CHAPTER 5, QUESTION 9

9. Determine $\alpha \in \mathbb{C}$ such that

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) = \mathbb{Q}(\alpha).$$

Solution. We have already seen in Example 5.6.1 that

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3}).$$

Hence

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{2} + \sqrt{3}, \sqrt{5}).$$

The conjugates of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ are $\sqrt{2} + \sqrt{3}$, $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$ and $-\sqrt{2} - \sqrt{3}$. The conjugates of $\sqrt{5}$ over $\mathbb{Q}$ are $\sqrt{5}$ and $-\sqrt{5}$. The eight numbers

$$\begin{array}{llll} (\sqrt{2} + \sqrt{3}) + \sqrt{5}, & (\sqrt{2} - \sqrt{3}) + \sqrt{5}, & (-\sqrt{2} + \sqrt{3}) + \sqrt{5}, & (-\sqrt{2} - \sqrt{3}) + \sqrt{5}, \\ (\sqrt{2} + \sqrt{3}) - \sqrt{5}, & (\sqrt{2} - \sqrt{3}) - \sqrt{5}, & (-\sqrt{2} + \sqrt{3}) - \sqrt{5}, & (-\sqrt{2} - \sqrt{3}) - \sqrt{5}, \end{array}$$

are all distinct. Hence, by the comments following the proof of Theorem 5.6.2, we have

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{2} + \sqrt{3} + \sqrt{5}).$$ ■

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