{2m-1}{2k+1}\left(\frac{\theta}{2}\right)^{2k+1}B^{2m-2k-2}\\
&=\frac{2m-1}{2}\sum_{k=0}^{m-2}\frac{(-1)^k\binom{2m-2}{2k}}{(2k+1)2^{2k}}\theta^{2k+1}B^{2m-2k-2}\\
&\quad+\frac{(-1)^{m-1}\theta^{2m-1}}{2^{2m-1}},
\end{aligned}
$$

we have

$$
\begin{aligned}
(3.9)\qquad &\left(1-\frac{1}{2^{2m-1}}\right)\zeta(2m)\\
&=\frac{(-1)^{m-1}\pi^{2m}}{2(2m)!3^{2m-1}}-\frac{2^{2m-1}}{(2m-2)!}\int_0^{\frac{\pi}{3}}\sum_{k=0}^{m-2}\frac{(-1)^k\binom{2m-2}{2k}}{(2k+1)2^{2k}}\theta^{2k+1}B^{2m-2k-2}d\theta.
\end{aligned}
$$

Recalling the formula (2.17), (3.9) gives the following result.

THEOREM 2. *For* $m \geq 2$,

(3.10)
$$\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-2} \frac{(-1)^k \binom{2m-2}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} \log^{2m-2k-2}\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{(-1)^m \pi^{2m}}{4m(2m-1)}\left[\left(\frac{1}{6}\right)^{2m-1} - 2\left(1 - \frac{1}{2^{2m-1}}\right)B_{2m}\right].$$

Taking $m = 2, 3$ in (3.10) we obtain

$$\int_0^{\frac{\pi}{3}} \theta \log^2\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{17\pi^4}{2^4 \cdot 3^4 \cdot 5}, \tag{3.11}$$

(3.12)
$$\int_0^{\frac{\pi}{3}} \left[\theta \log^4\left(2\sin\frac{\theta}{2}\right) - \frac{\theta^3}{2}\log\left(2\sin\frac{\theta}{2}\right)\right] d\theta = \frac{313\pi^6}{2^4 \cdot 3^6 \cdot 5 \cdot 7}.$$

Taking the imaginary part of (3.7) yields

(3.13)
$$\int_0^{\frac{\pi}{3}} \operatorname{Re}\left(B + \frac{i\theta}{2}\right)^{2m} d\theta = (2m)!S(2m+1) + \frac{(-1)^m}{2(2m+1)}\left(\frac{\pi}{6}\right)^{2m+1}.$$

Since

$$Re\left(B + \frac{i\theta}{2}\right)^{2m} = \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m}{2k}}{2^{2k}} \theta^{2k} B^{2m-2k} + \frac{(-1)^m \theta^{2m}}{2^{2m}}$$

and

$$(2m)!S(2m+1) = \frac{(-1)^m}{2}\left(\frac{\pi}{2}\right)^{2m+1} E_{2m}, \tag{3.14}$$

where the E_{2m} are the Euler numbers, we have from (3.13)

THEOREM 3. *For* $m \geq 1$,

(3.15)
$$\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m}{2k}}{2^{2k}} \theta^{2k} \log^{2m-2k}\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{(-1)^m \pi^{2m+1}}{2^{2m+2}}\left[E_{2m} - \frac{1}{(2m+1)3^{2m}}\right].$$

Taking $m = 1, 2, 3$ in (3.15), we have

(3.16) $$\int_0^{\frac{\pi}{3}} \log^2\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{7\pi^3}{2^2 \cdot 3^3},$$

(3.17) $$\int_0^{\frac{\pi}{3}} \left[\log^4\left(2\sin\frac{\theta}{2}\right) - \frac{3}{2}\theta^2 \log^2\left(2\sin\frac{\theta}{2}\right)\right] d\theta = \frac{253\pi^5}{2^3 \cdot 3^4 \cdot 5},$$

(3.18) $$\int_0^{\frac{\pi}{3}} \left[\log^6\left(2\sin\frac{\theta}{2}\right) - \frac{15}{4}\theta^2 \log^4\left(2\sin\frac{\theta}{2}\right) + \frac{15}{16}\theta^4 \log^2\left(2\sin\frac{\theta}{2}\right)\right] d\theta = \frac{77821\pi^7}{2^6 \cdot 3^6 \cdot 7}.$$

Taking the imaginary part of (3.6), we obtain

(3.19) $$\int_0^{\frac{\pi}{3}} Re\left(B + \frac{i\theta}{2}\right)^{2m-1} d\theta = -(2m-1)!S(2m) + \frac{(-1)^m}{2}\left(\frac{\pi}{6}\right)^{2m-1} \log 2 + \frac{(-1)^m(2m-1)}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-2} \log\left(1 - \sin\frac{\theta}{2}\right) d\theta.$$

Since

$$\operatorname{Re}\left(B + \frac{i\theta}{2}\right)^{2m-1} = \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{2^{2k}} \theta^{2k} B^{2m-2k-1},$$

we have

(3.20) $$S(2m) = \frac{(-1)^m}{2(2m-1)!}\left(\frac{\pi}{6}\right)^{2m-1} \log 2 + \frac{(-1)^m}{(2m-2)!2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-2} \log\left(1 - \sin\frac{\theta}{2}\right) d\theta + \frac{1}{(2m-1)!} \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{2^{2k}} \theta^{2k} \cdot \log^{2m-2k-1}\left(2\sin\frac{\theta}{2}\right) d\theta.$$

Taking $m = 1$ in (3.20) we obtain

$$S(2) = -\frac{\pi}{12}\log 2 - \frac{1}{4}\int_0^{\frac{\pi}{3}} \log\left(1-\sin\frac{\theta}{2}\right)d\theta - \int_0^{\frac{\pi}{3}} \log\left(2\sin\frac{\theta}{2}\right)d\theta,$$

where $S(2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965\ldots$ is Catalan's constant.

Taking the real part of (3.7), we have

$$\int_0^{\frac{\pi}{3}} \operatorname{Im}\left(B+\frac{i\theta}{2}\right)^{2m} d\theta = \frac{1}{2^{2m+1}}\left(\frac{1}{2^{2m}}-1\right)(2m)!\zeta(2m+1) + \frac{(-1)^m}{2}\left[\left(\frac{\pi}{6}\right)^{2m}\log 2 + \frac{2m}{2^{2m}}\int_0^{\frac{\pi}{3}} \theta^{2m-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta\right].$$

In view of

$$\operatorname{Im}\left(B+\frac{i\theta}{2}\right)^{2m} = m\sum_{k=0}^{m-1} \frac{(-1)^k\binom{2m-1}{2k}}{(2k+1)2^{2k}}\theta^{2k+1}B^{2m-2k-1},$$

we obtain

(3.21)

$$\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k\binom{2m-1}{2k}}{(2k+1)2^{2k}}\theta^{2k+1}\log^{2m-2k-1}\left(2\sin\frac{\theta}{2}\right)d\theta + \frac{(-1)^{m-1}}{2^{2m}}\int_0^{\frac{\pi}{3}} \theta^{2m-1}\log\left(1-\sin\frac{\theta}{2}\right)d\theta = \frac{1}{2^{2m}}\left(\frac{1}{2^{2m}}-1\right)(2m-1)!\zeta(2m+1) + \frac{(-1)^m}{2m}\left(\frac{\pi}{6}\right)^{2m}\log 2.$$

Taking $m = 1, 2$ in (3.21), we obtain

(3.22)

$$\int_0^{\frac{\pi}{3}} \theta\left[\log\left(2\sin\frac{\theta}{2}\right) + \frac{1}{4}\log\left(1-\sin\frac{\theta}{2}\right)\right]d\theta = -\frac{3}{16}\zeta(3) - \frac{\pi^2}{72}\log 2,$$

(3.23)

$$\int_0^{\frac{\pi}{3}} \left[\theta\log^3\left(2\sin\frac{\theta}{2}\right) - \frac{1}{4}\theta^3\log\left(2\sin\frac{\theta}{2}\right)\right]d\theta - \frac{1}{16}\int_0^{\frac{\pi}{3}} \theta^3\log\left(1-\sin\frac{\theta}{2}\right)d\theta = -\frac{45}{128}\zeta(5) + \frac{1}{4}\left(\frac{\pi}{6}\right)^4\log 2.$$

4. Relations between integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$ and $\log\left(2\sinh\frac{\theta}{2}\right)$ and certain series. The power series expansion of $\frac{\arcsin x}{\sqrt{1-x^2}}$ is given by

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{2^{2n}x^{2n-1}}{2n\binom{2n}{n}}, \qquad |x| < 1.$$

Integrating and differentiating this equality, we have

$$(4.1) \qquad (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2\binom{2n}{n}},$$

$$(4.2) \qquad \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n}} = \frac{x^2}{1-x^2} + \frac{x\arcsin x}{(1-x^2)^{3/2}}.$$

Next, we apply the method of constructing polylogarithms to the function

$$K_0(x) = \frac{2x\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}.$$

We set

$$K_1(x) = \int_0^x \frac{K_0(x)}{x}dx = (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2\binom{2n}{n}},$$

$$K_2(x) = \int_0^x \frac{K_1(x)}{x}dx = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2^2n^3\binom{2n}{n}}, \qquad \text{etc.}$$

In general, we have

(4.3)

$$K_m(x) = \int_0^x \frac{K_{m-1}(x)}{x}dx = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2^m n^{m+1}\binom{2n}{n}}, \quad m = 1, 2, 3, \dots .$$

Taking $x = \frac{1}{2}$ in (4.3) gives

$$\sum_{n=1}^{\infty} \frac{1}{2^m n^{m+1}\binom{2n}{n}} = K_m\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{K_{m-1}(x)}{x}dx.$$

Then, using integration by parts $(m-1)$ times, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{2^m n^{m+1}\binom{2n}{n}} = \int_0^{\frac{1}{2}} K_{m-1}(x) d\log(2x)$$

$$= -\int_0^{\frac{1}{2}} \log(2x) \frac{K_{m-2}(x)}{x} dx$$

$$= -\frac{1}{2!}\int_0^{\frac{1}{2}} K_{m-2}(x) d\log^2(2x) = \frac{1}{2!}\int_0^{\frac{1}{2}} \log^2(2x)\frac{K_{m-3}(x)}{x} dx$$

$$= \ldots = \frac{(-1)^{m-1}}{(m-1)!}\int_0^{\frac{1}{2}} \log^{m-1}(2x)\frac{2\arcsin x}{\sqrt{1-x^2}} dx \ \left(x = \sin\frac{\theta}{2}\right)$$

$$= \frac{(-1)^{m-1}}{2(m-1)!}\int_0^{\frac{\pi}{3}} \theta \log^{m-1}\left(2\sin\frac{\theta}{2}\right) d\theta,$$

that is

(4.4)

$$\sum_{n=1}^{\infty} \frac{1}{n^{m+2}\binom{2n}{n}} = \frac{(-1)^m 2^m}{m!}\int_0^{\frac{\pi}{3}} \theta \log^m\left(2\sin\frac{\theta}{2}\right) d\theta, \ m = 0, 1, 2, \ldots .$$

Taking $x = \frac{1}{2}$ in (4.2), (4.3) and $m = 0, 1, 2$ in (4.4), we obtain

(4.5) $$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\sqrt{3}}{27}\pi,$$

(4.6) $$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\sqrt{3}}{9}\pi,$$

(4.7) $$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18},$$

(4.8) $$\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} = -2\int_0^{\frac{\pi}{3}} \theta \log\left(2\sin\frac{\theta}{2}\right) d\theta,$$

(4.9)

$$\sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}} = 2\int_0^{\frac{\pi}{3}} \theta \log^2\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{17\pi^4}{2^3 \cdot 3^4 \cdot 5},$$

(4.10)

$$\sum_{n=1}^{\infty} \frac{1}{n^5\binom{2n}{n}} = -\frac{4}{3}\int_0^{\frac{\pi}{3}} \theta \log^3\left(2\sin\frac{\theta}{2}\right) d\theta.$$

Changing x into ix in (4.1) (4.2) and (4.3) with $m = 0$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{\binom{2n}{n}} = \frac{x}{1+x^2} + \frac{x \sinh^{-1} x}{(1+x^2)^{3/2}}, \tag{4.11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{n\binom{2n}{n}} = \frac{2x \sinh^{-1} x}{\sqrt{1+x^2}}, \tag{4.12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{2n^2\binom{2n}{n}} = (\sinh^{-1} x)^2. \tag{4.13}$$

Now, taking $x = \frac{1}{2}$ in (4.11), (4.12), (4.13), we deduce (as $\sinh^{-1} \frac{1}{2} = \log \tau$)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4}{5\sqrt{5}} \log \tau, \tag{4.14}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\binom{2n}{n}} = \frac{1}{\sqrt{5}} \log \tau, \tag{4.15}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\binom{2n}{n}} = 2 \log^2 \tau. \tag{4.16}$$

Analogous to the construction of $K_m(x)$, we set

$$F_0(x) = \frac{2x \sinh^{-1} x}{\sqrt{1+x^2}}, \ F_1(x) = \int_0^x \frac{F_0(x)}{x} dx = (\sinh^{-1} x)^2,$$

and, from (4.11) and (4.12), we have

(4.17)
$$\begin{aligned} F_m(x) &= \int_0^x \frac{F_{m-1}(x)}{x} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{2^m n^{m+1}\binom{2n}{n}}, \quad m = 1, 2, 3, \dots . \end{aligned}$$

After integration by parts, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{m+2}\binom{2n}{n}} = \frac{(-1)^m 2^{m+1}}{m!} \int_0^{\frac{1}{2}} \log^m(2x) \frac{F_0(x)}{x} dx,$$

that is

(4.18)
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{m+2}\binom{2n}{n}}=\frac{(-1)^m 2^m}{m!}\int_0^{2\log\tau}\theta\log^m\left(2\sinh\frac{\theta}{2}\right)d\theta,$$
$$m=0,1,2,\ldots.$$

Taking $m=1$ in (4.18) and appealing to (2.18), we have the well-known formula

(4.19)
$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Taking $m=3$ in (4.18) yields

(4.20)
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5\binom{2n}{n}}=-\frac{4}{3}\int_0^{2\log\tau}\theta\log^3\left(2\sinh\frac{\theta}{2}\right)d\theta$$

and substituting into (2.18) gives

(4.21)
$$\zeta(5)=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5\binom{2n}{n}}-\frac{5}{6}\int_0^{2\log\tau}\theta^3\log\left(2\sinh\frac{\theta}{2}\right)d\theta.$$

Since

$$\log\left(2\sinh\frac{\theta}{2}\right)=\log\theta-\sum_{n=1}^{\infty}\frac{(-1)^n\zeta(2n)}{n(2\pi)^{2n}}\theta^{2n},\quad 0<\theta<\pi/2,$$

from (4.21) we obtain

(4.22)
$$\zeta(5)=-\frac{5}{6}\Big(4\log\log\tau+4\log 2-1\Big)\log^4\tau+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5\binom{2n}{n}}$$
$$-\frac{20}{3}\sum_{n=1}^{\infty}\frac{(-1)^n\zeta(2n)}{n(n+2)\pi^{2n}}\log^{2n+4}\tau.$$

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