CHAPTER 4, QUESTION 12

12. Prove that $\mathbb{Z}+\mathbb{Z}\sqrt{2}+\mathbb{Z}\sqrt{5}+\mathbb{Z}\sqrt{10}$ is not integrally closed in its quotient field. [HINT: Consider $\left(\dfrac{1+\sqrt{5}}{\sqrt{2}}\right)^2$.]

Solution. Let

$D = \mathbb{Z}+\mathbb{Z}\sqrt{2}+\mathbb{Z}\sqrt{5}+\mathbb{Z}\sqrt{10} = \{a+b\sqrt{2}+c\sqrt{5}+d\sqrt{10} \mid a,b,c,d \in \mathbb{Z}\}.$

The field of quotients of D is

$$K = \mathbb{Q}(\sqrt{2},\sqrt{5}) = \{r+s\sqrt{2}+t\sqrt{5}+u\sqrt{10} \mid r,s,t,u \in \mathbb{Q}\}.$$

Let

$$\theta = \left(\frac{1+\sqrt{5}}{\sqrt{2}}\right).$$

Clearly

$$\theta = \frac{\sqrt{2}+\sqrt{10}}{2}$$

so

$$\theta \in K, \quad \theta \notin D.$$

Now

$$\theta^2 = \left(\frac{1+\sqrt{5}}{\sqrt{2}}\right)^2 = \frac{6+2\sqrt{5}}{2} = 3+\sqrt{5}$$

so that θ is integral over $\mathbb{Z}+\mathbb{Z}\sqrt{5}$ and thus over D. Hence D is not integrally closed in its quotients field. ∎

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