REMARK ON AN ASSERTION OF CHOWLA

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Williams, K.S. 1979. Remark on an Assertion of Chowla.
K. norske Vidensk. Selsk. Skr. 1, 3-4.

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In [1] Chowla asserts without proof that if $p \equiv 1 \pmod 4$ is a prime satisfying $\left(\frac{6}{p}\right) = 1$, then

$$S = \sum_{n=0}^{p-1} \left(\frac{6n^4 + 11n^3 + 6n^2 + n}{p}\right) = 2a - 1,$$

where $p = a^2 + b^2$, $a \equiv -\left(\frac{2}{p}\right) \pmod 4$, $b \equiv 0 \pmod 2$. We give a simple proof of the evaluation $S = 2a - \left(\frac{6}{p}\right)$, for any prime $p \equiv 1 \pmod 4$. Let P denote a complete residue system modulo p and define w by $w^2 \equiv -1 \pmod p$. The mapping $n \to \frac{1}{-wn - 2}$ (taken modulo p) is a bijection from $P - \{2w\}$ to $P - \{0\}$, which sends $6n^4 + 11n^3 + 6n^2 + n \to \frac{wn(n^2+1)}{(wn+2)^4}$. Hence we have

$$S = \sum_{n=1}^{p-1} \left(\frac{6n^4 + 11n^3 + 6n^2 + n}{p}\right) = \sum_{\substack{n=0 \\ n \neq 2w}}^{p-1} \left(\frac{\frac{wn(n^2+1)}{(wn+2)^4}}{p}\right) = \left(\frac{w}{p}\right) \sum_{\substack{n=0 \\ n \neq 2w}}^{p-1} \left(\frac{n(n^2+1)}{p}\right),$$

that is

$$S = \left(\frac{2}{p}\right) \sum_{n=0}^{p-1} \left(\frac{n(n^2+1)}{p}\right) - \left(\frac{6}{p}\right),$$

as $\left(\frac{w}{p}\right) = \left(\frac{2}{p}\right)$ (since $2w \equiv (1+w)^2 \pmod p$). The sum $T = \sum_{n=0}^{p-1} \left(\frac{n(n^2+1)}{p}\right)$ is a Jacobsthal sum whose value is given by $T = 2a_1$, where $p = a_1^2 + b_1^2$, $a_1 \equiv -1 \pmod 4$, $b_1 \equiv 0 \pmod 2$ (see [2: (6.1), (6.2)]). Clearly $a = \left(\frac{2}{p}\right)a_1$ and the result follows.

REFERENCES

1. S. Chowla, On the class number of the function field $y^2 = f(x)$ over GF(p), Norske Vid. Selks. Forh., 39(1966), 86-88.

2. A.L. Whiteman, Cyclotomy and Jacobsthal sums, Amer. J. Math., 54(1952), 89-99.

Submitted to the Secretary General 15th October 1977 by Mr. S. Selberg.
Printed January 1979.

DKNVS/Universitetsforlaget 1979.