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SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

NOTE ON BREWER'S CHARACTER SUM

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ABSTRACT. A very short proof is given of the result

$$\sum_{x=0}^{p-1}\left(\frac{(x+2)(x^2-2)}{p}\right)=0,$$

if $p\equiv 5$ or 7 (mod 8).

If p is an odd prime, the Brewer sum B defined by

$$B=\sum_{x=0}^{p-1}\left(\frac{(x+2)(x^2-2)}{p}\right),$$

where $(\frac{\cdot}{p})$ is the Legendre symbol, has been evaluated by a number of authors (see for example, Brewer [1], Leonard and Williams [2], Rajwade [3], Whiteman [4]). The following very elementary proof that $B=0$ when $p\equiv 5$ or 7 (mod 8) appears to have been overlooked.

Mapping $x\to x-2 \pmod p$ we obtain

$$B=\sum_{x=0}^{p-1}\left(\frac{x^3-4x^2+2x}{p}\right)=\sum_{x=1}^{p-1}\left(\frac{x^3-4x^2+2x}{p}\right),$$

and defining $\bar{x}$, for $1\leqslant x\leqslant p-1$, by $x\bar{x}\equiv 1 \pmod p$, $1\leqslant \bar{x}\leqslant p-1$, we have

$$B=\sum_{x=1}^{p-1}\left(\frac{x-4+2\bar{x}}{p}\right).$$

In this sum we collect together those terms having the same value y $(0\leqslant y\leqslant p-1)$ for $x+2\bar{x}$. The number of solutions x with $1\leqslant x\leqslant p-1$ of $x+2\bar{x}\equiv y \pmod p$ is the same as the number of solutions x with $0\leqslant x\leqslant p-1$ of $x^2-yx+2\equiv 0 \pmod p$. This latter number is given by

$$1+\left(\frac{y^2-8}{p}\right).$$

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Since

$$\sum_{y=0}^{p-1} \left(\frac{y-4}{p} \right) = 0,$$

we obtain

$$B = \sum_{y=0}^{p-1} \left(\frac{(y-4)(y^2-8)}{p} \right).$$

The mapping $y \to -2y$ (mod 4) then gives

$$B = \left(\frac{-2}{p} \right) B,$$

so that $B = 0$, when $p \equiv 5$ or 7 (mod 8).

References

1. B. W. Brewer, *On certain character sums*, Trans. Amer. Math. Soc. **99** (1961), 241–245.

2. P. A. Leonard and K. S. Williams, *Jacobi sums and a theorem of Brewer*, Rocky Mountain J. Math. **5** (1975), 301–308; erratum, ibid. **6** (1976), 509.

3. A. R. Rajwade, *Certain classical congruences via elliptic curves*, J. London Math. Soc. **8** (1974), 60–62.

4. A. L. Whiteman, *A theorem of Brewer on character sums*, Duke Math. J. **30** (1963), 545–552.

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