CHAPTER 5, QUESTION 14

14. Let $\theta \in \mathbb{C}$ be a root of $x^5 + x + 1 = 0$. If $\theta \notin \mathbb{Q}(\sqrt{-3})$, what is $\mathrm{irr}_{\mathbb{Q}}\theta$?

Solution. As

$$x^5 + x + 1 = (x^2 + x + 1)(x^3 - x^2 + 1),$$

θ is a root of $x^2 + x + 1$ or $x^3 - x^2 + 1$. In the former case $\theta = \dfrac{-1 \pm \sqrt{-3}}{2} \in \mathbb{Q}(\sqrt{-3})$, contradicting $\theta \notin \mathbb{Q}(\sqrt{-3})$. Hence θ is a root of $x^3 - x^2 + 1$. As $(\pm 1)^3 - (\pm 1)^2 + 1 \neq 0$, $x^3 - x^2 + 1$ has no linear factors in $\mathbb{Z}[x]$, and thus irreducible in $\mathbb{Z}[x]$. This proves that $\mathrm{irr}_{\mathbb{Q}}\theta = x^3 - x^2 + 1$. ∎

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