CHAPTER 1, QUESTION 33

33. Give an example of an ideal in $\mathbb{Z}+\mathbb{Z}\sqrt{10}$ which is not principal.

Solution. We show that the ideal $<2,\sqrt{10}>$ of $\mathbb{Z}+\mathbb{Z}\sqrt{10}$ is not principal.
Suppose that there exists $x+y\sqrt{10}\in\mathbb{Z}+\mathbb{Z}\sqrt{10}$ such that

$$<2,\sqrt{10}> = <x+y\sqrt{10}> .$$

Then $2\in<x+y\sqrt{10}>$ and $\sqrt{10}\in<x+y\sqrt{10}>$ so there exist $a+b\sqrt{10}\in\mathbb{Z}+\mathbb{Z}\sqrt{10}$ and $c+d\sqrt{10}\in\mathbb{Z}+\mathbb{Z}\sqrt{10}$ such that

$$2=(x+y\sqrt{10})(a+b\sqrt{10})$$

and

$$\sqrt{10}=(x+y\sqrt{10})(c+d\sqrt{10}).$$

As $\sqrt{10}\notin\mathbb{Q}$ we obtain

$$\begin{aligned}2=xa+10yb,\ 0=ya+xb,\\ 0=xc+10yd,\ 1=yc+xd.\end{aligned}$$

Thus

$$(x^2-10y^2)(a^2-10b^2)=(xa+10yb)^2-10(ya+xb)^2=4$$

and

$$(x^2-10y^2)(c^2-10d^2)=(xc+10yd)^2-10(yc+xd)^2=-10.$$

The first of these equations tells us that

$$x^2-10y^2=\pm1,\ \ \pm2 \text{ or } \pm4$$

and the second that

$$x^2-10y^2=\pm1,\ \ \pm2,\ \ \pm5 \text{ or } \pm10.$$

Hence

$$x^2-10y^2=\pm1 \text{ or } \pm2.$$

If $x^2 - 10y^2 = \pm 1$ we see from Question 17 that $x + y\sqrt{10} \in U(\mathbb{Z} + \mathbb{Z}\sqrt{10})$ so that by Question 7

$$< 2, \sqrt{10} >=< x + y\sqrt{10} >=< 1 > .$$

Hence there exist $s + t\sqrt{10} \in \mathbb{Z} + \mathbb{Z}\sqrt{10}$ and $u + v\sqrt{10} \in \mathbb{Z} + \mathbb{Z}\sqrt{10}$ such that

$$(s + t\sqrt{10})2 + (u + v\sqrt{10})\sqrt{10} = 1.$$

Thus

$$2s + 10v = 1, \; 2t + u = 0.$$

The first of these equations clearly cannot hold so that $x^2 - 10y^2 \neq \pm 1$. Hence $x^2 - 10y^2 = \pm 2$. Thus $x^2 \equiv \pm 2 \pmod 5$, which is impossible as the squares modulo 5 are $0, \; \pm 1$. Hence no such element $x + y\sqrt{10}$ can exist and $< 2, \sqrt{10} >$ is not a principal ideal. ∎

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