

On the Imaginary Bicyclic Biquadratic Fields With Class-Number 2

By D. A. Buell, H. C. Williams and K. S. Williams*

Abstract. Assuming that the list of imaginary quadratic number fields of class-number 4 is complete, a determination is made of all imaginary bicyclic biquadratic number fields of class-number 2.

1. Introduction. Recently, Brown and Parry [2] have determined all imaginary bicyclic biquadratic fields K with class-number $H = 1$, using results of Stark [11], [12] and Montgomery and Weinberger [10] giving all imaginary quadratic fields with class-numbers 1 and 2. Assuming that the list of imaginary quadratic fields with class-number 4 given by the first author [3], [4] is complete, we determine all imaginary bicyclic biquadratic fields with class-number 2. Available evidence suggests that this list is indeed complete, for if there were an imaginary quadratic field with class-number 4 and discriminant D with $-D > 4 \times 10^6$, then, by Dirichlet's class-number formula, we would have

$$0 < L(1, \chi_D) < \frac{4\pi}{2000} < 0.0065.$$

However, the observed minimum of $L(1, \chi_D)$ for $0 < -D < 4 \times 10^6$ is 0.1988 (see [4]).

We let k_1, k_2 and k be the three quadratic subfields of K , where we take k to be the real field. We write h for the class-number of k and h_i for the class-number of k_i ($i = 1, 2$). The fundamental unit of k is denoted by ϵ . From the work of Herglotz [6] we have

$$H = \frac{\sigma h_1 h_2 h}{\lambda_0},$$

where $\sigma = 2$ or 1 according as $K = Q(\sqrt{-1}, \sqrt{-2})$ or not, and λ_0 is defined by $N_{K/k}(E) = \epsilon^{\lambda_0}$, where E denotes a fundamental unit of K . Herglotz [6] has noted that $\lambda_0 = 1$ or 2, and Brown and Parry [2] have remarked that if the norm of ϵ is -1 , then $\lambda_0 = 2$. If $K = Q(\sqrt{-1}, \sqrt{-2})$, then $h_1 = h_2 = h = 1$, $\sigma = 2$, $\epsilon = 1 + \sqrt{2}$, $\lambda_0 = 2$; and we have $H = 1$. This field can thus be omitted from all future considerations, and we take $\sigma = 1$ from this point on.

The determination of those fields with $H = 2$ falls naturally into 4 cases:

I. $h_1 = h_2 = 1$,

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- II. $h_1 = 1, h_2 = 2,$
- III. $h_1 = 1, h_2 = 4,$
- IV. $h_1 = h_2 = 2.$

For $H = 2$ in case I, we must have $h = 2, \lambda_0 = 1,$ or $h = 4, \lambda_0 = 2;$ in case II, $h = 1, \lambda_0 = 1,$ or $h = 2, \lambda_0 = 2;$ in case III, $h = 1, \lambda_0 = 2;$ and in case IV, $h = 1, \lambda_0 = 2.$

Stark [11] has shown that the only imaginary quadratic fields with class-number 1 are the nine fields

$$Q(\sqrt{-n}): n = 1, 2, 3, 7, 11, 19, 43, 67, 163.$$

In case I, direct verification, using tables of class-numbers of real quadratic fields, shows that neither $h = 2$ nor $h = 4$ ever occurs; case I yields no fields K with $H = 2.$

2. Determination of $\lambda_0.$ In this section we develop a criterion for determining the value of λ_0 in the case when $N(\epsilon) = +1.$ Our first lemma, giving the roots of unity in $K,$ is well known.

LEMMA 1. *Let m and n be positive squarefree integers with $m > 1.$ Let ϵ be the fundamental unit of $Q(\sqrt{m}).$*

(a) *If $\sqrt{-1} \in Q(\sqrt{m}, \sqrt{-n}),$ then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are $\pm 1, \pm \sqrt{-1},$ with the additional roots $\frac{1}{2}(\pm \sqrt{2} \pm \sqrt{-2}),$ if $m = 2,$ and $\frac{1}{2}(\pm 1 \pm \sqrt{-3}), \frac{1}{2}(\pm \sqrt{3} \pm \sqrt{-1}),$ if $m = 3.$*

(b) *If $\sqrt{-2} \in Q(\sqrt{m}, \sqrt{-n}),$ then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are $\pm 1,$ with the additional roots $\pm \sqrt{-1}, \frac{1}{2}(\pm \sqrt{2} \pm \sqrt{-2}),$ if $m = 2,$ and $\frac{1}{2}(\pm 1 \pm \sqrt{-3}),$ if $m = 6.$*

(c) *If $\sqrt{-3} \in Q(\sqrt{m}, \sqrt{-n}),$ then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are $\pm 1, \frac{1}{2}(\pm 1 \pm \sqrt{-3}),$ with the additional roots $\pm \sqrt{-1}, \frac{1}{2}(\pm \sqrt{3} \pm \sqrt{-1}),$ if $m = 3.$*

(d) *If none of $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n}),$ then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are $\pm 1.$*

Our next lemma occurs in the work of Kuroda [9] and of Kubota [8].

LEMMA 2. *Suppose $N(\epsilon) = +1.$*

(a) *If either $\sqrt{-1}$ or $\sqrt{-2}$ belongs to $Q(\sqrt{m}, \sqrt{-n}),$ then ϵ is a fundamental unit of $Q(\sqrt{m}, \sqrt{-n}),$ equivalently $\lambda_0 = 2,$ if and only if there do NOT exist rational integers A and B such that*

$$2\epsilon = (A + B\sqrt{m})^2.$$

(This condition is equivalent to the condition that $2\epsilon = \mu^2,$ for some algebraic integer μ of $Q(\sqrt{m}).$)

(b) *If $\sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n}),$ and $Q(\sqrt{m}, \sqrt{-n}) \neq Q(\sqrt{3}, \sqrt{-1}),$ then ϵ is a fundamental unit of $Q(\sqrt{m}, \sqrt{-n}),$ equivalently $\lambda_0 = 2,$ if and only if there does NOT exist an integer μ of $Q(\sqrt{m})$ such that*

$$3\epsilon = \mu^2.$$

(c) *If none of $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n}),$ then ϵ is a*

fundamental unit of $Q(\sqrt{m}, \sqrt{-n})$, equivalently $\lambda_0 = 2$, if and only if there does NOT exist an integer μ of $Q(\sqrt{m})$ such that

$$n\epsilon = \mu^2.$$

Lemma 2 provides us with a criterion for determining λ_0 . We next develop an effective method for applying it.

We define rational integers x and y by setting

$$\epsilon = \begin{cases} \frac{1}{2}(x + y\sqrt{m}), & x \equiv y \pmod{2}, \text{ if } m \equiv 1 \pmod{4}, \\ x + y\sqrt{m}, & \text{ if } m \equiv 2, 3 \pmod{4}. \end{cases}$$

Further, in order to simplify the statements of the following lemmas, we shall use the term “representable” to mean “representable as the square of an integer of $Q(\sqrt{m})$ ”.

For the next two lemmas, see [1, Theorem 3.1].

LEMMA 3. Let r be a squarefree rational integer such that re is representable. Then

(a) if $m \equiv 1$ or $2 \pmod{4}$, or $m \equiv 3 \pmod{4}$ and y even, we have $r|m$, and $m\epsilon/r$ representable, and

(b) if $m \equiv 3 \pmod{4}$ and y odd, we have r even, $r/2|m$, and $4m\epsilon/r$ representable.

LEMMA 4. Let r and s be squarefree rational integers such that re and se are both representable. By Lemma 3, in case (a) we have $r|m$ and $s|m$, and, in case (b) r and s are even and $r/2|m$ and $s/2|m$. Then in case (a) either $r = s$ or $rs = m$, and in case (b) either $r = s$ or $rs = 4m$.

Lemmas 3 and 4 imply that if there are any squarefree integers r such that re is representable, then there are exactly two such integers, both of which must be factors of m or $2m$. We now specify these factors.

Let the principal cycle of binary quadratic forms of discriminant d ($d = m$ if $m \equiv 1 \pmod{4}$, and $d = 4m$ if $m \equiv 2, 3 \pmod{4}$) be $(a_0, b_0, -a_1) \sim \dots \sim (\pm a_i, b_i, \mp a_{i+1}) \sim \dots \sim (-a_{2k-1}, b_{2k-1}, a_{2k})$, where $a_0 = a_{2k} = 1$. Halfway through this cycle we find

$$(\mp a_{k-1}, b_{k-1}, \pm a_k) \sim (\pm a_k, b_k, \mp a_{k-1}),$$

with $a_k | b_k, b_{k-1} = b_k, a_{k-1} = a_{k+1}$.

The form $(\pm a_k, b_k, \mp a_{k+1})$ is ambiguous, and the second half of the cycle contains the opposites of the forms of the first half in reverse order.

LEMMA 5. The two squarefree integers r_1 and r_2 such that $r_1\epsilon$ and $r_2\epsilon$ are representable are, in case (a), a_k and m/a_k and, in case (b), a_k and $4m/a_k$.

Proof. Defining recursively the integers α_i of $Q(\sqrt{m})$ by

$$\alpha_0 = u_0 + v_0\sqrt{m} = \frac{b_0 + \sqrt{m}}{2a_1},$$

$$\alpha_i = u_i + v_i\sqrt{m} = \alpha_{i-1} \left(\frac{b_i + \sqrt{m}}{2a_{i+1}} \right) \quad (i \geq 1),$$

we have the following well-known result [7]:

$$\epsilon = \alpha_{2k-1} = \left(\frac{b_0 + \sqrt{m}}{2a_1}\right) \left(\frac{b_1 + \sqrt{m}}{2a_2}\right) \cdots \left(\frac{b_{2k-1} + \sqrt{m}}{2a_{2k}}\right).$$

But, as $b_{k+i} = b_{k-1-i}$, $a_{k+i} = a_{k-i}$, $a_{2k} = 1$, the above product is

$$\epsilon = \left(\frac{b_0 + \sqrt{m}}{2}\right)^2 \left(\frac{b_1 + \sqrt{m}}{2a_1}\right)^2 \cdots \left(\frac{b_{k-1} + \sqrt{m}}{2a_{k-1}}\right)^2 \frac{1}{a_k}.$$

Thus $a_k\epsilon$ is the square of an integer of $Q(\sqrt{m})$. Since $a_k \mid b_k$, $a_k \mid d$, and thus a_k is squarefree.

The following useful lemma is a consequence of Lemma 3.

LEMMA 6. *If n_1 and n_2 are distinct positive squarefree integers, and $g = (n_1, n_2) > 1$, then $\lambda_0 = 2$ for $Q(\sqrt{-n_1}, \sqrt{-n_2})$.*

Proof. Let $m = n_1n_2/g^2$, so that m is a squarefree integer > 1 . The real quadratic subfield of $Q(\sqrt{-n_1}, \sqrt{-n_2})$ is $Q(\sqrt{m})$. As $g > 1$, we have $n_1 \nmid m$ and $n_2 \nmid m$ so that by Lemma 3, $n_1\epsilon$ and $n_2\epsilon$ are not representable. Hence the fundamental unit ϵ of $Q(\sqrt{m})$ is not a fundamental unit of $Q(\sqrt{-n_1}, \sqrt{-n_2})$, and we must have $\lambda_0 = 2$.

3. Consideration of Case II. In this case we consider those fields K for which $h_1 = 1$ and $h_2 = 2$. Montgomery and Weinberger [10] and Stark [12] have shown that there are exactly 18 imaginary quadratic fields with class-number 2, namely,

$$Q(\sqrt{-n}): n = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$$

Thus, we have $9 \times 18 = 162$ fields to consider. In order to have $H = 2$ we must have $h = \lambda_0 = 1$ or $h = \lambda_0 = 2$. Of these, nineteen have $h = 1$, of which four have $\lambda_0 = 1$

$$(3.1) \quad Q(\sqrt{-1}, \sqrt{-6}), \quad Q(\sqrt{-1}, \sqrt{-22}), \quad Q(\sqrt{-2}, \sqrt{-6}), \quad Q(\sqrt{-2}, \sqrt{-22}).$$

These fields have $H = 2$. The other 15 fields appear in the list given by Brown and Parry [2] and have $H = 1$. It should be noted that for each of these $N(\epsilon) = -1$. There were no fields with $h = 1$, $N(\epsilon) = 1$, $\lambda_0 = 2$, $H = 1$. Of the remaining 143 fields, eight have $h > 2$ and so can be excluded. Five of the remaining fields have $N(\epsilon) = -1$, $\lambda_0 = 2$, hence $H = 2$. They are

$$(3.2) \quad \begin{aligned} &Q(\sqrt{-1}, \sqrt{-10}), \quad Q(\sqrt{-1}, \sqrt{-58}), \quad Q(\sqrt{-2}, \sqrt{-5}), \\ &Q(\sqrt{-2}, \sqrt{-13}), \quad Q(\sqrt{-2}, \sqrt{-37}). \end{aligned}$$

Finally, using the criterion given in Lemma 5 we determine whether $\lambda_0 = 1$ or 2 for the remaining 130 fields. We find that there are exactly 85 with $\lambda_0 = 2$, so that $H = 2$:

$$\begin{aligned}
 (3.3) \quad & Q(\sqrt{-1}, \sqrt{-n}), & n = 15, 35, 91, 115, 403, \\
 & Q(\sqrt{-2}, \sqrt{-n}), & n = 15, 35, 91, 115, 235, 403, 427, \\
 & Q(\sqrt{-3}, \sqrt{-n}), & n = 5, 10, 22, 35, 58, 115, 187, 235, \\
 & Q(\sqrt{-7}, \sqrt{-n}), & n = 5, 10, 13, 15, 51, 115, 123, 187, 235, 267, 403, \\
 & Q(\sqrt{-11}, \sqrt{-n}), & n = 6, 13, 51, 58, 91, 123, 403, 427, \\
 & Q(\sqrt{-19}, \sqrt{-n}), & n = 6, 13, 22, 37, 58, 91, 123, 403, \\
 & Q(\sqrt{-43}, \sqrt{-n}), & n = 5, 6, 10, 15, 22, 35, 37, 58, 115, 235, 267, 427, \\
 & Q(\sqrt{-67}, \sqrt{-n}), & n = 5, 6, 10, 13, 15, 22, 35, 123, 235, 403, \\
 & Q(\sqrt{-163}, \sqrt{-n}), & n = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 187, \\
 & & 235, 267, 403.
 \end{aligned}$$

Case II yields 94 fields with $H = 2$.

4. Consideration of Case III. In this case we consider those fields K with $h_1 = 1$, $h_2 = 4$. In order for $H = 2$ to hold we must have $h = 1$, $\lambda_0 = 2$. The class-number of $Q(\sqrt{-n})$ is 4 for the 54 values of n shown in Table 1; it is conjectured that this list is complete. We thus have $9 \times 54 = 486$ fields to consider.

We begin by considering the 54 fields K containing $Q(\sqrt{-1})$ as a subfield. Direct examination of tables and genus considerations show that $h \geq 2$ (so that $H > 2$) except in 13 cases. Of these, four have $N(\epsilon) = -1$, so that $\lambda_0 = 2$, $H = 2$:

$$(4.1) \quad Q(\sqrt{-1}, \sqrt{-17}), \quad Q(\sqrt{-1}, \sqrt{-73}), \quad Q(\sqrt{-1}, \sqrt{-97}), \quad Q(\sqrt{-1}, \sqrt{-193}).$$

The condition that 2ϵ be representable is satisfied for $n = 14$ and 46 , so $\lambda_0 = 1$, $H = 4$. The discriminant of the real quadratic subfield in the remaining 7 cases is odd; therefore, 2ϵ cannot be represented.

Thus, the seven fields

$$(4.2) \quad Q(\sqrt{-1}, \sqrt{-21}), \quad Q(\sqrt{-1}, \sqrt{-33}), \quad Q(\sqrt{-1}, \sqrt{-57}), \quad Q(\sqrt{-1}, \sqrt{-93}), \\ Q(\sqrt{-1}, \sqrt{-133}), \quad Q(\sqrt{-1}, \sqrt{-177}), \quad Q(\sqrt{-1}, \sqrt{-253}),$$

all have $H = 2$.

Next we consider the fields K having $Q(\sqrt{-2})$ as a subfield. Again, direct examination of tables and appeal to genus considerations show that $h \geq 2$ (so that $H > 2$) except in six cases. Of these six fields, two have $N(\epsilon) = -1$, so that $\lambda_0 = 2$, $H = 2$, namely the fields

$$(4.3) \quad Q(\sqrt{-2}, \sqrt{-34}), \quad Q(\sqrt{-2}, \sqrt{-82}).$$

Of the remaining four fields $Q(\sqrt{-2}, \sqrt{-n})$ ($n = 14, 42, 46, 142$) the condition that 2ϵ be representable is satisfied for the three values $n = 14, 46, 142$. Thus, just the field

TABLE 1

Imaginary quadratic fields of discriminant d with class-number 4					
Odd discriminants	O_1		39	323	1027
		$-d = pq$	55	355	1227
			155	667	1243
			203	723	1387
			219	763	1411
			259	955	1507
			291	1003	1555
			O_2		195
			$-d = pqr$	435	715
				483	795
				555	1435
				595	
	Even discriminants	E_1	$-d = 4p$	4.17	4.97
		4.73		4.193	
E_2		$-d = 8p$	8.7	8.41	
			8.17	8.71	
			8.23		
E_3		$-d = 4pq$	4.21	4.93	
			4.33	4.133	
			4.57	4.177	
			4.85	4.253	
E_4		$-d = 8pq$	8.15	8.51	
			8.21	8.65	
			8.35	8.95	
			8.39		

(4.4) $Q(\sqrt{-2}, \sqrt{-42})$

has $H = 2$.

Finally, it remains to consider the $7 \times 54 = 378$ fields K which do not possess either $Q(\sqrt{-1})$ or $Q(\sqrt{-2})$ as a subfield. These are all of the form $Q(\sqrt{-p}, \sqrt{-n})$, where

p is a prime $\equiv 3 \pmod{4}$ (indeed $p = 3, 7, 11, 19, 43, 67, 163$) and n is one of the integers listed in Table 1.

We begin by looking at those fields for which $p \nmid n$. Genus considerations show that in cases O_1, E_1, E_2 (see Table 1) we have $2 \mid h$, and in cases O_2, E_3, E_4 we have $4 \mid h$, so that certainly $h \neq 1$. In many cases this can also be directly verified from tables. Thus, we need only consider those fields for which $p \mid n$. However, when $p \mid n$ we know from Lemma 6 that $\lambda_0 = 2$, giving $H = 2h$. However, the only such fields with $h = 1$ are given by:

$$\begin{aligned}
 p = 3, \quad n = 21, 33, 39, 42, 57, 93, 177, 219, 291, 483, 627, 723, 1227, \\
 p = 7, \quad n = 14, 21, 42, 133, 203, 259, 483, 763, \\
 (4.5) \quad p = 11, \quad n = 33, 55, 253, 627, 1243, 1507, \\
 p = 19, \quad n = 57, 133, 323, 627, 1387.
 \end{aligned}$$

(There are none corresponding to $p = 43, 67, 163$.) Those listed in (4.5) all have $H = 2$. Case III yields 46 fields with $H = 2$.

5. Consideration of Case IV. In this case we consider those fields $K = Q(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_1 = h_2 = 2$. There are $17 + 16 + \dots + 2 + 1 = 153$ cases to consider. For $H = 2$ to hold, we must have $h = 1, \lambda_0 = 2$. If $(n_1, n_2) = 1$, genus considerations show that $2 \mid h$. If $(n_1, n_2) = 2$, genus considerations show that $2 \mid h$ except when $n_1 = 2p_1, n_2 = 2p_2, p_1, p_2$ distinct primes with $p_1 \equiv p_2 \pmod{4}$. There are just 2 fields to be considered individually, namely, $Q(\sqrt{-6}, \sqrt{-22})$ and $Q(\sqrt{-10}, \sqrt{-58})$. The second of these is ruled out as $h = 4$. The first, on the other hand, has $h = 1, N(\epsilon) = 1$, and Lemma 6 shows that $\lambda_0 = 2$, so that $H = 2$ for

$$(5.1) \quad Q(\sqrt{-6}, \sqrt{-22}).$$

The only fields left to be considered are those for which $(n_1, n_2) > 2$. We need only consider those 19 fields for which $h = 1$, namely,

$$\begin{aligned}
 n_1 = 5, \quad n_2 = 10, 15, 35, 115, 235, \\
 n_1 = 10, \quad n_2 = 15, 35, 115, 235, \\
 n_1 = 13, \quad n_2 = 91, 403, \\
 n_1 = 15, \quad n_2 = 35, 115, 235, \\
 n_1 = 35, \quad n_2 = 115, 235, \\
 n_1 = 51, \quad n_2 = 187, \\
 n_1 = 91, \quad n_2 = 403, \\
 n_1 = 115, \quad n_2 = 235.
 \end{aligned}$$

The first of these, $Q(\sqrt{-5}, \sqrt{-10})$, has $N(\epsilon) = -1, \lambda_0 = 2, H = 2$. The remaining eighteen have $N(\epsilon) = +1$, and by Lemma 6, $\lambda_0 = 2, H = 2$. Thus all 19 fields listed in (5.2) have $H = 2$. Case IV yields 20 fields with $H = 2$.

6. Main Theorem and Concluding Remarks.

THEOREM. *If the list of imaginary quadratic fields with class-number 4 given in Table 1 is complete, then there are exactly 160 imaginary bicyclic biquadratic fields with class-number 2. These are listed in (3.1), (3.2), (3.3), (4.1), (4.2), (4.3), (4.4), (4.5), (5.1), (5.2).*

The following tables were used in the proof of the Theorem:

(i) that of E. L. Ince [7] giving the class-number, the cycles of binary quadratic forms, the fundamental unit ϵ and its norm, and the representation of a suitable multiple of ϵ as the square of an integer of $Q(\sqrt{m})$, for fields $Q(\sqrt{m})$ of radicand m , $2 \leq m \leq 2025$;

(ii) that of M. N. and G. Gras [5] giving the class-number and the norm of ϵ of $Q(\sqrt{m})$ for radicands m , $2 \leq m < 10^4$;

(iii) that of H. C. Williams and J. Broere [13] giving (among other things) the class-number of $Q(\sqrt{m})$ for radicands m , $2 \leq m < 1.5 \times 10^5$;

(iv) an unpublished table of A. O. L. Atkin giving the class-number, the cycle of forms, and the norm of ϵ , for fields $Q(\sqrt{d})$ of (odd and even) discriminant d , $4 \leq d < 4000$;

(v) an unpublished table of A. O. L. Atkin giving the class-number and the norm of ϵ , for fields $Q(\sqrt{d})$ of odd discriminant d , $5 \leq d < 900000$;

(vi) that of D. A. Buell [3] giving the class-numbers and class groups of the imaginary quadratic fields $Q(\sqrt{d})$ of discriminant d , $0 < -d < 4000000$.

Where necessary, further machine computations were carried out by the first and second authors independently on different computers. No numerical results in this paper are presented on the strength of a single source; needed information contained in only one of the tables (this applied particularly in determining λ_0) was verified by a separate computation. The determination of λ_0 was made by expanding \sqrt{m} as a continued fraction to find the integer n such that $n\epsilon_m$ is representable in $Q(\sqrt{m})$. We remark that no discrepancies were found among the various tables.

Department of Mathematics
Carleton University
Ottawa, Ontario K1S 5B6, Canada

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba R3T 2N2, Canada

Department of Mathematics
Carleton University
Ottawa, Ontario K1S 5B6, Canada

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