I wish to thank the referee for his comments.

References

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A SIMPLE NORM INEQUALITY

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The following useful norm inequality seems to be new. For nonzero vectors x and y in a normed linear space,

$$(*)\qquad \|x - y\| \geqq \tfrac{1}{4}(\|x\| + \|y\|)\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|.$$

It yields, for example, a simple proof that if X is a Banach Space, and M_1, M_2 are closed independent subspaces of X, then if there exists a $d>0$ such that $\|x_1-x_2\|\geqq d$, whenever $x_1\in M_1$, $x_2\in M_2$ and $\|x_1\|=\|x_2\|=1$, then $M_1\oplus M_2$ is closed.

Proof of Inequality:

$$(1)\qquad \begin{aligned} \|x\|\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &\leqq \|x\|\left\|\frac{x}{\|x\|} - \frac{y}{\|x\|}\right\| + \|x\|\left\|\frac{y}{\|x\|} - \frac{y}{\|y\|}\right\| \leqq \|x - y\| \\ &+ \frac{\|(\|y\| - \|x\|)y\|}{\|y\|} \leqq \|x - y\| + |\,\|y\| - \|x\|\,| \leqq 2\|x - y\|. \end{aligned}$$

Similarly by adding and subtracting $x/\|y\|$ we have:

$$(2)\qquad \|y\|\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leqq 2\|x - y\|.$$

The result now follows by adding (1) and (2).

The authors originally conjectured the inequality (*) in the stronger form

$$\|x - y\| \geqq \tfrac{1}{2}(\|x\| + \|y\|)\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|,$$

but this is not true in general. Consider, for example, the normed linear space consisting of ordered pairs of real numbers, say (x_1, x_2), with norm equal to $|x_1|+|x_2|$. Take $x=(1, \epsilon)$ and $y=(1, 0)$, where ϵ is positive and small. Then the inequality (*) becomes

$$\epsilon \geqq \frac{4(1 + \epsilon/2)}{4(1 + \epsilon)}\,\epsilon.$$

Thus it is obvious that the constant $\frac{1}{4}$ is the best possible. We have been unable, however, to answer this question: Does equality ever hold if $\|x\|+\|y\|\neq 0$?

We show that in a complex inner-product space with $\sqrt{\{(x, x)\}}$ as norm the inequality does hold with $\frac{1}{2}$ instead of $\frac{1}{4}$. For then

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 = \left(\frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\right)$$

$$= 2 - 2 \operatorname{Re}\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)$$

$$= \frac{1}{\|x\| \|y\|} [2\|x\| \|y\| - 2 \operatorname{Re}(x, y)]$$

$$= \frac{1}{\|x\| \|y\|} [2\|x\| \|y\| - (\|x\|^2 + \|y\|^2 - \|x - y\|^2)]$$

$$= \frac{[\|x - y\|^2 - (\|x\| - \|y\|)^2]}{\|x\| \|y\|}.$$

Hence

$$\|x - y\|^2 - \left(\frac{\|x\| + \|y\|}{2}\right)^2 \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2$$

$$= \frac{[\|x\| - \|y\|]^2}{4\|x\| \|y\|} [(\|x\| + \|y\|)^2 - \|x - y\|^2] \geqq 0.$$

One further question, is the converse true—Does the inequality with the constant $\frac{1}{2}$ imply that X is an inner-product space?

ON A NOTE BY Q. G. MOHAMMAD

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Theorem 2 in the Note [1] is based on the fact that all roots of $p(z) = z^n + a_q z^{n-q} + \cdots + a_n$ are also roots of $P(z) = (z^q - a_q)p(z) = z^{n+q} + b_{q+1}z^{n-1} + \cdots + b_{n+q}$, and on an estimate of the influence of the gap-size q on bounds for the roots. Dr. Mohammad's argument on top of p. 903 leads to an improvement of my statements in the Note [2] in the same issue of the MONTHLY. Especially, the bound 2 is now replaced by a bound 1.

Let be $m(k, M)$ the unique root >1 of

$$x^k - x^{k-1} - M = 0$$

defined for positive integer k and positive M. In

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0 \tag{1}$$

let be a_q the first nonzero coefficient, a_p the coefficient greatest in absolute value. *All roots of* (1) *lie in the circle* $|z| < m(q, |a_p|)$.