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3029. A generalisation of Cardan's solution of the cubic

I first recall Cardan's solution, presenting it in a different form from the usual, which lends itself to generalization.

Consider $x^3 - \lambda x + a = 0$.

Let the roots of this equation be α and β so

$$\alpha + \beta = \lambda \qquad \alpha\beta = a$$

Now $\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \lambda^3 - 3a\lambda$
and $\alpha^3\beta^3 \equiv (\alpha\beta)^3 = a^3$

Thus the equation with roots α^3 and β^3 is

$$y^2 - (\lambda^3 - 3a\lambda)y + a^3 = 0$$

Let $\lambda^3 - 3a\lambda = b$ i.e. $\lambda^3 - 3a\lambda - b = 0$
so $y^2 - by + a^3 = 0$

$$\therefore \quad y = \tfrac{1}{2}\{b \pm \sqrt{b^2 - 4a^3}\}$$

$$\therefore \quad \alpha^3 = \tfrac{1}{2}\{b + \sqrt{b^2 - 4a^3}\} \quad \text{and} \quad \beta^3 = \tfrac{1}{2}\{b - \sqrt{b^2 - 4a^3}\}$$

$$\therefore \quad \lambda = \alpha + \beta = [\tfrac{1}{2}\{b + \sqrt{b^2 - 4a^3}\}]^{1/3} + [\tfrac{1}{2}\{b - \sqrt{b^2 - 4a^3}\}]^{1/3}$$

which is Cardan's solution of the cubic,

$$\lambda^3 - 3a\lambda - b = 0$$

Every cubic equation $Ax^3 + Bx^2 + Cx + D = 0$ can be put into this form $\lambda^3 - 3a\lambda - b = 0$ by the transformation $x = \lambda - \dfrac{B}{3A}$ yielding $a = \dfrac{B^2}{9A^2} - \dfrac{C}{3A}$ and $b = \dfrac{BC}{3A^2} - \dfrac{2B^3}{27A^3} - \dfrac{D}{A}$, so the solution is perfectly general.

Now we proceed to the general case. Consider

$$x^2 - \lambda x + a = 0$$

Let the roots of this equation be α and β so $\alpha + \beta = \lambda$ and $\alpha\beta = a$.
Thus $(1 - \alpha x)(1 - \beta x) \equiv 1 - \lambda x + ax^2$

$$\begin{aligned}\therefore \quad \log(1 - \lambda x + ax^2) &= \log(1 - \alpha x)(1 - \beta x) \\ &= \log(1 - \alpha x) + \log(1 - \beta x) \\ &= -\sum_{m=1}^{\infty} \frac{\alpha^m x^m}{m} - \sum_{m=1}^{\infty} \frac{\beta^m x^m}{m} \qquad (1) \\ &= -\sum_{m=1}^{\infty} \frac{(\alpha^m + \beta^m)}{m} x^m \qquad (2)\end{aligned}$$

$$\begin{aligned}\text{Now } \log(1 - \lambda x + ax^2) &= \log(1 - x(\lambda - ax)) \\ &= -\sum_{r=1}^{\infty} \frac{x^r(\lambda - ax)^r}{r} \qquad (3) \\ &= -\sum_{r=1}^{\infty} \frac{x^r}{r} \sum_{s=0}^{s=r} (-1)^s \binom{r}{s} \lambda^{r-s} a^s x^s \\ &= -\sum_{r=1}^{\infty} \sum_{s=0}^{s=r} \frac{(-1)^s}{r} \binom{r}{s} \lambda^{r-s} a^s x^{r+s}\end{aligned}$$

(Now change the independent variables r and s to m and s by the transformation $r = m - s$.)

$$= -\sum_{m=1}^{\infty} \sum_{s=0}^{[m/2]} \frac{(-1)^s}{m-s} \binom{m-s}{s} \lambda^{m-2s} a^s x^m \quad (4)$$

Equating coefficients of x^m we have

$$\frac{\alpha^m + \beta^m}{m} = \sum_{s=0}^{[m/2]} \frac{(-1)^s}{m-s} \binom{m-s}{s} \lambda^{m-2s} a^s$$

Taking $m = 2n + 1$

$$\therefore \quad \alpha^{2n+1} + \beta^{2n+1} = \sum_{s=0}^{n} (-1)^s \frac{2n+1}{2n-s+1} \binom{2n-s+1}{s} \lambda^{2n-2s+1} a^s$$

Also $\alpha^{2n+1}\beta^{2n+1} \equiv (\alpha\beta)^{2n+1} = a^{2n+1}$

So the equation with roots α^{2n+1} and β^{2n+1} is

$$y^2 - \left\{ \sum_{s=0}^{n} (-1)^s \frac{2n+1}{2n-s+1} \binom{2n-s+1}{s} \lambda^{2n-2s+1} a^s \right\} y + a^{2n+1} = 0$$

Let

$$\sum_{s=0}^{n} (-1)^s \frac{2n+1}{2n-s+1} \binom{2n-s+1}{s} \lambda^{2n-2s+1} a^s = b$$

so $y^2 - by + a^{2n+1} = 0$

and exactly as before

$$\lambda = [\tfrac{1}{2}(b + \sqrt{b^2 - 4a^{2n+1}})]^{\frac{1}{2n+1}} + [\tfrac{1}{2}(b - \sqrt{b^2 - 4a^{2n+1}})]^{\frac{1}{2n+1}}$$

is the solution of

$$\lambda^{2n+1} - (2n+1)a\lambda^{2n-1} + \frac{(2n+1)(2n-2)}{2} a^2\lambda^{2n-3} - \dots$$
$$+ (-1)^n(2n+1)a^n\lambda - b = 0$$

Thus we can solve such equations as

$$\lambda^5 - 5a\lambda^3 + 5a^2\lambda - b = 0 \qquad (n = 2)$$

and

$$\lambda^7 - 7a\lambda^5 + 14a^2\lambda^3 - 7a^3\lambda - b = 0 \quad (n = 3)$$

Many mathematicians tried to reduce the general quintic equation to the form $\lambda^5 - 5a\lambda^3 + 5a^2\lambda - b = 0$ which is known as the reducible quintic. The best result was obtained by the Swedish mathematician, E. S. Bring, who, in 1786, reduced the general quintic to the trinomial form $\lambda^5 - A\lambda - B = 0$. It is now known however, that in general the reduction is impossible.

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