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THE RECURRENCE RELATION $a_n = (An+B)a_{n-1} + (Cn+D)a_{n-2}$

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Suppose n objects are labelled $1, 2, \cdots, n$. A permutation of these objects in which object i is not placed in the ith place for any i is called a *derangement*. It is well-known (see for example [3, p. 204]) that the number D_n of derangements of n objects satisfies the recurrence relation

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2} \quad (n = 3, 4, \cdots)$$

with $D_1 = 0$, $D_2 = 1$. Less well-known is the fact that the number R_n of permutations of the n objects in which object $i+1$ is not positioned immediately to the right of object i for any i satisfies the recurrence relation

$$R_n = (n-1)R_{n-1} + (n-2)R_{n-2} \quad (n = 3, 4, \cdots)$$

with $R_1 = 1, R_2 = 1$, see for example [2, p. 115].

Both of these recurrence relations are of the type

$$a_n = (An+B)a_{n-1} + (Cn+D)a_{n-2} \quad (n = 3, 4, \cdots) \tag{1a}$$

with

$$a_1 = \alpha, \; a_2 = \beta \tag{1b}$$

for suitable real numbers A, B, C, D, α and β. If both A and C are zero the recurrence relation (1a) reduces to $a_n = Ba_{n-1} + Da_{n-2}$ whose solution is well-known (see for example [1, p. 250]) so we can exclude this possibility. Similarly if A and B are both zero or C and D are both zero, the recurrence relation is easy to solve, so we can exclude these possibilities as well. However, it is very difficult to solve the recurrence relation (1ab) in general. We identify four situations when a simple explicit solution can be given.

To solve the above recurrence relation for the number of derangements D_n, we usually rewrite the recurrence as

$$D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2}) \quad (n = 3, 4, \cdots)$$

so that

$$D_n - nD_{n-1} = (-1)^n(D_2 - 2D_1) = (-1)^n \quad (n = 2, 3, \cdots).$$

Thus

$$\frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!} \quad (n = 2, 3, \cdots)$$

and so

$$\frac{D_n}{n!} - \frac{D_1}{1!} = \sum_{k=2}^{n} \left(\frac{D_k}{k!} - \frac{D_{k-1}}{(k-1)!} \right) = \sum_{k=2}^{n} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots)$$

giving

$$D_n = n!\sum_{k=2}^{n}\frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots),$$

which is the well-known expression for D_n (see for example 2, p. 114], [3, p. 225]).

We now attempt to apply this technique to the general recurrence relation (1ab) by trying to express it in the form

$$a_n - (An + B + X)a_{n-1} = Y(a_{n-1} - (A(n-1) + B + X)a_{n-2}) \quad (n = 3, 4, \cdots) \qquad (2)$$

for suitable constants X and Y. By comparing (2) with (1a) we see that this is possible if and only if

$$X = \frac{C}{A}, \quad Y = -\frac{C}{A}, \quad D = -C + \frac{BC}{A} + \frac{C^2}{A^2}.$$

From (2) with these values, we obtain

$$a_n - \left(An + B + \frac{C}{A}\right)a_{n-1} = \left(\frac{-C}{A}\right)^{n-2}\left(a_2 - \left(2A + B + \frac{C}{A}\right)a_1\right) \quad (n = 2, 3, \cdots).$$

Appealing to (1b) we see that

$$a_n - \left(An + B + \frac{C}{A}\right)a_{n-1} = \left(\frac{-C}{A}\right)^{n-2}\left(\beta - \left(2A + B + \frac{C}{A}\right)\alpha\right) \quad (n = 2, 3, \cdots). \qquad (3)$$

We solve the two-term recurrence relation (3) by making use of the following result.

LEMMA: Let $f(n)$ and $g(n)$ be real-valued functions defined for $n = 2, 3, 4, \ldots$. If b_n $(n = 1, 2, 3, \cdots)$ satisfies the recurrence relation

$$b_n = f(n)b_{n-1} + g(n) \quad (n = 2, 3, \cdots)$$

then

$$b_n = b_1\prod_{i=2}^{n} f(i) + \sum_{k=2}^{n} g(k)\prod_{i=k+1}^{n} f(i) \quad (n = 1, 2, 3, \cdots).$$

Remark: We remark that the empty sum is always understood to be 0 and the empty product to be 1.

Proof: For $k = 1, 2, \cdots, n$ we set

$$p(n,k) = \prod_{i=k+1}^{n} f(i),$$

so that is particular we have $p(n,n) = 1$. Multiplying the relation

$$b_k - f(k)b_{k-1} = g(k) \quad (k = 2, 3, \cdots)$$

by $p(n,k)$, we obtain

$$p(n,k)b_k - p(n,k-1)b_{k-1} = p(n,k)g(k) \quad (k = 2, 3, \cdots).$$

Summing over $k = 2, \cdots, n$, we have

$$\sum_{k=2}^{n}(p(n,k)b_k - p(n,k-1)b_{k-1}) = \sum_{k=2}^{n} p(n,k)g(k),$$

so that

$$b_n - p(n,1)b_1 = \sum_{k=2}^{n} p(n,k)g(k),$$

that is

$$b_n = b_1 \prod_{i=2}^{n} f(i) + \sum_{k=2}^{n} g(k) \prod_{i=k+1}^{n} f(i) \quad (n = 1, 2, \cdots),$$

as asserted. □

Applying the Lemma with

$$f(n) = An + B + \frac{C}{A} \quad \text{and} \quad g(n) = \left(\frac{-C}{A}\right)^{n-2} \left(\beta \left(2A + B + \frac{C}{A}\right) \alpha\right),$$

we obtain the following explicit formula for the solution a_n of the recurrence relation (1ab) when $D = -C + BC/A + C^2/A^2$.

THEOREM 1: Let A, B, C, D be real numbers such that

$$D = -C + \frac{BC}{A} + \frac{C^2}{A^2}\,.$$

Then the solution of the recurrence relation (1ab) is

$$a_n = \alpha \prod_{i=2}^{n} \left(Ai + \left(B + \frac{C}{A}\right)\right) + \left(\beta - \left(2A + B + \frac{C}{A}\right) \alpha\right) \sum_{k=2}^{n} \left(\frac{-C}{A}\right)^{k-2} \prod_{i=k+1}^{n} \left(Ai + \left(B + \frac{C}{A}\right)\right) \tag{4}$$

$(n = 1, 2, \cdots)$. □

The recurrence relation for the number D_n of derangements is the special case of Theorem 1 with $A = 1$, $B = -1$, $C = 1$, $D = -1$, $\alpha = 0$, $\beta = 1$. With these values (4) gives

$$D_n = \sum_{k=2}^{n}(-1)^{k-2} \prod_{i=k+1}^{n} i = \sum_{k=2}^{n}(-1)^k \frac{n!}{k!} = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots).$$

The recurrence relation for R_n given above has $A = 1$, $B = -1$, $C = 1$, $D = -2$ and so does not satisfy the condition $D = -C + BC/A + C^2/A^2$. However, $b_n = nR_n$ is a solution of the recurrence relation $b_n = nb_{n-1} + nb_{n-2}$ which does satisfy this condition. Theorem 1 then gives an explicit expression for $R_n = b_n/n$.

More generally if we try to find real numbers X and Y such that

$$b_n = (Xn + Y)a_n \tag{5}$$

satisfies a recurrence relation of the type (1a) where A, B, C, D satisfy $D = -C + BC/A + C^2/A^2$, we can obtain the following result.

THEOREM 2: Let A, B, C, D be real numbers such that

$$C = A^2, \quad D = AB - A^2. \tag{6}$$

Then the solution of the recurrence relation (1ab) is

$$a_n = \frac{1}{An + (A+B)} \left\{ \alpha(2A+B) \prod_{i=2}^{n} (Ai + 2A + B) \right.$$
$$\left. + ((3A+B)\beta - (2A+B)(4A+B)\alpha) \sum_{k=2}^{n} (-A)^{k-2} \prod_{i=k+1}^{n} (Ai + (2A+B)) \right\} \tag{7}$$

$(n = 1, 2, 3, \cdots)$. □

The recurrence relation for the numbers R_n has $A = 1$, $B = -1$, $C = 1$, $D = -2$ so that (6) is satisfied. With these values of A, B, C, D and $\alpha = \beta = 1$, (7) gives

$$R_n = \frac{1}{n} \left\{ \prod_{i=2}^{n} (i+1) - \sum_{k=2}^{n} (-1)^{k-2} \prod_{i=k+1}^{n} (i+1) \right\} = \frac{(n+1)!}{n} \left\{ \frac{1}{2!} + \sum_{k=2}^{n} \frac{(-1)^{k-1}}{(k+1)!} \right\},$$

that is

$$R_n = \frac{(n+1)!}{n} \sum_{k=2}^{n+1} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots).$$

Another method of trying to solve the recurrence relation (1a) is to try and express it in the form

$$a_n + Xa_{n-1} = (Yn + Z)(a_{n-1} + Xa_{n-2}) \quad (n = 3, 4, \cdots)$$

for suitable constants X, Y, Z. Again with some effort we may obtain the following result.

THEOREM 3: Let A, B, C, D be real numbers such that

$$D = \frac{BC}{A} + \frac{C^2}{A^2}. \tag{8}$$

Then the solution of the recurrence relation (1ab) is

$$a_n = \alpha \left(\frac{-C}{A} \right)^{n-1} + \left(\beta + \frac{C}{A}\alpha \right) \sum_{k=2}^{n} \left(\frac{-C}{A} \right)^{n-k} \prod_{i=3}^{k} \left(Ai + B + \frac{C}{A} \right) \quad (n = 1, 2, \cdots). \quad □$$

The recurrence relation

$$\begin{cases} a_n = (2n+1)a_{n-1} + (4n+6)a_{n-2} & (n = 3, 4, \cdots), \\ a_1 = 1, a_2 = 2, \end{cases}$$

satisfies the condition (8) of Theorem 3. Hence

$$a_n = (-2)^{n-1} + 4\sum_{k=2}^{n}(-2)^{n-k}\prod_{i=3}^{k}(2i+3) \quad (n = 1, 2, \cdots).$$

Expressing $\prod_{i=3}^{k}(2i+3)$ in the form

$$\frac{(2k+3)!}{105 \cdot 2^{k+1}(k+1)!} \quad (k = 2, 3, \cdots),$$

we obtain

$$a_n = (-1)^n 2^{n-1}\left\{\frac{4}{105}\sum_{k=2}^{n}\frac{(-1)^k(2k+3)!}{2^{2k}(k+1)!} - 1\right\} \quad (n = 1, 2, \cdots).$$

Finally, if we consider the recurrence relation (1ab) with $A = 0$, $D = -C$, $\alpha = B$, $\beta = B^2 + C$, that is

$$\begin{cases} a_n = Ba_{n-1} + C(n-1)a_{n-2} & (n = 3, 4, \cdots), \\ a_1 = B, a_2 = B^2 + C, \end{cases}$$

and apply the standard generating function method, we obtain the following result.

THEOREM 4: The solution of the recurrence relation

$$\begin{cases} a_n = Ba_{n-1} + C(n-1)a_{n-2} & (n = 2, 3, \cdots) \\ a_0 = 1, a_1 = B, \end{cases} \tag{9}$$

is given by

$$a_n = n!\sum_{s=0}^{[n/2]}\frac{B^{n-2s}C^s}{2^s(n-2s)!s!} \quad (n = 0, 1, 2, \cdots). \qquad \square$$

As a concluding example we consider the number S_n of $n \times n$ symmetric $(0,1)$ matrices all of whose columns sum to 1. When $n = 1$ there is exactly one such matrix, namely [1], so that $S_1 = 1$. When $n = 2$ there are exactly two such matrices, namely,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $S_2 = 2$. When $n = 3$ there are exactly four such matrices, namely,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $S_3 = 4$. It is also easy to show that

$$S_n = S_{n-1} + (n-1)S_{n-2} \quad (n = 3, 4, \cdots). \tag{10}$$

If we set $S_0 = 1$ the recurrence relation (10) is (9) with $B = C = 1$. Thus, by Theorem 4, we deduce

$$S_n = n! \sum_{s=0}^{[n/2]} \frac{1}{2^s (n-2s)! s!} \quad (n = 0, 1, \cdots).$$

References:

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[2] Kenneth P. Bogart, *Introductory Combinatorics*, Pitman Publishing Inc. (1983).

[3] Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall Inc. (1984).

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