CHAPTER 5, QUESTION 10

10. Prove that

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 8.$$

Solution. We have

$$\begin{aligned}&[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}]\\ &\quad = [\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{2}, \sqrt{3})][\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}), \mathbb{Q}].\end{aligned}$$

The minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$ so that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. The minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$ is $x^2 - 3$ so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$. The minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $x^2 - 5$ so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{2}, \sqrt{3})] = 2$. Hence

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 2 \cdot 2 \cdot 2 = 2^3 = 8.$$

To check that $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$ we have only to check that it is irreducible in $\mathbb{Q}(\sqrt{2})[x]$, that is there do not exist $a, b \in \mathbb{Q}$ such that

$$(a + b\sqrt{2})^2 - 3 = 0.$$

This is clear as this equation gives the equations

$$a^2 + 2b^2 = 3, \quad ab = 0,$$

which have no rational solutions.

similarly we can check that $x^2 - 5$ is the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ ∎

June 22, 2004