CHAPTER 3, QUESTION 11

11. Suppose that D is a unique factorization domain and $a(\neq 0)$ and $b(\neq 0)$ are coprime nonunits in D. Prove that if $ab = c^n$ for some $c \in D$ and some $n \in \mathbb{N}$ then there is a unit $e \in D$ such that ea and $e^{-1}b$ are n th powers in D.

Solution. Let $\{\pi_1, \ldots, \pi_m\}$ be all the nonassociated irreducibles dividing a or b or c. Then,

$$\begin{aligned} a &= \varepsilon \pi_1^{\alpha_1} \cdots \pi_m^{\alpha_m}, \\ b &= \eta \pi_1^{\beta_1} \cdots \pi_m^{\beta_m}, \\ c &= \delta \pi_1^{\gamma_1} \cdots \pi_m^{\gamma_m}, \end{aligned}$$

where $\varepsilon,\ \eta,\ \delta \in U(D)$ and $\alpha_i,\ \beta_i,\ \gamma_i$ are nonnegative integers. Then

$$\varepsilon \eta \pi_1^{\alpha_1+\beta_1} \cdots \pi_m^{\alpha_m+\beta_m} = \delta^n \pi^{n\gamma_1} \cdots \pi_m^{n\gamma_m}.$$

As D is a unique factorization domain, we have

$$\alpha_i + \beta_i = n\gamma_i,\ i = 1, 2, \ldots, m.$$

As a and b are coprime in D, either $\alpha_i = 0$ or $\beta_i = 0$ for each i. Hence $\alpha_i = 0$ or $\alpha_i = n\gamma_i$ for each i. Thus α_i is a multiple of n for all i, so

$$a = \varepsilon d^n,\ \text{where}\ d = \pi_1^{\alpha_1/n} \cdots \pi_m^{\alpha_m/n} \in D,$$

that is

$$ea = d^n,\ \text{where}\ e = 1/\varepsilon \in U(D).$$

Then

$$e^{-1} = e^{-1}c^n/a = \frac{c^n}{ea} = \frac{c^n}{d^n} = (\frac{c}{d})^n.$$

Finally

$$\frac{c}{d} = \frac{\delta \pi_1^{\gamma_1} \cdots \pi_m^{\gamma_m}}{\pi_1^{\alpha_1/n} \cdots \pi_m^{\alpha_m/n}} = \delta \pi_1^{\beta_1/n} \cdots \pi_m^{\beta_m/n} \in D,$$

as each $\beta_i = 0$ or $n\gamma_i$. ∎