CHAPTER 3, QUESTION 8

8. If $M_1, \ldots, M_n$ are $n(\geq 1)$ nonempty subsets of an R-module M, we define

$$M_1 + \cdots + M_n = \{m_1 + \cdots + m_n \mid m_i \in M_i\}.$$

If $M_1, \ldots, M_n$ are submodules of M prove that $M_1 + \cdots + M_n$ is a submodule of M.

Solution. First, as $M_1, \ldots, M_n$ are submodules of the R-module M, $M_1, \ldots, M_n$ are nonempty subsets of M. Thus

$$M_1 + \cdots + M_n \text{ is a nonempty subset of } M. \tag{1}$$

Secondly, as $M_1, \ldots, M_n$ are submodules of M, we have $0 \in M_i$, $i = 1, 2, \ldots, n$. Thus

$$0 = 0 + \cdots + 0 \in M_1 + \cdots + M_n. \tag{2}$$

Thirdly, let $m, m' \in M_1 + \cdots + M_n$. Then

$$m = m_1 + \cdots + m_n, \quad m_i \in M_i, \quad i = 1, 2, \ldots, n,$$
$$m' = m'_1 + \cdots + m'_n, \quad m'_i \in M_i, \quad i = 1, 2, \ldots, n.$$

Hence

$$m - m' = (m_1 - m'_1) + \cdots + (m_n - m'_n).$$

As M_i is a submodule of M, we have $m_i - m'_i \in M_i$, $i = 1, 2, \ldots, n$. Thus

$$m - m' \in M_1 + \cdots + M_n, \text{ for all } m, m' \in M_1 + \cdots + M_n. \tag{3}$$

Finally let $r \in R$ and $m \in M_1 + \cdots + M_n$. Then

$$m = m_1 + \cdots + m_n, \quad m_i \in M_i, \quad i = 1, 2, \ldots, n.$$

Hence

$$rm = r(m_1 + \cdots + m_n) = rm_1 + \cdots + rm_n.$$

As M_i is a submodule of the R-module M_i, we have $rm_i \in M_i$, $i = 1, 2, \ldots, n$. Thus

$$rm \in M_1 + \cdots + M_n \text{ for all } r \in R \text{ and all } m \in M_1 + \cdots + M_n. \tag{4}$$

From (1), (2), (3), (4) and the result of the Question 6, we deduce that $M_1 + \cdots + M_n$ is a submodule of M. ∎

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