

NOTE ON INTEGERS REPRESENTABLE BY BINARY QUADRATIC FORMS

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Let B be the set of positive integers prime to d which are representable by some primitive, positive, integral binary quadratic form of discriminant d . It is the purpose of this note to show that the following asymptotic estimate for the number of integers in B less than or equal to x can be proved using only elementary arguments:

$$(1) \quad B(x) = \sum_{m \leq x, m \in B} 1 = c_1 \frac{x}{(\log x)^{1/2}} \{1 + \mathcal{O}((\log \log x)^{-1})\} \quad (x \rightarrow \infty),$$

where c_1 is the positive constant given in (17) below. (Using the deeper methods of complex analysis James [2] has proved this result with the error term $\mathcal{O}((\log x)^{-1/2})$ replacing $\mathcal{O}((\log \log x)^{-1})$. Heupel [1] using a transcendental method as in James [2] improved this to $\mathcal{O}((\log x)^{-1})$.)

We follow closely the ideas of Rieger [6] and set $M = \{n:p \mid n \Rightarrow (d/p) = 1\}$ and $N = \{n:p \mid n \Rightarrow (d/p) = -1\}$. From the work of Selberg [7] we have

$$(2) \quad \sum_{p \leq x, (d/p) = 1} \frac{\log p}{p} = \frac{1}{2} \log x + \mathcal{O}(1).$$

Appealing to (2) and a result of Rieger [5] we obtain

$$(3) \quad m(x) = \sum_{m \leq x, m \in M} \frac{1}{m} = \frac{e^{-c/2}}{\Gamma(\frac{3}{2})} \prod_{\substack{p \leq x \\ (d/p) = 1}} \left(1 - \frac{1}{p}\right)^{-1} \{1 + \mathcal{O}((\log \log x)^{-1})\},$$

where

$$c = - \int_0^\infty e^{-t} \log t \, dt.$$

Next we recall Merten's theorem ([3], p. 139)

$$(4) \quad \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = e^{-c} (\log x)^{-1} + \mathcal{O}((\log x)^{-2}),$$

and a result of Landau ([3], §109)

$$(5) \quad \prod_{p \leq x} \left(1 - \frac{(d/p)}{p}\right) = \frac{1}{L(1)} + \mathcal{O}((\log x)^{-1}),$$

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where

$$L(1) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-1} > 0.$$

Using (4) and (5) and an argument of Uchiyama [8] we obtain

$$(6) \quad \prod_{p \leq x, (d/p)=1} \left(1 - \frac{1}{p}\right) = e^{-c/2} \prod_{p|d} \left(1 - \frac{1}{p}\right)^{-1/2} \prod_{(d/p)=-1} \left(1 - \frac{1}{p}\right)^{-1/2} L(1)^{-1/2} (\log x)^{-1/2} + \mathcal{O}((\log x)^{-3/2}).$$

Putting (6) into (3) we obtain

$$(7) \quad m(x) = c_2 (\log x)^{1/2} (1 + \mathcal{O}((\log \log x)^{-1})),$$

where

$$(8) \quad c_2 = \frac{2}{\sqrt{\pi}} \prod_{p|d} \left(1 - \frac{1}{p}\right)^{1/2} \prod_{(d/p)=-1} \left(1 - \frac{1}{p}\right)^{1/2} L(1)^{1/2}.$$

Selberg [7] (see also Wirsing [9]) has shown that if $(l, k)=1$ the following form of the prime number theorem for arithmetic progressions can be proved by elementary means

$$(9) \quad \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \log p = \frac{x}{\phi(k)} + \mathcal{O}\left(\frac{x}{(\log x)^\alpha}\right),$$

where α is a positive constant. Since $p \in M$ if and only if $(d/p)=1$ and this latter condition means that p lies in one of $(\frac{1}{2})\phi(4 \prod_{p|d} p)$ residue classes mod $4 \prod_{p|d} p$ which are prime to $4 \prod_{p|d} p$, we have using (9)

$$(10) \quad \sum_{p \leq x, p \in M} \log p = \frac{x}{2} (1 + \mathcal{O}((\log \log x)^{-1})).$$

Now using (7) and (10) in an argument of Rieger [6] (p. 199) we obtain

$$(11) \quad M(x) = \sum_{m \leq x, m \in M} 1 = c_2 \frac{x}{2(\log x)^{1/2}} (1 + \mathcal{O}((\log \log x)^{-1})).$$

Finally, noting that if k is prime to d , then k is represented by some primitive positive integral binary quadratic form of discriminant d , if and only if $k=mn^2$, where $m \in M, n \in N$, we have

$$(12) \quad B(x) = \sum_{\substack{mn^2 \leq x \\ m \in M, n \in N}} 1 = \sum_{n \leq \sqrt{x}, n \in N} M(xn^{-2}),$$

and, since $M(t) \leq t$, (12) gives

$$(13) \quad B(x) = \sum_{n \leq \log x, n \in N} M(xn^{-2}) + \mathcal{O}(x(\log x)^{-1}).$$

From (11) we have

$$(14) \quad M(xn^{-2}) = c_2 \frac{x}{2n^2(\log x)^{1/2}} (1 + \mathcal{O}((\log \log x)^{-1})) \quad (1 \leq n < \log x),$$

and as

$$(15) \quad \sum_{n \leq y, n \in N} n^{-2} = \prod_{(d/p)=-1} (1 - p^{-2})^{-1} + \mathcal{O}(y^{-1}) \quad (y \rightarrow \infty),$$

from (13), (14) and (15) we obtain (1) with

$$(16) \quad c_1 = \frac{1}{2} c_2 \prod_{(d/p)=-1} (1 - p^{-2})^{-1}.$$

(16) together with (8) gives

$$(17) \quad c_1 = \left(\frac{L(1)}{\pi} \right)^{1/2} \prod_{(d/p)=-1} (1 - p^{-2})^{-1/2} \prod_{p|d} (1 - p^{-1})^{1/2}.$$

(1) can be extended to all positive integers k by following Pall's argument in [4].

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