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Solution to a conjecture by Hofmeier–Wittstock

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Abstract

For a locally compact (non-compact) group $\mathcal{G}$, consider $L_\infty(\mathcal{G})$ as a left module over the algebra $L_1(\mathcal{G})^{**}$, endowed with the first Arens product. We answer in the affirmative a question raised by Hofmeier–Wittstock (Math. Ann. 308 (1) (1997) 141) concerning the automatic boundedness of the corresponding module homomorphisms on $L_\infty(\mathcal{G})$. In fact, we prove that an even stronger assertion holds true. Furthermore, we show that the theorem, first obtained by Ghahramani–McClure (Canad. Math. Bull. 35 (2) (1992) 180), on the automatic w^*-continuity of the (bounded) $L_\infty(\mathcal{G})^*$-module homomorphisms on $L_\infty(\mathcal{G})$, can be sharpened in a similar fashion. To this end, a general factorization theorem for bounded families in $L_\infty(\mathcal{G})$ is proved from which we shall derive both results.

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1. Introduction and preliminaries

The central aim of the present article is to to solve a conjecture made in [2], which is done by showing an even stronger result. In Section 2.6 of [2], the authors

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consider, for an infinite discrete group $\mathscr{G}$, the space $\ell_\infty(\mathscr{G})$ as a left module over the algebra $\ell_1(\mathscr{G})^{**}$, equipped with the first Arens product (stemming from the convolution product in the algebra $\ell_1(\mathscr{G})$; cf. definitions below). As an application of the main (technical) result, it is shown that every linear left $\ell_1(\mathscr{G})^{**}$-module homomorphism $\Phi \in \mathfrak{L}(\ell_\infty(\mathscr{G}))$ is automatically bounded and even normal (i.e., w^*-w^*-continuous). The authors remark that under the *assumption of boundedness*, the corresponding result on automatic normality has been obtained independently by Ghahramani and McClure [1, Theorem 1.8] for an *arbitrary locally compact* group $\mathscr{G}$. They thus ask whether the result on automatic boundedness as well could be achieved in the general setting of locally compact groups. (We wish to remark at this point that Fereidoun Ghahramani has kindly informed us about himself having raised this problem at the end of several conference talks.)

Our main objective is to give an affirmative answer to this question for all locally compact non-compact groups $\mathscr{G}$. Moreover, we shall in this situation derive Ghahramani–McClure's theorem on automatic normality, which constitutes the main result of [1], by a completely different method; the original proof relies on the deep result, due to Lau, stating that the topological centre of the algebra $\mathrm{LUC}(\mathscr{G})^*$ is precisely the measure algebra $\mathrm{M}(\mathscr{G})$—cf. the main theorem of [5, Theorem 1]. Indeed, we shall deduce automatic boundedness and normality from a remarkable general factorization theorem which we shall prove below (Theorem 2.1); the latter states that bounded families in $L_\infty(\mathscr{G})$ (with a certain cardinality bound) can be factorized through the module action of $L_\infty(\mathscr{G})^*$ on $L_\infty(\mathscr{G})$ as mentioned above.

We stress that, as we shall see, the assertions of both Hofmeier–Wittstock's conjecture and Ghahramani–McClure's result can be (formally) sharpened. Namely, we will show that for a linear operator Φ on $L_\infty(\mathscr{G})$ in order to be bounded and even w^*-continuous, it suffices that Φ commutes with the module action of a small set of functionals consisting of w^*-limits of point evaluations (cf. Theorems 3.1 and 3.2).

Let us now fix our notation and present some basic results that will be needed in the sequel. We write $\mathfrak{k}(\mathscr{G})$ for the *compact covering number* of the group $\mathscr{G}$, i.e., the least cardinality of a covering of $\mathscr{G}$ by compact subsets. For a Banach space X, we denote by $\mathfrak{L}(X)$ the space of all linear (not necessarily bounded) operators on X; we write $\mathscr{B}(X)$ for the space of bounded linear operators on X. Furthermore, $\mathscr{B}^\sigma(X^*)$ denotes the space of normal (i.e., w^*-w^*-continuous) linear operators on X^*. If $\mathscr{A}$ is a Banach algebra and X^* is a (left) Banach module over $\mathscr{A}$, we write $\mathfrak{L}_{\mathscr{A}}(X^*)$, $\mathscr{B}_{\mathscr{A}}(X^*)$ and $\mathscr{B}^\sigma_{\mathscr{A}}(X^*)$ for the corresponding spaces of (left) module homomorphisms on X^*, i.e., the corresponding operators Φ satisfying $\Phi(ah) = a\Phi(h)$ for all $a \in \mathscr{A}$ and $h \in X^*$.

We shall regard $L_1(\mathscr{G})^{**}$ as equipped with the first Arens multiplication whose construction we briefly recall here. It arises from convolution (denoted by "$*$") in $L_1(\mathscr{G})$ via the different module actions which constitute the structure to be investigated in the present paper. For $m, n \in L_1(\mathscr{G})^{**}$, $h \in L_1(\mathscr{G})^*$ and $f, g \in L_1(\mathscr{G})$ one defines:

$$\langle h \odot f, g \rangle := \langle h, f * g \rangle,$$

$$\langle n \odot h, f \rangle := \langle n, h \odot f \rangle,$$

$$\langle m \odot n, h \rangle := \langle m, n \odot h \rangle.$$

The reader may find a detailed exposition of the basic theory of Arens products in [11, Section 1.4]. We recall the well-known fact that with the first module operation defined above, we have the equality $L_\infty(\mathscr{G}) \odot L_1(\mathscr{G}) = \mathrm{LUC}(\mathscr{G})$, where $\mathrm{LUC}(\mathscr{G})$ denotes the space of all complex-valued bounded left uniformly continuous functions on the group $\mathscr{G}$. Thus one naturally introduces the following module operation of $\mathrm{LUC}(\mathscr{G})^*$ (which inherits a Banach algebra structure as a quotient of $L_\infty(\mathscr{G})^*$) on $L_\infty(\mathscr{G})$:

$$\langle m \diamond h, g \rangle = \langle m, h \odot g \rangle,$$

where $m \in \mathrm{LUC}(\mathscr{G})^*$, $h \in L_\infty(\mathscr{G})$, $g \in L_1(\mathscr{G})$. For instance, we have $\delta_y \diamond h = r_y h$ for all $y \in \mathscr{G}$ and $h \in L_\infty(\mathscr{G})$, where r_y denotes the operator of right translation, i.e., $(r_y h)(x) = h(xy)$ for $x \in \mathscr{G}$. We remark that this module operation obviously is w^*-w^*-continuous (i.e., $\sigma(\mathrm{LUC}(\mathscr{G})^*, \mathrm{LUC}(\mathscr{G}))$-$\sigma(L_\infty(\mathscr{G}), L_1(\mathscr{G}))$-continuous) in the left variable.

Moreover, it is readily verified that $m \diamond h = \widetilde{m} \odot h$, where $\widetilde{m}$ is an arbitrary Hahn–Banach extension of m to $L_\infty(\mathscr{G})^*$. Conversely, we trivially have $m \odot h = (m|_{\mathrm{LUC}(\mathscr{G})}) \diamond h$ for all $m \in L_\infty(\mathscr{G})^*$. These observations immediately yield the following simple observation that we shall tacitly use in the sequel.

Remark 1.1. Every left $L_\infty(\mathscr{G})^*$-module homomorphism on $L_\infty(\mathscr{G})$ is an $\mathrm{LUC}(\mathscr{G})^*$-module homomorphism, and vice versa. In other words: $\mathfrak{L}_{L_\infty(\mathscr{G})^*}(L_\infty(\mathscr{G})) = \mathfrak{L}_{\mathrm{LUC}(\mathscr{G})^*}(L_\infty(\mathscr{G}))$.

2. The factorization theorem

We now come to our main tool which is of interest in its own right.

Theorem 2.1. *Let $\mathscr{G}$ be a locally compact non-compact group with compact covering number $\mathfrak{k}(\mathscr{G})$. Let further $(h_j)_{j \in J} \subseteq L_\infty(\mathscr{G})$ be a bounded family of functions where $|J| \leqslant \mathfrak{k}(\mathscr{G})$. Then there exist a family $(\psi_j)_{j \in J}$ of functionals in $\overline{\delta_{\mathscr{G}}}^{w^*} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathscr{G})^*)$ and a function $h \in L_\infty(\mathscr{G})$ such that the factorization formula*

$$h_j = \psi_j \diamond h$$

holds for all $j \in J$. Here, the functionals ψ_j, $j \in J$, do not depend, except for the index set, on the given family $(h_j)_{j \in J}$; they are universal in the sense that they are obtained intrinsically from the group $\mathscr{G}$.

Proof. There is a covering of $\mathscr{G}$ by open sets whose closure is compact, of minimal cardinality, i.e., of cardinality $\mathfrak{k}(\mathscr{G})$, and closed under finite unions; we write $(K_\alpha)_{\alpha\in I}$ for the corresponding family of compacta. Set $\widetilde{I} := I \times J$. For $\widetilde{\alpha} = (\alpha, i)\in\widetilde{I}$, put $K_{\widetilde{\alpha}} = K_{(\alpha,i)} := K_\alpha$. Then $(K_{\widetilde{\alpha}})_{\widetilde{\alpha}\in\widetilde{I}}$ is a covering of $\mathscr{G}$ with the same properties as the original one. It is easy to see (cf. part of Lemma 3 in [6]) that there exists a family $(y_{\widetilde{\alpha}})_{\widetilde{\alpha}\in\widetilde{I}}\subseteq\mathscr{G}$ such that

$$K_{\widetilde{\alpha}}y_{\widetilde{\alpha}}\cap K_{\widetilde{\beta}}y_{\widetilde{\beta}} = \varnothing\,\forall\,\widetilde{\alpha},\widetilde{\beta}\in\widetilde{I},\widetilde{\alpha}\neq\widetilde{\beta}. \tag{1}$$

We define partial orderings on $\widetilde{I}$ and I by setting, for $(\alpha, i), (\beta,j)\in\widetilde{I}$:

$$(\alpha, i)\preceq(\beta,j) :\Leftrightarrow K_{(\alpha,i)}\subseteq K_{(\beta,j)}\Leftrightarrow K_\alpha\subseteq K_\beta\Leftrightarrow :\alpha\preceq'\beta. \tag{2}$$

Let $\mathfrak{F}$ be an ultrafilter on I which dominates the order filter. Define, for $j\in J$,

$$\psi_j{}' := w^* - \lim_{\beta\to\mathfrak{F}}\delta_{y_{(\beta,j)}}\in\overline{\delta_{\mathscr{G}}}^{w^*}\subseteq\mathrm{Ball}(\mathrm{LUC}(\mathscr{G})^*),$$

and let ψ_j be arbitrary Hahn–Banach extensions of $\psi_j{}'$ to $L_\infty(\mathscr{G})^*$.

Since the family $(h_i)_{i\in J}$ is bounded and, by (1), the projections

$$r_{y^{-1}_{(\alpha,i)}}\chi_{K_{(\alpha,i)}} = \chi_{K_{(\alpha,i)}y_{(\alpha,i)}}, (\alpha, i)\in\widetilde{I},$$

are pairwise orthogonal, we have

$$h := \sum_{\alpha\in I}\sum_{i\in J} r_{y^{-1}_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}h_i)\in L_\infty(\mathscr{G}) \quad (w^*\text{-limits}).$$

By (1), we see that for all (α, i), (β,j), $(\gamma, k)\in\widetilde{I}$, where $(\gamma, k)\preceq(\beta,j)$:

$$\begin{aligned}
&\chi_{K_{(\gamma,k)}}r_{y_{(\beta,j)}}r_{y^{-1}_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}h_i)\\
&\qquad = \chi_{K_{(\gamma,k)}}\chi_{K_{(\beta,j)}}r_{y_{(\beta,j)}}r_{y^{-1}_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}h_i)\\
&\qquad = \chi_{K_{(\gamma,k)}}[r_{y_{(\beta,j)}}((r_{y^{-1}_{(\beta,j)}}\chi_{K_{(\beta,j)}})r_{y^{-1}_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}h_i))]\\
&\qquad = \delta_{(\alpha,i),(\beta,j)}\chi_{K_{(\gamma,k)}}h_j.
\end{aligned}$$

Using (2), we thus obtain for all $j\in J$ and $(\gamma, k)\in\widetilde{I}$:

$$\begin{aligned}
\chi_{K_{(\gamma,k)}}(\psi_j\diamond h) &= w^* - \lim_{\beta\to\mathfrak{F}}\sum_{\alpha\in I}\sum_{i\in J}\underbrace{\chi_{K_{(\gamma,k)}}r_{y_{(\beta,j)}}r_{y^{-1}_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}h_i)}_{=\delta_{(\alpha,i),(\beta,j)}\chi_{K_{(\gamma,k)}}h_j}\\
&= \chi_{K_{(\gamma,k)}}h_j,
\end{aligned}$$

which entails the desired factorization. $\square$

3. Automatic boundedness and w^*-continuity of module homomorphisms

We will now apply our factorization theorem to solve the problem on automatic boundedness posed in [2, Section 2.6], by deriving an even stronger assertion.

Theorem 3.1. *Let $\mathcal{G}$ be a locally compact non-compact group. Then every linear operator on $L_\infty(\mathcal{G})$ which commutes with the (left) module action of functionals in the set $\overline{\delta_{\mathcal{G}}}^{w^*} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ on $L_\infty(\mathcal{G})$ is automatically bounded. In particular, every linear (left) $L_\infty(\mathcal{G})^*$-module homomorphism on $L_\infty(\mathcal{G})$ is bounded.*

Proof. Our procedure is indirect, following a gliding hump argument. Consider an operator $\Phi \in \mathfrak{L}(L_\infty(\mathcal{G}))$ which commutes with the action of elements in $S := \overline{\delta_{\mathcal{G}}}^{w^*} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$, and suppose Φ is unbounded. Then there exists a sequence $(h_n)_{n\in\mathbf{N}} \subseteq L_\infty(\mathcal{G})$, $\|h_n\|_\infty = 1$, such that for all $n \in \mathbf{N}$:

$$\|\Phi(h_n)\|_\infty \geqslant n.$$

Since $\mathcal{G}$ is non-compact, we have $|\mathbf{N}| \leqslant \mathfrak{k}(\mathcal{G})$, whence Theorem 2.1 yields the factorization:

$$h_n = \psi_n \diamond h,$$

where $(\psi_n)_{n\in\mathbf{N}} \subseteq S$ and $h \in L_\infty(\mathcal{G})$.

It follows that for all $n \in \mathbf{N}$ we must have

$$n \leqslant \|\Phi(h_n)\|_\infty = \|\Phi(\psi_n \diamond h)\|_\infty = \|\psi_n \diamond \Phi(h)\|_\infty \leqslant \|\Phi(h)\|_\infty < \infty,$$

a contradiction. $\square$

We shall now present an alternative proof of the automatic normality result obtained by Ghahramani–McClure, which is based on our factorization theorem (and in particular does not rely on the topological centre theorem for the algebra $\mathrm{LUC}(\mathcal{G})^*$). As above, our approach will even yield a sharpening of the original result.

Theorem 3.2. *Let $\mathcal{G}$ be a locally compact non-compact group. Then every bounded linear operator on $L_\infty(\mathcal{G})$ which commutes with the (left) module action of functionals in the set $\overline{\delta_{\mathcal{G}}}^{w^*} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ on $L_\infty(\mathcal{G})$ is automatically normal. In particular, every bounded linear (left) $L_\infty(\mathcal{G})^*$-module homomorphism on $L_\infty(\mathcal{G})$ is normal.*

In the proof of Theorem 3.2, we shall refer to the concept of Mazur's property of higher cardinal level, as introduced in [9], and use a result obtained in [9] as a technical device; we recall both here for the convenience of the reader.

Definition 3.3. Let X be a Banach space and $\kappa \geqslant \aleph_0$ a cardinal number.

(i) A functional $f \in X^{**}$ is called w^*-κ-continuous if for all nets $(x_\alpha)_{\alpha \in I} \subseteq \mathrm{Ball}(X^*)$ of cardinality $\aleph_0 \leqslant |I| \leqslant \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have: $\langle f, x_\alpha \rangle \to 0$.

(ii) We say that X has Mazur's property of level κ if every w^*-κ-continuous functional $f \in X^{**}$ actually is w^*-continuous, i.e., defines an element of X.

The result which is important in our situation is the following.

Theorem 3.5. *Let $\mathscr{G}$ be a locally compact group with compact covering number $\mathfrak{k}(\mathscr{G})$. Then $L_1(\mathscr{G})$ has Mazur's property of level $\mathfrak{k}(\mathscr{G}) \cdot \aleph_0$.*

Proof. This is Theorem 4.4 in [9]. $\square$

Remark 3.6. We would like to stress that essentially the proof of Theorem 3.5 only uses a certain stability result for Mazur's property of higher level (cf. Theorem 4.6 in [9]), as well as the fact that the L_1 space of a σ-finite measure space has the classical Mazur's property (cf. Example 3.14 in [9]). The proof does not rely on various other, more sophisticated results contained in [9] that involve geometry of Banach spaces, von Neumann algebras, measurable cardinals, Gode froy—Talagrand's property (X), etc.

We are now ready to prove Theorem 3.2.

Proof. Let $\Phi \in \mathscr{B}_{L_\infty(\mathscr{G})^*}(L_\infty(\mathscr{G}))$. In order to prove that Φ is normal, we only have to show, due to Theorem 3.5, that if $(h_i)_{i \in I} \subseteq \mathrm{Ball}(L_\infty(\mathscr{G}))$, where $|I| \leqslant \mathfrak{k}(\mathscr{G})$, is a net which converges w^* to 0, then $\Phi(h_i)$ converges w^* to 0 as well. Fix $g \in L_1(\mathscr{G})$. We have to show that the bounded net $\langle \Phi(h_i), g \rangle$ converges to 0. It clearly suffices to prove that any convergent subnet, say $(\langle \Phi(h_{i_\alpha}), g \rangle)_\alpha$, converges to 0. By Theorem 2.1, the net $(h_i)_{i \in I}$ can be factorized via

$$h_i = \psi_i \diamond h \quad (i \in I),$$

where $(\psi_i)_{i \in I} \subseteq \overline{\delta_{\mathscr{G}}}^{w^*} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathscr{G})^*)$ and $h \in L_\infty(\mathscr{G})$.

The subnet $(\psi_{i_\alpha})_\alpha \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathscr{G})^*)$ has itself a w^*-convergent subnet $(\psi_{i_{\alpha_\beta}})_\beta$ with limit

$$E := w^* - \lim_\beta \psi_{i_{\alpha_\beta}} \in \overline{\delta_{\mathscr{G}}}^{w^*}.$$

Using the w^*-continuity of the module operation of $\mathrm{LUC}(\mathscr{G})^*$ on $L_\infty(\mathscr{G})$ in the left variable, we obtain:

$$\begin{aligned} E \diamond \Phi(h) &= \left(w^* - \lim_\beta \psi_{i_{\alpha_\beta}} \right) \diamond \Phi(h) \\ &= w^* - \lim_\beta [\psi_{i_{\alpha_\beta}} \diamond \Phi(h)] \\ &= w^* - \lim_\beta \Phi(\psi_{i_{\alpha_\beta}} \diamond h) \\ &= w^* - \lim_\beta \Phi(h_{i_{\alpha_\beta}}). \end{aligned}$$

This shows that we have

$$\begin{aligned} \lim_\alpha \langle \Phi(h_{i_\alpha}), g \rangle &= \lim_\beta \langle \Phi(h_{i_{\alpha_\beta}}), g \rangle \\ &= \langle E \diamond \Phi(h), g \rangle. \end{aligned}$$

But on the other hand we get (again using the one-sided w^*-continuity of our module operation):

$$\begin{aligned} E \diamond \Phi(h) &= \Phi(E \diamond h) \\ &= \Phi\left(\left(w^* - \lim_\beta \psi_{i_{\alpha_\beta}} \right) \diamond h \right) \\ &= \Phi\left(w^* - \lim_\beta (\psi_{i_{\alpha_\beta}} \diamond h) \right) \\ &= \Phi\left(w^* - \lim_\beta h_{i_{\alpha_\beta}} \right) \\ &= \Phi(0) = 0. \end{aligned}$$

We conclude that

$$\lim_\alpha \langle \Phi(h_{i_\alpha}), g \rangle = \langle E \diamond \Phi(h), g \rangle = 0,$$

as desired. $\square$

Corollary 3.7. *Let $\mathscr{G}$ be a locally compact non-compact group. Then we have*:

$$\mathfrak{L}_{\overline{\delta_{\mathscr{G}}}^{w^*}}(L_\infty(\mathscr{G})) = \mathscr{B}^\sigma_{\overline{\delta_{\mathscr{G}}}^{w^*}}(L_\infty(\mathscr{G})) = \mathscr{B}^\sigma_{\mathrm{LUC}(\mathscr{G})^*}(L_\infty(\mathscr{G})) = \mathscr{B}^\sigma_{L_\infty(\mathscr{G})^*}(L_\infty(\mathscr{G})).$$

Proof. The first equality is a combination of Theorems 3.1 and 3.2. The second equality holds since $\overline{\mathrm{lin}}^{w^*} \delta_{\mathscr{G}} = \mathrm{LUC}(\mathscr{G})^*$ and the module operation of $\mathrm{LUC}(\mathscr{G})^*$ on $L_\infty(\mathscr{G})$ is $\sigma(\mathrm{LUC}(\mathscr{G})^*, \mathrm{LUC}(\mathscr{G}))$–$\sigma(L_\infty(\mathscr{G})^*, L_\infty(\mathscr{G}))$-continuous. Finally, Remark 1.1 yields the last equality. $\square$

We finish by deducing a result which completely describes the structure of the $L_\infty(\mathscr{G})^*$-module homomorphisms on $L_\infty(\mathscr{G})$; we shall obtain, via a different approach, the assertion of Theorem 1.8 in [1] without the assumption of boundedness. We use the notation $\mathrm{conv}(\mu)$ for the convolution operator

$$\mathrm{conv}(\mu) : L_1(\mathscr{G}) \to L_1(\mathscr{G})$$

$$f \mapsto \mu * f.$$

Corollary 3.8. *Let $\mathscr{G}$ be a locally compact non-compact group, and let $\Phi : L_\infty(\mathscr{G}) \to L_\infty(\mathscr{G})$ be a linear mapping. Then the following are equivalent*:

(i) *Φ is a left $L_\infty(\mathscr{G})^*$–module homomorphism*;
(ii) *there exists a measure $\mu \in \mathrm{M}(\mathscr{G})$ such that $\Phi = \mathrm{conv}(\mu)^*$*.

Proof. (i) $\Rightarrow$ (ii): By Corollary 3.7, Φ is bounded and even normal. Hence there is $\Phi_0 \in \mathscr{B}(L_1(\mathscr{G}))$ such that $\Phi = \Phi_0^*$. Owing to the classical Theorem of Wendel, we have thus only to show that Φ_0 is a left multiplier (i.e., a right $L_1(\mathscr{G})$-module homomorphism) on $L_1(\mathscr{G})$.

Denoting by $\imath$ the canonical embedding of $L_1(\mathscr{G})$ in $L_\infty(\mathscr{G})^*$, we obtain for arbitrary $f, g \in L_1(\mathscr{G})$ and $h \in L_\infty(\mathscr{G})$:

$$\begin{aligned}
\langle \Phi_0(f * g), h \rangle &= \langle f * g, \Phi(h) \rangle \\
&= \langle \imath(f) \odot \imath(g), \Phi(h) \rangle \\
&= \langle \imath(f), \imath(g) \odot \Phi(h) \rangle \\
&= \langle \imath(f), \Phi(\imath(g) \odot h) \rangle \\
&= \langle \Phi(\imath(g) \odot h), f \rangle \\
&= \langle \imath(g) \odot h, \Phi_0(f) \rangle \\
&= \langle h \odot \Phi_0(f), g \rangle \\
&= \langle \Phi_0(f) * g, h \rangle,
\end{aligned}$$

which yields the claim.

(ii) $\Rightarrow$ (i): Evident (note that this implication of course holds as well in case $\mathscr{G}$ is compact). $\square$

Remark 3.9. (i) Besides the applications given in the present article, our factorization principle, Theorem 2.1, has already proven to be extremely useful in connection with the topological centre problem in abstract harmonic analysis. It is the key in order to provide a unified and elegant solution to the topological centre problem (even in a stronger form than classically stated) for both the algebras $\mathrm{LUC}(\mathscr{G})^*$ and $L_1(\mathscr{G})^{**}$; see [8].

(ii) Moreover, it is as well the crucial tool in order to give a short and completely parallel proof of the two theorems stating that the topological centre of $\mathrm{LUC}(\mathscr{G})^*$ is the measure algebra $\mathrm{M}(\mathscr{G})$, and the topological centre of the semigroup compactification $\mathscr{G}^{\mathrm{LUC}}$ of $\mathscr{G}$ is the group $\mathscr{G}$ itself (and even of the stronger result, first obtained by Protasov–Pym in [12], Theorem 2, that the topological centre of the remainder $\mathscr{G}^{\mathrm{LUC}} \backslash \mathscr{G}$ is empty); see [10]. Thus, revealing the perfectly analogous nature of these two results, the latter proof may provide an answer to a question by Lau–Pym—cf. [7, p. 568]—on the connection between the two theorems (the one regarding $\mathscr{G}^{\mathrm{LUC}}$ being the main result of the paper [7]).

(iii) Finally, it was used recently in [4] (see Remarks 4.4) to answer a question arising from [3]. It is shown in [3, Proposition 4.1] that if $\mathscr{G}$ is a non-discrete locally compact group and $\mathscr{H}$ is an open subgroup of $\mathscr{G}$, then the support of each element in the group von Neumann algebra $VN(\mathscr{G})$ can be covered by at most $\chi(\mathscr{G})$ many left cosets of $\mathscr{H}$ in $\mathscr{G}$, where $\chi(\mathscr{G})$ denotes the local weight of $\mathscr{G}$ (i.e., the minimal cardinality of a neighbourhood basis at the neutral element of $\mathscr{G}$). It was not known whether this estimate is optimal in case $\chi(\mathscr{G}) > \aleph_0$. Our Theorem 2.1 shows that even for abelian groups, this upper bound can indeed be attained.

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