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EVALUATION OF THE INFINITE SERIES $\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1}$

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ABSTRACT. Let p be a prime $\equiv 1 \pmod 4$. The value of the infinite series $\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} (\frac{n}{p})_4 n^{-1}$ is given in finite terms.

Let p be a prime $\equiv 1 \pmod 4$. If n is an integer such that the Legendre symbol $(\frac{n}{p})$ has the value $+1$, then the Dirichlet symbol $(\frac{n}{p})_4$ of quartic residuacity is defined by

$$\left(\frac{n}{p}\right)_4 = \begin{cases} +1, & \text{if } x^4 \equiv n \pmod p \text{ is solvable}, \\ -1, & \text{if } x^4 \equiv n \pmod p \text{ is unsolvable}. \end{cases}$$

In this note we use Mathews' evaluation [3] of the quartic Gauss sum to determine the sum of the infinite series

$$(1) \qquad \sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1}.$$

We define integers a and b uniquely by

$$(2) \qquad \begin{cases} p = a^2 + b^2, \\ a \equiv 1 \pmod 4,\ b \equiv -((p-1)/2)!a \pmod p. \end{cases}$$

We set

$$(3) \qquad \omega = a + ib, \quad (\text{so that } \omega\overline{\omega} = p),$$

and define a quartic character χ_ω on the ring of Gaussian integers $Z[i]$ mod ω by

$$\chi_\omega(\nu) \equiv \nu^{(p-1)/4} \pmod \omega.$$

Thus, for an integer $n \not\equiv 0 \pmod p$, we have

$$\chi_\omega(n) = \left(\frac{n}{p}\right)_4, \quad \text{if } \left(\frac{n}{p}\right) = 1,$$

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and

$$\frac{\chi_\omega(n)}{i} = \pm 1, \quad \text{if } \left(\frac{n}{p}\right) = -1.$$

We also let

$$LS_\omega(i^k) = \sum_{\substack{s=1 \\ \chi_\omega(s)=i^k}}^{p-1} \log\sin(\pi s/p) \qquad (k = 0, 1, 2, 3) \tag{4}$$

and

$$F_\omega(i^k) = \sum_{\substack{s=1 \\ \chi_\omega(s)=i^k}}^{p-1} s \qquad (k = 0, 1, 2, 3). \tag{5}$$

With the above notation, we have

Theorem. (a) *For* $p \equiv 1 \pmod 8$ *we have*

$$\begin{aligned}\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} &= (2p)^{-1}(-1)^{(p+2b+7)/8}\left(\frac{2|b|}{|a|}\right) \\ &\times\left(-\frac{b}{|b|}\sqrt{2p-2a\sqrt{p}}(LS_\omega(1)-LS_\omega(-1))\right. \\ &\left.+\sqrt{2p+2a\sqrt{p}}(LS_\omega(i)-LS_\omega(-i))\right),\end{aligned} \tag{6}$$

and for $p \equiv 5 \pmod 8$ *we have*

$$\begin{aligned}\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} &= (2p^2)^{-1}(-1)^{(p+2b+7)/8}\left(\frac{2|b|}{|a|}\right) \\ &\times\pi\left(\sqrt{2p+2a\sqrt{p}}(F_\omega(1)-F_\omega(-1))\right. \\ &\left.+\frac{b}{|b|}\sqrt{2p-2a\sqrt{p}}(F_\omega(i)-F_\omega(-i))\right).\end{aligned} \tag{7}$$

(b) *For* $p \equiv 1 \pmod 8$ *we have*

$$\begin{aligned}\sum_{\substack{n=1 \\ (\frac{n}{p})=-1}}^{\infty} \left(\frac{\chi_\omega(n)}{i}\right) n^{-1} &= (2p)^{-1}(-1)^{(p+2b+7)/8}\left(\frac{2|b|}{|a|}\right) \\ &\times\left(\sqrt{2p+2a\sqrt{p}}(LS_\omega(1)-LS_\omega(-1))\right. \\ &\left.+\frac{b}{|b|}\sqrt{2p-2a\sqrt{p}}(LS_\omega(i)-LS_\omega(-i))\right),\end{aligned} \tag{8}$$

and for $p \equiv 5 \pmod 8$ *we have*

$$\sum_{\substack{n=1\\(\frac{n}{p})=-1}}^{\infty} \left(\frac{\chi_\omega(n)}{i}\right) n^{-1} = (2p^2)^{-1}(-1)^{(p+2b+7)/8}\left(\frac{2|b|}{|a|}\right) \times \pi\left(\frac{b}{|b|}\sqrt{2p-2a\sqrt{p}}(F_\omega(1)-F_\omega(-1)) - \sqrt{2p+2a\sqrt{p}}(F_\omega(i)-F_\omega(-i))\right). \tag{9}$$

(*The large parentheses denote the usual Jacobi symbol.*)

We remark that

$$\sum_{\substack{n=1\\(\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} = \mathrm{Re}(L(1,\chi_\omega)), \qquad \sum_{\substack{n=1\\(\frac{n}{p})=-1}}^{\infty} \left(\frac{\chi_\omega(n)}{i}\right) n^{-1} = \mathrm{Im}(L(1,\chi_\omega)),$$

where

$$L(1,\chi_\omega) = \sum_{n=1}^{\infty} \chi_\omega(n) n^{-1}.$$

Example 1. $p = 5$. Here $a = 1$, $b = -2$, $\omega = 1 - 2i$, and part (a) of the Theorem gives

$$\begin{aligned} 1 - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{14} + \cdots &= \sum_{\substack{n=1\\(\frac{n}{5})=1}}^{\infty} \left(\frac{n}{5}\right)_4 n^{-1} \\ &= -\frac{\pi}{50}\left(\sqrt{10+2\sqrt{5}}(F_{1-2i}(1)-F_{1-2i}(-1)) - \sqrt{10-2\sqrt{5}}(F_{1-2i}(i)-F_{1-2i}(-i))\right) \\ &= \frac{\pi}{50}\left(3\sqrt{10+2\sqrt{5}}+\sqrt{10-2\sqrt{5}}\right) \\ &\simeq 0.8648. \end{aligned}$$

Part (b) of the Theorem gives

$$-\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{1}{12} + \tfrac{1}{13} - \cdots = \sum_{\substack{n=1\\(\frac{n}{5})=-1}}^{\infty} \left(\frac{\chi_{1-2i}(n)}{i}\right) n^{-1}$$

(continues)

$$= -\frac{\pi}{50}\left(-\sqrt{10-2\sqrt{5}}(F_{1-2i}(1)-F_{1-2i}(-1)) \right.$$
$$\left. -\sqrt{10+2\sqrt{5}}(F_{1-2i}(i)-F_{1-2i}(-i))\right)$$
$$= \frac{\pi}{50}\left(-3\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}\right)$$
$$\simeq -0.2041\,.$$

Example 2. $p = 17$. Here $a = 1, b = 4, \omega = 1 + 4i$, and part (a) of the Theorem gives

$$1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}-\tfrac{1}{9}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{16}+\tfrac{1}{18}-\tfrac{1}{19}+\tfrac{1}{21}-\tfrac{1}{25}-\cdots$$

$$= \sum_{\substack{n=1\\ (\frac{n}{17})=1}}^{\infty}\left(\frac{n}{17}\right)_4 n^{-1}$$
$$= \frac{1}{34}\left(-\sqrt{34-2\sqrt{17}}(LS_{1+4i}(1)-LS_{1+4i}(-1)) \right.$$
$$\left. +\sqrt{34+2\sqrt{17}}(LS_{1+4i}(i)-LS_{1+4i}(-i))\right)$$
$$= \frac{1}{17}\left(\sqrt{34-2\sqrt{17}}\log\left(4\cos\frac{\pi}{17}\cos\frac{4\pi}{17}\right)\right.$$
$$\left. +\sqrt{34+2\sqrt{17}}\log\left(4\cos\frac{3\pi}{17}\cos\frac{5\pi}{17}\right)\right)$$
$$\simeq 0.5927\,.$$

Part (b) of the Theorem gives

$$-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{10}+\tfrac{1}{11}-\tfrac{1}{12}-\tfrac{1}{14}-\tfrac{1}{20}-\tfrac{1}{22}+\tfrac{1}{23}+\tfrac{1}{30}+\cdots$$

$$= \sum_{\substack{n=1\\ (\frac{n}{17})=-1}}^{\infty}\left(\frac{\chi_{1+4i}(n)}{i}\right) n^{-1}$$
$$= \frac{1}{34}\left(\sqrt{34+2\sqrt{17}}(LS_{1+4i}(1)-LS_{1+4i}(-1)) \right.$$
$$\left. +\sqrt{34-2\sqrt{17}}(LS_{1+4i}(i)-LS_{1+4i}(-i))\right)$$
$$= \frac{1}{17}\left(-\sqrt{34+2\sqrt{17}}\log\left(4\cos\frac{\pi}{17}\cos\frac{4\pi}{17}\right)\right.$$
$$\left. +\sqrt{34-2\sqrt{17}}\log\left(4\cos\frac{3\pi}{17}\cos\frac{5\pi}{17}\right)\right)$$
$$\simeq -0.1936\,.$$

Proof. The quartic Gauss sum $G(\chi_\omega)$ is defined by

$$G(\chi_\omega) = \sum_{r=1}^{p-1} \chi_\omega(r) \exp(2\pi i r/p). \tag{10}$$

The value of $G(\chi_\omega)$ was conjectured by Loxton [2] in 1978 and proved by Mathews [3] in 1979. Mathews showed that

$$G(\chi_\omega) = i^{(p+2b+3)/4} \left(\frac{2|b|}{|a|}\right) p^{1/4} \omega^{1/2}, \tag{11}$$

where the large brackets denote the usual Jacobi symbol and it is understood throughout that fractional powers take their principal values. We note that

$$G(\overline{\chi}_\omega) = (-1)^{(p-1)/4} \overline{G(\chi_\omega)},$$

so that by (11) we have

$$G(\overline{\chi}_\omega) = (-1)^{(p+3)/4} i^{(p+2b+3)/4} \left(\frac{2|b|}{|a|}\right) p^{1/4} \overline{\omega}^{1/2}. \tag{12}$$

For convenience we set

$$X = (-1)^{(p+2b+7)/8}, \qquad Y = \frac{b}{|b|}, \qquad Z = \left(\frac{2|b|}{|a|}\right), \tag{13}$$

$$\alpha = \sqrt{2p + 2a\sqrt{p}}, \qquad \beta = \sqrt{2p - 2a\sqrt{p}}, \tag{14}$$

so that

$$\alpha\beta = 2|b|p^{1/2}, \tag{15}$$

$$\omega^{1/2} - \overline{\omega}^{1/2} = iY\beta/p^{1/4}, \tag{16}$$

$$\omega^{1/2} + \overline{\omega}^{1/2} = \alpha/p^{1/4}. \tag{17}$$

For $m \not\equiv 0 \pmod{p}$, appealing to (11), (12), (13), (14), (16), (17), we obtain

$$\begin{aligned} &G(\chi_\omega)\overline{\chi}_\omega(m) + G(\overline{\chi}_\omega)\chi_\omega(m) \\ &\quad = i^{(p+2b+3)/4} Z p^{1/4} (\overline{\chi}_\omega(m)\omega^{1/2} + (-1)^{(p+3)/4} \chi_\omega(m)\overline{\omega}^{1/2}) \end{aligned}$$

$$= \begin{cases} \chi_\omega(m) XYZ\beta, & \text{if } \chi_\omega(m) = \pm 1,\ p \equiv 1 \pmod 8, \\ -i\chi_\omega(m) XZ\alpha, & \text{if } \chi_\omega(m) = \pm 1,\ p \equiv 5 \pmod 8, \\ i\chi_\omega(m) XZ\alpha, & \text{if } \chi_\omega(m) = \pm i,\ p \equiv 1 \pmod 8, \\ -\chi_\omega(m) XYZ\beta, & \text{if } \chi_\omega(m) = \pm i,\ p \equiv 5 \pmod 8, \end{cases} \tag{18}$$

and

$$\begin{aligned} &G(\chi_\omega)\overline{\chi}_\omega(m) - G(\overline{\chi}_\omega)\chi_\omega(m) \\ &\quad = i^{(p+2b+3)/4} Z p^{1/4} (\overline{\chi}_\omega(m)\omega^{1/2} - (-1)^{(p+3)/4} \chi_\omega(m)\overline{\omega}^{1/2}) \end{aligned}$$

(19)
$$= \begin{cases} -i\chi_\omega(m)XZ\alpha, & \text{if } \chi_\omega(m) = \pm 1,\ p \equiv 1 \pmod 8, \\ \chi_\omega(m)XYZ\beta, & \text{if } \chi_\omega(m) = \pm 1,\ p \equiv 5 \pmod 8, \\ -\chi_\omega(m)XYZ\beta, & \text{if } \chi_\omega(m) = \pm i,\ p \equiv 1 \pmod 8, \\ i\chi_\omega(m)XZ\alpha, & \text{if } \chi_\omega(m) = \pm i,\ p \equiv 5 \pmod 8. \end{cases}$$

Next, from the work of Dirichlet (see for example [1, Theorem 1.3, p. 137]), we have

(20)
$$L(1, \chi_\omega) = \begin{cases} -(G(\chi_\omega)/p)\sum_{m=1}^{p-1} \overline{\chi}_\omega(m)\log\left(\sin\frac{m\pi}{p}\right), & \text{if } p \equiv 1 \pmod 8, \\ (\pi i G(\chi_\omega)/p^2)\sum_{m=1}^{p-1} m\overline{\chi}_\omega(m), & \text{if } p \equiv 5 \pmod 8, \end{cases}$$

and

(21)
$$L(1, \overline{\chi}_\omega) = \begin{cases} -(G(\overline{\chi}_\omega)/p)\sum_{m=1}^{p-1} \chi_\omega(m)\log\left(\sin\frac{m\pi}{p}\right), & \text{if } p \equiv 1 \pmod 8, \\ (\pi i G(\overline{\chi}_\omega)/p^2)\sum_{m=1}^{p-1} m\chi_\omega(m), & \text{if } p \equiv 5 \pmod 8. \end{cases}$$

Hence, from (20) and (21), we obtain

$$\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} = \frac{1}{2}\sum_{n=1}^{\infty}(\chi_\omega(n) + \overline{\chi}_\omega(n))n^{-1}$$

$$= \frac{1}{2}(L(1, \chi_\omega) + L(1, \overline{\chi}_\omega))$$

$$= \begin{cases} \frac{-1}{2p}\sum_{m=1}^{p-1}(G(\chi_\omega)\overline{\chi}_\omega(m) + G(\overline{\chi}_\omega)\chi_\omega(m))\log\left(\sin\frac{m\pi}{p}\right), \\ \qquad\qquad \text{if } p \equiv 1 \pmod 8, \\ \frac{\pi i}{2p^2}\sum_{m=1}^{p-1}(G(\chi_\omega)\overline{\chi}_\omega(m) + G(\overline{\chi}_\omega)\chi_\omega(m))m, \\ \qquad\qquad \text{if } p \equiv 5 \pmod 8, \end{cases}$$

and

$$\sum_{\substack{n=1 \\ (\frac{n}{p})=-1}}^{\infty} \left(\frac{\chi_\omega(n)}{i}\right) n^{-1} = \frac{1}{2i}\sum_{n=1}^{\infty}(\chi_\omega(n) - \overline{\chi}_\omega(n))n^{-1}$$

$$= \frac{1}{2i}(L(1, \chi_\omega) - L(1, \overline{\chi}_\omega))$$

$$= \begin{cases} \frac{i}{2p}\sum_{m=1}^{p-1}(G(\chi_\omega)\overline{\chi}_\omega(m) - G(\overline{\chi}_\omega)\chi_\omega(m))\log(\sin\frac{m\pi}{p}), \\ \qquad\qquad \text{if } p \equiv 1 \pmod 8, \\ \frac{\pi}{2p^2}\sum_{m=1}^{p-1}(G(\chi_\omega)\overline{\chi}_\omega(m) - G(\overline{\chi}_\omega)\chi_\omega(m))m, \\ \qquad\qquad \text{if } p \equiv 5 \pmod 8. \end{cases}$$

The theorem now follows using (18) and (19).

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