CHAPTER 2, QUESTION 26

26. Let m be a positive integer with $m \equiv 1 \pmod 4$. Show that the solvability of the equation $T^2 + TU + \frac{1}{4}(1-m)U^2 = -1$ in integers T and U (see Theorem 1.4.5) is equivalent to the solvability of the equation $X^2 - mY^2 = -4$ in integers X and Y.

Solution. Let m be a positive integer with $m \equiv 1 \pmod 4$ so that $\frac{1}{4}(1-m) \in \mathbb{Z}$.

Suppose that there exist integers T and U such that

$$T^2 + TU + \frac{1}{4}(1-m)U^2 = -1.$$

Then

$$(2T+U)^2 - mU^2 = 4(T^2 + TU + \frac{1}{4}(1-m)U^2) = -4.$$

Hence $X^2 - mY^2 = -4$ is solvable in integers X and Y with $X = 2T + U$ and $Y = U$.

Conversly suppose that there exist integers X and Y such that

$$X^2 - mY^2 = -4.$$

As m is odd we see that $X \equiv Y \pmod 2$. Define $T,\ U \in \mathbb{Z}$ by

$$T = \frac{X-Y}{2}, \quad U = Y.$$

Then, $2T + U = X$ and

$$T^2 + TU + \frac{1}{4}(1-m)U^2) = \frac{1}{4}((2T+U)^2 - mU^2) = \frac{X^2 - mY^2}{4} = -1. \qquad \blacksquare$$

June 20, 2004