Chapter 6, Question 11

11. Determine the discriminant of the ideal $< 5 + \sqrt{2}, 7 + 2\sqrt{2} >$ in O_K, where $K = \mathbb{Q}(\sqrt{2})$.

Solution. Let $I =< 5 + \sqrt{2}, 7 + 2\sqrt{2} >$. We have

$$1 = -13(5 + \sqrt{2}) + (10 - \sqrt{2})(7 + 2\sqrt{2}) \in I$$

so that

$$I =< 1 > .$$

Now $\{1, \sqrt{2}\}$ is an integral basis for $\mathbb{Q}(\sqrt{2})$ so that

$$I = \mathbb{Z} + \mathbb{Z}\sqrt{2}.$$

Hence

$$D(I) = \begin{vmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{vmatrix}^2 = (-2\sqrt{2})^2 = 8. \qquad \blacksquare$$

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