Linear Quadratic Mean Field Games: Asymptotic Solvability and Relation to the Fixed Point Approach

Minyi Huang and Mengjie Zhou

Abstract—**Mean field game theory has been developed largely following two routes. One of them, called the direct approach, starts by solving a large-scale game and next derives a set of limiting equations as the population size tends to infinity. The second route is to apply mean field approximations and formalize a fixed point problem by analyzing the best response of a representative player. This paper addresses the connection and difference of the two approaches in a linear quadratic (LQ) setting. We first introduce an asymptotic solvability notion for the direct approach, which means for all sufficiently large population sizes, the corresponding game has a set of feedback Nash strategies in addition to a mild regularity requirement. We provide a necessary and sufficient condition for asymptotic solvability and show that in this case the solution converges to a mean field limit. This is accomplished by developing a re-scaling method to derive a low dimensional ordinary differential equation (ODE) system, where a non-symmetric Riccati ODE has a central role. We next compare with the fixed point approach which determines a two point boundary value (TPBV) problem, and show that asymptotic solvability implies feasibility of the fixed point approach, but the converse is not true. We further address non-uniqueness in the fixed point approach and examine the long time behavior of the non-symmetric Riccati ODE in the asymptotic solvability problem.**

Index Terms—**Asymptotic solvability, direct approach, fixed point approach, linear quadratic, mean field game, re-scaling, Riccati differential equation.**

I. Introduction

MEAN field game (MFG) theory has undergone a phenomenal growth. It provides a powerful methodology for tackling complexity in large-population noncooperative decision problems. The readers are referred to [4], [7], [9], [12], [20] for an overview of the theory and applications. The past developments have largely followed two routes [28], [29], [36] which are called, respectively, the bottom-up and top-down approaches in [7].

One route starts by formally solving an N-player game to obtain a large coupled solution equation system. The next step is to derive a limit for the solution by taking $N \to \infty$ [36], which can be called the direct (or bottom-up) approach; see route one in Fig. 1. Another route is to solve an optimal control problem of a single agent based on consistent mean field approximations and formalize a fixed point problem to determine the mean field, and this is called the fixed point (or top-down) approach [28], [29] and also called Nash certainty equivalence in [29]; see route two in Fig. 1. The solution of the fixed point problem may be used to design decentralized strategies in the original large but finite population model to achieve an ε-Nash equilibrium [28]. Under such a set of strategies, each player can further improve little even if it can access centralized information of all players. Compared with Nash strategies determined under centralized information, the above solution has much lower complexity in its computation and implementation.

The reader may consult further literature on the direct approach [9] and the fixed point approach [4], [5], [12], [32], [37]. Also see [17], [34] for the direct approach in a probabilistic framework. We note that the diagram in Fig. 1 displays the basic theoretic framework of mean field games with all players being comparably small, called peers. When the model involves a major player or common noise, the analysis has been extended for the direct approach [10] and the fixed point approach [4], [8], [12], [13], [27], [41].

So far the investigation of the connection and difference between the two approaches regarding their scope of applicability is scarce. Their systematic comparison is generally difficult since in the literature very often the analysis in each approach is carried out under various sufficient conditions. In this work we contribute in this direction within the framework of linear-quadratic (LQ) mean field games with a finite time horizon. The analysis of mean field games in the LQ setting has attracted substantial interest due to their appealing analytical structure [5], [8], [25], [28], [37], [40], [42], [48], [51], [52]. Specifically, the decentralized strategy of an individual player may be determined in a linear feedback form. Partial state information is considered in [8], [25], and [25] adopts linear backward stochastic differential equations to model state dynamics.

In this paper we first study an asymptotic solvability problem initially introduced in [31], which may be viewed as an instance of the direct (i.e., bottom-up) approach. We adopt an appropriately defined asymptotic solvability notion for the sequence of LQ games with increasing population sizes so that a neat necessary and sufficient condition can be derived. This will on one hand further our understanding of the direct approach and on the other offer a foundation for a thorough comparison with the fixed point approach.

This work was supported in part by Natural Sciences and Engineering Research Council (NSERC) of Canada under a Discovery Grant and a Discovery Accelerator Supplements Program.

M. Huang and M. Zhou are with the School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada (mhuang@math.carleton.ca, mengjiezhou@cmail.carleton.ca).

IEEE Trans. Autom. Control, submitted Apr. 2018; revised Dec. 2018; accepted May 2019.

We start with an entirely conventional solution of the game by dynamic programming, which leads to a set of coupled Riccati ODEs. It turns out that the necessary and sufficient condition for asymptotic solvability is characterized by a low dimensional non-symmetric Riccati ODE derived by a novel re-scaling technique. The methodology of identifying low dimensional dynamics to capture essential information on high dimensional dynamical behavior shares similarity to the statistical physics literature on mean field oscillator models [38], [43], [45]. This approach is also closely related to an early problem of mean field social optimization, which studies a high dimensional algebraic Riccati equation (ARE) and uses symmetry for dimension reduction [26, Sec. 6.3]. Other related works include [24], [44], [47]. An optimal control problem for a set of symmetric agents with mean field coupling is solved in [24] by a large-scale Riccati ODE, and a mean field limit is derived. An LQ Nash game of infinite time horizon is analyzed in [44] where the number of players increases to infinity. By postulating the strategies of all players and examining the control problem of a fixed player, a family of low dimensional control problems and their parametrized AREs are solved by applying an implicit function theorem for which sufficient conditions are obtained for large population sizes. The solvability of LQ games with increasing population sizes in the set-up of [36] is studied in [47] analyzing $2N$ coupled steady-state Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck-Kolmogorov (FPK) equations under some algebraic conditions, where each player's control is restricted to be local state feedback from the beginning.

Subsequently the paper investigates the relation of the two fundamental approaches [28], [29], [36] shown in Fig. 1, which has been made possible by the solution of the asymptotic solvability problem. In so doing, we first revisit the fixed point approach for the mean field game, and determine the necessary and sufficient condition for the solvability of the resulting two point boundary value (TPBV) problem. It is shown that asymptotic solvability provides a sufficient condition for the TPBV problem to be solvable and in fact uniquely solvable in this case; this is due to the fact that one can use a non-symmetric Riccati ODE to decouple and solve a general linear TPBV problem [18]. However, there exist scenarios for our TPBV problem to be solvable but asymptotic solvability fails. This suggests non-equivalence of the two approaches in general. We make a further connection with the original work [28], which applies the fixed point approach under a contraction condition; we show in this case asymptotic solvability holds for the sequence of games.

Our study of the asymptotic solvability problem and the subsequent comparison of the two fundamental approaches provide new insights into the relation between the infinite population mean field game and large finite population games. Historically, the study of the relation between large finite population games and their infinite population limit has been a subject of great interest and importance [1], [11], [21], [23], [39] although this is usually for static games.

For the TPBV problem in the fixed point approach we further examine the non-uniqueness issue, which has been of interest in the MFG literature; see non-uniqueness results for nonlinear MFG models [2], [14], [19] and for an LQ example with a non-quadratic terminal cost [50]. Non-uniqueness has been well studied in the traditional literature of LQ dynamic games; see [15], [16]. Finally, we analyze the long time behavior of the non-symmetric Riccati ODE in the asymptotic solvability problem. The analysis is related to a non-symmetric algebraic Riccati equation (NARE) and faces the issue of solution selection. We introduce the notion of a stabilizing solution for the NARE and derive the necessary and sufficient condition for its existence and uniqueness.

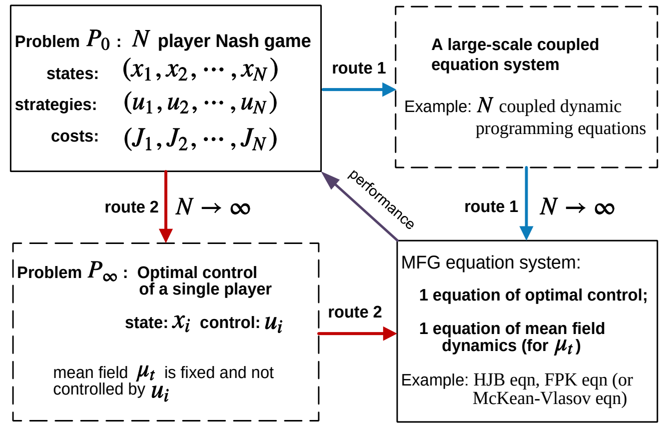

Fig. 1. The two fundamental approaches: The direct (or bottom-up) approach (see route 1) and the fixed point (or top-down) approach (see route 2)

The main contributions of the paper are outlined as follows:

1) We study an N-player LQ Nash game and introduce the notion of asymptotic solvability, which can be regarded as a direct approach in mean field games.
2) By a re-scaling technique, a necessary and sufficient condition for asymptotic solvability is obtained in terms of a non-symmetric Riccati ODE. This lays down a foundation to address the exact relation of two fundamental approaches in mean field games: the direct approach and the fixed point approach. We show asymptotic solvability implies unique solvability of the TPBV problem in the fixed point approach. We further show that a contraction condition of the fixed point approach introduced in the original work [28] implies asymptotic solvability. We further determine conditions for non-uniqueness to occur in the fixed point approach.
3) The long time behavior of the non-symmetric Riccati ODE in the direct approach is studied. A necessary and sufficient algebraic condition is obtained for it to have a stabilizing solution.

We make some convention on notation. Throughout the paper, E is reserved for denoting the mean of a random variable or a random vector. For symmetric matrix $S \geq 0$, we may write $x^T Sx = |x|_S^2$. We denote by $\mathbf{1}_{k\times l}$ a $k \times l$ matrix with all entries equal to 1, by $\otimes$ the Kronecker product, and by the column vectors $\{e_1^k, \dots, e_k^k\}$ the canonical basis of $\mathbb{R}^k$. We may use a subscript n to indicate the identity matrix I_n to be $n \times n$. For a vector or matrix Z, $|Z|$ stands for its Euclidean norm. For an $l \times m$ real matrix $Z = (z_{ij})_{1\leq i\leq l, 1\leq j\leq m}$, denote the l_1-norm $\|Z\|_{l_1} = \sum_{i,j} |z_{ij}|$.

The organization of the paper is as follows. Section II describes the LQ Nash game together with its solution via dynamic programming and Riccati ODEs. Section III presents the necessary and sufficient condition for asymptotic solvability and derives decentralized strategies. We revisit the fixed point approach in Section IV and examine its relation to asymptotic solvability. To further study the relation of the two approaches, Section V develops in-depth analysis of the scalar individual state case. The long time behavior of the non-symmetric Riccati ODE is examined in Section VI. Illustrative examples are provided in Section VII. Section VIII concludes the paper.

II. The LQ Nash Game

Consider a population of N players (or agents) denoted by $\mathscr{A}_i$, $1 \le i \le N$. The state process $X_i(t)$ of $\mathscr{A}_i$ satisfies the following stochastic differential equation (SDE)

$$dX_i(t) = \big(AX_i(t) + Bu_i(t) + GX^{(N)}(t)\big)dt + DdW_i(t), \qquad 1 \le i \le N, \tag{1}$$

where we have state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^{n_1}$, and the coupling term $X^{(N)} = \frac{1}{N}\sum_{k=1}^N X_k$. The constant matrices A, B, G, D have compatible dimensions. The initial states $\{X_i(0), 1 \le i \le N\}$ are independent with $EX_i(0) = x_i(0)$ and finite second moment. The N standard n_2-dimensional Brownian motions $\{W_i, 1 \le i \le N\}$ are independent and also independent of the initial states. The cost of player $\mathscr{A}_i$ in the Nash game is given by

$$J_i = E\int_0^T \Big(|X_i(t) - \Gamma X^{(N)}(t) - \eta|_Q^2 + u_i^T(t)Ru_i(t)\Big)dt + E|X_i(T) - \Gamma_f X^{(N)}(T) - \eta_f|_{Q_f}^2. \tag{2}$$

The constant matrices (or vectors) Γ, Q, R, Γ_f, Q_f, η, η_f above have compatible dimensions, and we have $Q \ge 0$, $R > 0$, $Q_f \ge 0$ for these symmetric matrices. For notational simplicity, we only consider constant parameters for the model. Except for long time behavior in Section VI, our analysis and results can be easily extended to the case of time-dependent parameters.

Define

$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{Nn}, \quad W(t) = \begin{bmatrix} W_1(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{Nn_2},$$
$$\widehat{\boldsymbol{A}} = \mathrm{diag}[A, \cdots, A] + \mathbf{1}_{n\times n} \otimes \frac{G}{N} \in \mathbb{R}^{Nn\times Nn},$$
$$\widehat{\boldsymbol{D}} = \mathrm{diag}[D, \cdots, D] \in \mathbb{R}^{Nn\times Nn_2},$$
$$\boldsymbol{B}_k = e_k^N \otimes B \in \mathbb{R}^{Nn\times n_1}, \qquad 1 \le k \le N.$$

Now we write system of SDEs in (1) in the form

$$dX(t) = \Big(\widehat{\boldsymbol{A}}X(t) + \sum_{k=1}^N \boldsymbol{B}_k u_k(t)\Big)dt + \widehat{\boldsymbol{D}}dW(t). \tag{3}$$

Under closed-loop perfect state (CLPS) information, we denote the value function of $\mathscr{A}_i$ by $V_i(t, \boldsymbol{x})$, $1 \le i \le N$, which corresponds to the initial condition $X(t) = \boldsymbol{x} = (x_1^T, \ldots, x_N^T)^T$ and a cost evaluated on $[t, T]$ in place of (2). The set of value functions is determined by the system of HJB equations

$$0 = \frac{\partial V_i}{\partial t} + \min_{u_i \in \mathbb{R}^{n_1}} \left(\frac{\partial^T V_i}{\partial \boldsymbol{x}}\big(\widehat{\boldsymbol{A}}\boldsymbol{x} + \sum_{k=1}^N \boldsymbol{B}_k u_k\big) + u_i^T R u_i + |x_i - \Gamma x^{(N)} - \eta|_Q^2 + \frac{1}{2}\mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T (V_i)_{\boldsymbol{x}\boldsymbol{x}} \widehat{\boldsymbol{D}}\big)\right), \tag{4}$$
$$V_i(T, \boldsymbol{x}) = |x_i - \Gamma_f x^{(N)} - \eta_f|_{Q_f}^2, \qquad 1 \le i \le N,$$

where $x^{(N)} = (1/N)\sum_{k=1}^N x_k$ and the minimizer is

$$u_i = -\frac{1}{2}R^{-1}\boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}}, \qquad 1 \le i \le N. \tag{5}$$

Next we substitute (5) into (4) to obtain

$$\begin{aligned} 0 =& \frac{\partial V_i}{\partial t} + \frac{\partial^T V_i}{\partial \boldsymbol{x}}\big(\widehat{\boldsymbol{A}}\boldsymbol{x} - \sum_{k=1}^N \frac{1}{2}\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \frac{\partial V_k}{\partial \boldsymbol{x}}\big) \\ &+ |x_i - \Gamma x^{(N)} - \eta|_Q^2 \\ &+ \frac{1}{4}\frac{\partial^T V_i}{\partial \boldsymbol{x}}\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}} + \frac{1}{2}\mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T (V_i)_{\boldsymbol{x}\boldsymbol{x}}\widehat{\boldsymbol{D}}\big). \end{aligned} \tag{6}$$

Denote

$$\boldsymbol{K}_i = [0, \cdots, 0, I_n, 0, \cdots, 0] - \frac{1}{N}[\Gamma, \Gamma, \cdots, \Gamma],$$
$$\boldsymbol{K}_{if} = [0, \cdots, 0, I_n, 0, \cdots, 0] - \frac{1}{N}[\Gamma_f, \Gamma_f, \cdots, \Gamma_f],$$
$$\boldsymbol{Q}_i = \boldsymbol{K}_i^T Q \boldsymbol{K}_i, \quad \boldsymbol{Q}_{if} = \boldsymbol{K}_{if}^T Q_f \boldsymbol{K}_{if},$$

where I_n is the ith submatrix. We write

$$|x_i - \Gamma x^{(N)} - \eta|_Q^2 = \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{K}_i^T Q\eta + \eta^T Q \eta, \tag{7}$$

and write $|x_i - \Gamma_f x^{(N)} - \eta_f|_{Q_f}^2$ in a similar form.

Suppose $V_i(t, \boldsymbol{x})$ has the following form

$$V_i(t, \boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_i(t)\boldsymbol{x} + 2\boldsymbol{S}_i^T(t)\boldsymbol{x} + \boldsymbol{r}_i(t), \tag{8}$$

where $\boldsymbol{P}_i$ is symmetric. Then

$$\frac{\partial V_i}{\partial \boldsymbol{x}} = 2\boldsymbol{P}_i(t)\boldsymbol{x} + 2\boldsymbol{S}_i(t), \quad \frac{\partial^2 V_i}{\partial \boldsymbol{x}^2} = 2\boldsymbol{P}_i(t). \tag{9}$$

We substitute (8) and (9) into (6) and derive the equation systems:

$$\begin{cases} \dot{\boldsymbol{P}}_i(t) = -\Big(\boldsymbol{P}_i(t)\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{A}}^T \boldsymbol{P}_i(t)\Big) + \\ \qquad \Big(\boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \boldsymbol{P}_k(t) \\ \qquad + \sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \boldsymbol{P}_i(t)\Big) \\ \qquad - \boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T \boldsymbol{P}_i(t) - \boldsymbol{Q}_i, \\ \boldsymbol{P}_i(T) = \boldsymbol{Q}_{if}, \end{cases} \tag{10}$$

$$\begin{cases} \dot{\boldsymbol{S}}_i(t) = -\widehat{\boldsymbol{A}}^T \boldsymbol{S}_i(t) - \boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T \boldsymbol{S}_i(t) \\ \qquad + \boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \boldsymbol{S}_k(t) \\ \qquad + \sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \boldsymbol{S}_i(t) \\ \qquad + \boldsymbol{K}_i^T Q\eta, \\ \boldsymbol{S}_i(T) = -\boldsymbol{K}_{if}^T Q_f \eta_f, \end{cases} \tag{11}$$

$$\begin{cases}\dot{\boldsymbol{r}}_i(t)=2\boldsymbol{S}_i^T(t)\sum_{k=1}^N \boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k(t)\\ \qquad\quad -\boldsymbol{S}_i^T(t)\boldsymbol{B}_iR^{-1}\boldsymbol{B}_i^T\boldsymbol{S}_i(t)\\ \qquad\quad -\eta^TQ\eta-\mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T\boldsymbol{P}_i(t)\widehat{\boldsymbol{D}}\big),\\ \boldsymbol{r}_i(T)=\eta_f^TQ_f\eta_f.\end{cases} \tag{12}$$

Remark 1: If (10) has a solution $(\boldsymbol{P}_1,\cdots,\boldsymbol{P}_N)$ on $[\tau,T]\subseteq[0,T]$, such a solution is unique due to the local Lipschitz continuity of the vector field [22]. Taking transpose on both sides of (10) gives an ODE system for $\boldsymbol{P}_i^T$, $1\le i\le N$, which shows that $(\boldsymbol{P}_1^T,\cdots,\boldsymbol{P}_N^T)$ still satisfies (10). So the ODE system (10) guarantees each $\boldsymbol{P}_i$ to be symmetric.

Remark 2: If (10) has a unique solution $(\boldsymbol{P}_1,\cdots,\boldsymbol{P}_N)$ on $[0,T]$, then we can uniquely solve $(\boldsymbol{S}_1,\cdots,\boldsymbol{S}_N)$ and $(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_N)$ by using linear ODEs.

For the N-player Nash game, we consider CLPS information, so that the state vector $X(t)$ is available to each player.

Theorem 1: Suppose that (10) has a unique solution $(\boldsymbol{P}_1,\cdots,\boldsymbol{P}_N)$ on $[0,T]$. Then we can uniquely solve (11)–(12), and the game of N players has a set of feedback Nash strategies given by

$$u_i=-R^{-1}\boldsymbol{B}_i^T(\boldsymbol{P}_iX(t)+\boldsymbol{S}_i),\qquad 1\le i\le N.$$

Proof: This theorem follows the standard results in [3, Theorem 6.16, Corollaries 6.5 and 6.12]. □

By Theorem 1, the solution of the feedback Nash strategies completely reduces to the study of (10). For this reason, our subsequent analysis starts by analyzing (10).

III. Asymptotic Solvability

Definition 2: The sequence of Nash games (1)–(2) with closed-loop perfect state information has asymptotic solvability if there exists N_0 such that for all $N\ge N_0$, $(\boldsymbol{P}_1,\cdots,\boldsymbol{P}_N)$ in (10) has a solution on $[0,T]$ and

$$\sup_{N\ge N_0}\sup_{1\le i\le N,0\le t\le T}\|\boldsymbol{P}_i(t)\|_{l_1}<\infty. \tag{13}$$

Definition 2 only involves the Riccati equations. This is sufficient due to Remark 2. The boundedness condition (13) is to impose certain regularity of the solutions, which is necessary for studying the asymptotic behavior of the system when $N\to\infty$.

Let the $Nn\times Nn$ identity matrix be partitioned in the form:

$$I_{Nn}=\begin{bmatrix}I_n & 0 & \cdots & 0\\ 0 & I_n & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & I_n\end{bmatrix}.$$

For $1\le i\ne j\le N$, exchanging the ith and jth rows of submatrices in I_{Nn}, let J_{ij} denote the resulting matrix. For instance, we have

$$J_{12}=\begin{bmatrix}0 & I_n & \cdots & 0\\ I_n & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & I_n\end{bmatrix}.$$

It is easy to check that $J_{ij}^T=J_{ij}^{-1}=J_{ij}$.

Theorem 3: We assume that (10) has a solution $(\boldsymbol{P}_1(t),\cdots,\boldsymbol{P}_N(t))$ on $[0,T]$. Then the following holds.

i) $\boldsymbol{P}_1(t)$ has the representation

$$\boldsymbol{P}_1(t)=\begin{bmatrix}\Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t)\\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t)\\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t)\end{bmatrix}, \tag{14}$$

where Π_1 and Π_3 are $n\times n$ symmetric matrices.

ii) For $i>1$, $\boldsymbol{P}_i(t)=J_{1i}^T\boldsymbol{P}_1(t)J_{1i}$.

Proof: See Appendix A. □

By Theorem 3, (13) is equivalent to the following condition:

$$\sup_{N\ge N_0,0\le t\le T}\big(|\Pi_1(t)|+N|\Pi_2(t)|+N^2|\Pi_3(t)|\big)<\infty. \tag{15}$$

We present some continuous dependence result of parametrized ODEs in Theorem 4 below. This will play a key role in establishing Theorem 5 later. Consider

$$\dot{x}=f(t,x),\quad x(0)=z\in\mathbb{R}^K, \tag{16}$$

$$\dot{y}=f(t,y)+g(\varepsilon,t,y), \tag{17}$$

where $y(0)=z_\varepsilon\in\mathbb{R}^K$, $0<\varepsilon\le 1$.

Let $\phi(t,x)=f(t,x)$, or $f(t,x)+g(\varepsilon,t,x)$. We introduce the following assumptions on (16) and (17).

(A1) $\sup_{\varepsilon,0\le t\le T}|f(t,0)|+|g(\varepsilon,t,0)|\le C_1$.

(A2) $\phi(\cdot,x)$ is Lebesgue measurable for each fixed $x\in\mathbb{R}^K$.

(A3) For each $t\in[0,T]$, $\phi(t,x):\mathbb{R}^K\to\mathbb{R}^K$ is locally Lipschitz in x, uniformly with respect to (t,ε), i.e., for any fixed $r>0$, and $x,y\in B_r(0)$ which is the open ball of radius r centering 0,

$$|\phi(t,x)-\phi(t,y)|\le \mathrm{Lip}(r)|x-y|,$$

where $\mathrm{Lip}(r)$ depends only on r, not on $\varepsilon\in(0,1],t\in[0,T]$.

(A4) $\lim_{\varepsilon\to 0}|z_\varepsilon-z|=0$, and for each fixed $r>0$,

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T,y\in B_r(0)}|g(\varepsilon,t,y)|=0.$$

If the solutions to (16) and (17), denoted by $x^z(t)$ and $y^\varepsilon(t)$, exist on $[0,T]$, they are unique by the local Lipschitz condition (A3); in this case denote $\delta_\varepsilon=\int_0^T|g(\varepsilon,\tau,x^z(\tau))|d\tau$, which converges to 0 as $\varepsilon\to 0$ due to (A4).

Theorem 4: Under Assumptions (A1)–(A4), we have the following assertions:

i) If (16) has a solution $x^z(t)$ on $[0,T]$, then there exists $0<\bar{\varepsilon}\le 1$ such that for all $0<\varepsilon\le\bar{\varepsilon}$, (17) has a solution $y^\varepsilon(t)$ on $[0,T]$ and

$$\sup_{0\le t\le T}|y^\varepsilon(t)-x^z(t)|=O(|z_\varepsilon-z|+\delta_\varepsilon). \tag{18}$$

ii) Suppose there exists a sequence $\{\varepsilon_i,i\ge 1\}$ where $0<\varepsilon_i\le 1$ and $\lim_{i\to\infty}\varepsilon_i=0$ such that (17) with $\varepsilon=\varepsilon_i$ has a solution y^{ε_i} on $[0,T]$ and $\sup_{i\ge 1,0\le t\le T}|y^{\varepsilon_i}(t)|\le C_2$ for some constant C_2. Then (16) has a solution on $[0,T]$.

Proof: See Appendix B. □

Remark 3: If (16) and (17) are replaced by matrix ODEs and (or) a terminal condition at T is used in each equation, the results in Theorem 4 still hold.

Let

$$M = BR^{-1}B^T.$$

Before presenting further results, we introduce two Riccati ODEs:

$$\begin{cases}\dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\ \Lambda_1(T) = Q_f,\end{cases} \tag{19}$$

and

$$\begin{cases}\dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\ \qquad -(\Lambda_1 G + \Lambda_2(A+G) + A^T \Lambda_2) + Q\Gamma, \\ \Lambda_2(T) = -Q_f \Gamma_f.\end{cases} \tag{20}$$

Note that (19) is the standard Riccati ODE in LQ optimal control and has a unique solution Λ_1 on $[0,T]$. Equation (20) is a non-symmetric Riccati ODE, where Λ_1 is now treated as a known function. We state the main theorem on asymptotic solvability.

Theorem 5: The sequence of games in (1)–(2) has asymptotic solvability if and only if (20) has a unique solution on $[0,T]$.

Proof: See Appendix C. □

We outline the key idea for identifying this necessary and sufficient condition of asymptotic solvability. By Theorem 3 and the ODE of $\boldsymbol{P}_1(t)$ in (10), we obtain an ODE system of the form

$$\begin{bmatrix}\dot{\Pi}_1 \\ \dot{\Pi}_2 \\ \dot{\Pi}_3\end{bmatrix} = \Psi_N(\Pi_1, \Pi_2, \Pi_3).$$

However, directly taking $N \to \infty$ is not useful because this method on one hand will not generate a meaningful limit of the vector field Ψ_N owing to terms such as $(N-1)\Pi_2 M \Pi_2$ in Ψ_N (see (A.3)) and on the other will cause a loss of dynamical information since (Π_2, Π_3) can vanish when $N \to \infty$. Our method is to re-scale by defining

$$\Lambda_1^N = \Pi_1(t),\ \Lambda_2^N = N\Pi_2(t),\ \Lambda_3^N = N^2 \Pi_3(t), \tag{21}$$

and examine their ODE system. This procedure leads to a new limiting ODE system which can preserve key information about the dynamics of (Π_1, Π_2, Π_3) and which consists of (19) and (20) together with another equation:

$$\begin{cases}\dot{\Lambda}_3 = \Lambda_2^T M \Lambda_2 + \Lambda_3 M \Lambda_1 + \Lambda_1 M \Lambda_3 + \Lambda_3 M \Lambda_2 + \Lambda_2^T M \Lambda_3 \\ \qquad - \left(\Lambda_2^T G + G^T \Lambda_2 + \Lambda_3(A+G) + (A^T + G^T)\Lambda_3\right) \\ \qquad - \Gamma^T Q \Gamma, \\ \Lambda_3(T) = \Gamma_f^T Q_f \Gamma_f.\end{cases} \tag{22}$$

Note that after (19) and (20) are solved on $[0,T]$ (or otherwise on a maximal existence interval for the latter), (22) becomes a linear ODE.

Theorem 6: Suppose (20) has a solution on $[0,T]$. Then we have

$$\sup_{0\le t\le T} (|\Pi_1 - \Lambda_1| + |N\Pi_2 - \Lambda_2| + |N^2\Pi_3 - \Lambda_3|) = O(1/N).$$

Proof: The bound follows from Theorem 4 i) by use of g_1, g_2, g_3 and the terminal conditions which appear in the equations of Λ_1^N, Λ_2^N, Λ_3^N in Appendix C. □

A. Decentralized Control

Proposition 7: Assume that (10) has a solution $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ on $[0,T]$. Then the assertions hold:

i) $\boldsymbol{S}_i(t)$ in (11) has the form

$$\boldsymbol{S}_i(t) = [\theta_2^T(t), \cdots, \theta_1^T(t), \cdots, \theta_2^T(t)]^T, \tag{23}$$

in which the ith sub-vector is $\theta_1(t) \in \mathbb{R}^n$ and the remaining sub-vectors are $\theta_2(t) \in \mathbb{R}^n$.

ii) Furthermore, $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \cdots = \boldsymbol{r}_N$ for $t \in [0,T]$.

Proof: See Appendix C. □

We introduce two ODEs:

$$\begin{cases}\dot{\chi}_1(t) = (\Lambda_1 M + \Lambda_2 M - A^T)\chi_1 + Q\eta, \\ \chi_1(T) = -Q_f \eta_f,\end{cases} \tag{24}$$

and

$$\begin{cases}\dot{\chi}_2(t) = ((\Lambda_2^T + \Lambda_3)M - G^T)\chi_1 \\ \qquad + ((\Lambda_1 + \Lambda_2^T)M - (A^T + G^T))\chi_2 - \Gamma^T Q\eta, \\ \chi_2(T) = \Gamma_f^T Q_f \eta_f.\end{cases} \tag{25}$$

Define

$$\chi_1^N(t) = \theta_1(t), \quad \chi_2^N(t) = N\theta_2(t). \tag{26}$$

In fact (24) and (25) can be derived as the limit of the ODEs satisfied by (χ_1^N, χ_2^N); see Appendix C.

Proposition 8: For $(\theta_1(t), \theta_2(t))$ specified in (23), we have

$$\sup_{0\le t\le T} (|\theta_1(t) - \chi_1(t)| + |N\theta_2(t) - \chi_2(t)|) = O(1/N). \tag{27}$$

Proof: See Appendix C. □

By Theorem 1, the strategy of player $\mathscr{A}_i$ is

$$u_i = -R^{-1}B^T\Big(\Pi_1(t)X_i + \Pi_2(t)\sum_{j\ne i} X_j + \theta_1(t)\Big). \tag{28}$$

The closed-loop equation of X_i is now given by

$$dX_i(t) = \Big(AX_i - M\Big(\Pi_1 X_i + \Pi_2 \sum_{j\ne i} X_j + \theta_1\Big) + GX^{(N)}\Big)dt + DdW_i,$$

which gives

$$dX^{(N)} = [(A - M(\Pi_1 + (N-1)\Pi_2) + G)X^{(N)} - M\theta_1]dt + \frac{D}{N}\sum_{i=1}^N dW_i. \tag{29}$$

To denote the limit of (29) when $N \to \infty$, we introduce the closed-loop mean field dynamics

$$\frac{d\bar{X}}{dt} = (A - M(\Lambda_1 + \Lambda_2) + G)\bar{X} - M\chi_1(t), \tag{30}$$

where $\bar{X}(0) = x_0$.

Proposition 9: Suppose $E\sup_{i\ge1}|X_i(0)|^2 \le C$ for some fixed constant C and $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N EX_i(0) = x_0$. Then

$$\sup_{0\le t\le T} E|X^{(N)}(t) - \bar{X}(t)|^2 = O(|\frac{1}{N}\sum_{i=1}^N EX_i(0) - x_0|^2 + 1/N).$$

Proof: By (29)–(30), we find the explicit expression of $X^{(N)}(t)-\bar{X}(t)$. The proposition follows from elementary estimates by use of Theorem 6 and Proposition 8. $\square$

When $N\to\infty$, from (28) we obtain the control law

$$u_i^d=-R^{-1}B^T\left(\Lambda_1X_i+\Lambda_2\bar{X}+\chi_1(t)\right), \tag{31}$$

which is decentralized since $\bar{X}$ and χ_1 do not depend on the sample path information of other players and can be computed off-line. Suppose Λ_1 and Λ_2 have been given on $[0,T]$. Then (31) can be determined by solving the decoupled ODE system (24) and (30), which has a unique solution. Note that (24) has its origin in dynamic programming.

IV. Relation to the Fixed Point Approach

The fixed point approach for solving the LQ mean field game consists of two steps (see e.g. [28]).

Step 1. We use $\overline{X}\in C([0,T],\mathbb{R}^n)$ to approximate $X^{(N)}$ in (1) and consider the optimal control problem with dynamics and cost:

$$dX_i^\infty(t)=(AX_i^\infty(t)+Bu_i(t)+G\overline{X}(t))dt+DdW_i, \tag{32}$$

$$\begin{aligned}\bar{J}_i(u_i)=&\,E\int_0^T\left(|X_i^\infty-\Gamma\overline{X}-\eta|_Q^2+u_i^TRu_i\right)dt\\&+E|X_i^\infty(T)-\Gamma_f\overline{X}(T)-\eta_f|_{Q_f}^2,\end{aligned}$$

where we set $X_i^\infty(0)=X_i(0)$. The Brownian motion is the same as in (1). Applying dynamic programming, the optimal control law is given by

$$\hat{u}_i=-R^{-1}B^T(\Lambda_1X_i^\infty(t)+s(t)), \tag{33}$$

where Λ_1 is solved from (19) and

$$\dot{s}(t)=-(A^T-\Lambda_1M)s(t)-\Lambda_1G\overline{X}(t)+Q(\Gamma\overline{X}(t)+\eta),$$

and $s(T)=-Q_f(\Gamma_f\overline{X}(T)+\eta_f)$.

Step 2. Let $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^NEX_i^\infty(t)$ be determined from the closed-loop system of (32) under the control law $\hat{u}_i$ and the given $\overline{X}$. By the standard consistency requirement in mean field games [28], we impose $\overline{X}(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^NEX_i^\infty(t)$ for all $t\in[0,T]$, which amounts to specifying $\overline{X}$ as a fixed point. This introduces the equation

$$\frac{d\overline{X}}{dt}=(A-M\Lambda_1+G)\overline{X}-Ms,$$

where $\overline{X}(0)=x_0$ and we assume $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^NEX_i(0)=x_0$ as in Section III.

Combining the ODEs of s and $\overline{X}$ gives the MFG solution equation system

$$\begin{cases}\frac{d\overline{X}}{dt}=(A-M\Lambda_1+G)\overline{X}-Ms,\\ \dot{s}=-(A^T-\Lambda_1M)s-\Lambda_1G\overline{X}+Q(\Gamma\overline{X}+\eta),\end{cases} \tag{34}$$

where $\overline{X}(0)=x_0$ and $s(T)=-Q_f(\Gamma_f\overline{X}(T)+\eta_f)$. The equation system (34) is a TPBV problem.

Remark 4: We introduce in (34) the new notation $\overline{X}$ instead of $\bar{X}$. It is necessary to maintain this distinction since the two functions coincide only under certain conditions as shown later.

A. Solving the TPBV Problem

Denote

$$\mathbb{A}(t)=\begin{bmatrix}A-M\Lambda_1(t)+G & -M\\ Q\Gamma-\Lambda_1(t)G & -A^T+\Lambda_1(t)M\end{bmatrix}. \tag{35}$$

The fundamental solution matrix of (34) is determined by the matrix ODE

$$\frac{\partial}{\partial t}\Phi(t,\tau)=\mathbb{A}\Phi(t,\tau),\quad \Phi(\tau,\tau)=I_{2n}. \tag{36}$$

Denote

$$\Phi(t,\tau)=\begin{bmatrix}\Phi_{11}(t,\tau) & \Phi_{12}(t,\tau)\\ \Phi_{21}(t,\tau) & \Phi_{22}(t,\tau)\end{bmatrix}, \tag{37}$$

where each submatrix is $n\times n$.

Denote

$$\begin{aligned}Z_1&=\Phi_{22}(T,0)+Q_f\Gamma_f\Phi_{12}(T,0)\in\mathbb{R}^{n\times n},\\ Z_2&=[\Phi_{21}(T,0)+Q_f\Gamma_f\Phi_{11}(T,0)]x_0+Q_f\eta_f\\&\quad+\int_0^T[\Phi_{22}(T,\tau)+Q_f\Gamma_f\Phi_{12}(T,\tau)]Q\eta d\tau\in\mathbb{R}^n.\end{aligned}$$

Proposition 10: i) (34) has a solution if and only if

$$Z_2\in\operatorname{span}\{Z_1\}. \tag{38}$$

ii) If $\det\big(Z_1\big)\neq0$, (34) has a unique solution.

Proof: i) We introduce $s(0)$ to be determined. By (34),

$$\begin{aligned}\begin{bmatrix}\overline{X}(T)\\ s(T)\end{bmatrix}&=\Phi(T,0)\begin{bmatrix}x_0\\ s(0)\end{bmatrix}+\int_0^T\Phi(T,\tau)\begin{bmatrix}0\\ Q\eta\end{bmatrix}d\tau\\&=\begin{bmatrix}\Phi_{11}(T,0) & \Phi_{12}(T,0)\\ \Phi_{21}(T,0) & \Phi_{22}(T,0)\end{bmatrix}\begin{bmatrix}x_0\\ s(0)\end{bmatrix}\\&\quad+\int_0^T\begin{bmatrix}\Phi_{11}(T,\tau) & \Phi_{12}(T,\tau)\\ \Phi_{21}(T,\tau) & \Phi_{22}(T,\tau)\end{bmatrix}\begin{bmatrix}0\\ Q\eta\end{bmatrix}d\tau.\end{aligned}$$

Then (34) has a solution if and only if there exists $s(0)$ such that

$$\begin{aligned}s(T)&=\Phi_{21}(T,0)x_0+\Phi_{22}(T,0)s(0)+\int_0^T\Phi_{22}(T,\tau)Q\eta d\tau,\\&=-Q_f(\Gamma_f\overline{X}(T)+\eta_f),\end{aligned} \tag{39}$$

which is equivalent to $Z_1s(0)+Z_2=0$. This proves part i).

ii) This part follows from part i). $\square$

For illustration, we consider the special case with $\Gamma_f=0$, $\eta_f=0$. Then (38) in Proposition 10 i) becomes

$$\Phi_{21}(T,0)x_0+\int_0^T\Phi_{22}(T,\tau)Q\eta d\tau\in\operatorname{span}\big\{\Phi_{22}(T,0)\big\}.$$

B. Direct Approach Solvability Implies Fixed Point Solvability

Theorem 11: Suppose Λ_2 has a solution on $[0,T]$. Then the following holds.

i) (34) has a unique solution $(\overline{X},s)$ given by

$$\begin{cases}\overline{X}(t)=\bar{X}(t),\\ s(t)=\Lambda_2(t)\bar{X}(t)+\chi_1(t),\end{cases}$$

where $(\bar{X},\chi_1)$ is solved from (24) and (30) in the direct approach.

ii) Asymptotic solvability of the sequence of games (1)–(2) implies that (34) has a unique solution.

Proof: i) For (34), we write

$$s = \Lambda_2 \overline{X} + \varphi(t), \tag{40}$$

where φ is a new unknown function. Now (34) is transformed into a new equation system in terms of $(\overline{X}, \varphi)$, where

$$\dot{\varphi} = (\Lambda_1 M + \Lambda_2 M - A^T)\varphi + Q\eta, \quad \varphi(T) = -Q_f \eta_f.$$

The terminal condition $\varphi(T)$ has been determined from (40) with $t = T$. We can uniquely solve φ and in fact $\varphi = \chi_1$. Subsequently, we further obtain $\overline{X} = \bar{X}$. It is clear the solution $(\overline{X}, s)$ is unique.

ii) This part follows from Theorem 5 and part i). □

Let (33) be applied by the N players in (1), and accordingly denote

$$\hat{u}_i^d = -R^{-1}B^T(\Lambda_1 X_i(t) + s(t)). \tag{41}$$

Under the asymptotic solvability condition, the two control laws $\hat{u}_i^d$ in (41) and u_i^d in (31) are equivalent by Theorem 11. Based on assumptions on the initial states as given in Proposition 9, one can apply the standard method in [28] to show that the set of strategies $(\hat{u}_1^d, \cdots, \hat{u}_N^d)$ in (41) is an ε-Nash equilibrium of the N-player game, where $\varepsilon \to 0$ as $N \to \infty$.

The existence and uniqueness condition in the TPBV problem is quite different from the condition for asymptotic solvability. It is possible that the Riccati equation of Λ_2 has a finite escape time in $[0, T)$ but the TPBV problem is still solvable. A detailed comparison will be developed in the next section for scalar models.

C. Fixed Point via A Contraction Mapping

The original analysis in [28] applies the fixed point approach to infinite time horizon LQ mean field games and establishes existence and uniqueness of a solution by specifying a contraction mapping. The procedure in [28] can be applied to (34) to derive a corresponding contraction condition as well. By Theorem 11, asymptotic solvability in the direct approach implies the fixed point solvability, but the converse may not hold (and is indeed not true as it turns out later). Now if the fixed point is determined from a contraction mapping as in [28], an intriguing question is what is its implication regarding asymptotic solvability. Below we show asymptotic solvability holds in this case.

To facilitate further analysis, we consider (34) on a general interval $[t_0, T]$ for $t_0 \in [0, T)$, and rewrite it as below:

$$\begin{cases} \frac{d\overline{X}}{dt} = (A - M\Lambda_1 + G)\overline{X} - Ms, \\ \dot{s} = -(A^T - \Lambda_1 M)s - \Lambda_1 G\overline{X} + Q(\Gamma\overline{X} + \eta). \end{cases} \tag{42}$$

The initial and terminal conditions are given by $\overline{X}(t_0) = x_{t_0}$ and $s(T) = -Q_f(\Gamma_f \overline{X}(T) + \eta_f)$.

Denote the linear ODEs

$$\dot{y}_1 = (A - M\Lambda_1(t) + G)y_1, \quad \dot{y}_2 = (-A^T + \Lambda_1(t)M)y_2,$$

where $t \in [0, T]$ and $y_i(t) \in \mathbb{R}^n$. Let Ψ_1 and Ψ_2 be their fundamental solution matrices so that

$$\frac{\partial \Psi_1(t,\tau)}{\partial t} = (A - M\Lambda_1(t) + G)\Psi_1(t,\tau), \quad \Psi_1(\tau,\tau) = I,$$
$$\frac{\partial \Psi_2(t,\tau)}{\partial t} = (-A^T + \Lambda_1(t)M)\Psi_2(t,\tau), \quad \Psi_2(\tau,\tau) = I.$$

Following the procedure in [28], we solve s from the second equation of (42) to obtain

$$\begin{aligned} s(t) = &-\Psi_2(t,T)Q_f\Gamma_f\overline{X}(T) \\ &- \int_t^T \Psi_2(t,r)(Q\Gamma - \Lambda_1(r)G)\overline{X}(r)dr + \zeta_1(t), \end{aligned} \tag{43}$$

where ζ_1 depends on (η, η_f) but not on $\overline{X}$. Substituting (43) into the first equation of (42), we have the expression

$$\begin{aligned} \overline{X}(t) = &\Psi_1(t,t_0)x_{t_0} + \int_{t_0}^t \Psi_1(t,\tau)M\Psi_2(\tau,T)Q_f\Gamma_f\overline{X}(T)d\tau \\ &+ \int_{t_0}^t \int_\tau^T \Psi_1(t,\tau)M\Psi_2(\tau,r)(Q\Gamma - \Lambda_1(r)G)\overline{X}(r)drd\tau \\ &+ \zeta_2(t), \end{aligned} \tag{44}$$

where ζ_2 depends on (η, η_f) but not on $(\overline{X}, x_{t_0})$. Denote the operator Υ_{t_0}: $C([t_0,T],\mathbb{R}^n) \to C([t_0,T],\mathbb{R}^n)$ as follows:

$$\begin{aligned} (\Upsilon_{t_0}\phi)(t) = &\int_{t_0}^t \Psi_1(t,\tau)M\Psi_2(\tau,T)Q_f\Gamma_f\phi(T)d\tau \\ &+ \int_{t_0}^t \int_\tau^T \Psi_1(t,\tau)M\Psi_2(\tau,r)(Q\Gamma - \Lambda_1(r)G)\phi(r)drd\tau. \end{aligned}$$

We take the norm $\|\phi\| = \sup_{t\in[t_0,T]} |\phi(t)|$ in $C([t_0,T],\mathbb{R}^n)$. Now (44) can be written as

$$\overline{X}(t) = \Psi_1(t,t_0)x_{t_0} + (\Upsilon_{t_0}\overline{X})(t) + \zeta_2(t), \qquad t \in [t_0, T].$$

Denote the constant

$$\begin{aligned} \kappa_{t_0} = \sup_{t_0 \le t \le T} \Big[&\int_{t_0}^t \int_\tau^T |\Psi_1(t,\tau)M\Psi_2(\tau,r)[Q\Gamma - \Lambda_1(r)G]| drd\tau \\ &+ \int_{t_0}^t |\Psi_1(t,\tau)M\Psi_2(\tau,T)Q_f\Gamma_f| d\tau \Big]. \end{aligned}$$

We have the estimate

$$\|\Upsilon_{t_0}\phi_1 - \Upsilon_{t_0}\phi_2\| \le \kappa_{t_0}\|\phi_1 - \phi_2\|, \quad \forall\ \phi_1, \phi_2 \in C([t_0,T],\mathbb{R}^n).$$

It is straightforward to check that $\kappa_{t_0} \le \kappa_0$ for all $t_0 \in [0, T]$.

Theorem 12: Suppose $\kappa_0 < 1$. Then asymptotic solvability holds for the sequence of games (1)–(2).

Proof: We prove by contradiction. Suppose asymptotic solvability doe not hold for (1)–(2), which implies Λ_2 has a maximal existence interval $(t^*, T]$ for $t^* \in [0, T)$. So there exists a strictly decreasing sequence $\{t_k, k \ge 1\}$ converging to t^* such that $\lim_{k\to\infty} |\Lambda_2(t_k)| = \infty$. We can find an appropriate subsequence, still denoted by $\{t_k, k \ge 1\}$, such that for some $(\hat{i}, \hat{j})$, we have

$$\lim_{k\to\infty} |\Lambda_2^{\hat{i},\hat{j}}(t_k)| = \infty, \quad |\Lambda_2^{\hat{i},\hat{j}}(t_k)| = \max_{1\le i,j\le n} |\Lambda_2^{i,j}(t_k)|, \tag{45}$$

where the superscripts indicate the (i, j)-th entry of $\Lambda_2(t_k)$.

Now for t_k in (45), we select $x_{t_k} = [0, \ldots, 0, 1, 0, \ldots, 0]^T = e_{\hat{j}}^n$, and solve a special form of (42) on $[t_k, T]$ as follows:

$$\begin{cases} \frac{d\overline{X}^*}{dt} = (A - M\Lambda_1 + G)\overline{X}^* - Ms^*, \\ \dot{s}^* = -(A^T - \Lambda_1 M)s^* + (Q\Gamma - \Lambda_1 G)\overline{X}^*, \end{cases} \tag{46}$$

which has initial condition $\overline{X}^*(t_k) = x_{t_k} = e_{\hat{j}}^n$ and terminal condition $s^*(T) = -Q_f\Gamma_f\overline{X}^*(T)$. By the relation

$$\overline{X}^*(t) = \Psi_1(t,t_k)x_{t_k} + (\Upsilon_{t_k}\overline{X}^*)(t), \qquad t \in [t_k, T]$$

and $\kappa_{t_k} \le \kappa_0$, we obtain a unique solution $(\overline{X}^*, s^*) \in C([t_k,T],\mathbb{R}^{2n})$ for (46) and have the bound

$$\|\overline{X}^*\| \le \frac{1}{1-\kappa_0} \sup_{t,\tau\in[0,T]} |\Psi_1(t,\tau)|.$$

In parallel to (43),

$$\begin{aligned} s^*(t) = &-\Psi_2(t,T)Q_f\Gamma_f\overline{X}^*(T) \\ &-\int_t^T \Psi_2(t,r)(Q\Gamma - \Lambda_1(r)G)\overline{X}^*(r)dr. \end{aligned}$$

We may further find a fixed constant C_0 independent of t_k such that

$$\sup_{t\in[t_k,T]} [|\overline{X}^*(t)| + |s^*(t)|] \le C_0. \tag{47}$$

On the other hand, for each t_k appearing in (45) and the resulting interval $[t_k,T]$, by the fact that Λ_2 exists on $(t^*,T] \supset [t_k,T]$, we may use the method in Theorem 11 to show the relation

$$s^*(t) = \Lambda_2(t)\overline{X}^*(t), \qquad t\in[t_k,T].$$

Hence $s^*(t_k) = \Lambda_2(t_k)e^n_j$, and by (45),

$$\lim_{k\to\infty} |s^*(t_k)| = \infty,$$

which contradicts (47).

We conclude that Λ_2 has a solution on $[0,T]$. Therefore, asymptotic solvability holds for (1)–(2). □

Remark 5: We use $\kappa_0 < 1$ to ensure a contraction condition for the TPBV problem defined on $[0,T]$. It is possible to have improved contraction estimates. Our method here is adequate for addressing the qualitative relation as shown in Theorem 12.

V. The Scalar Case: Explicit Solutions

A. Riccati Equations of Asymptotic Solvability

We analyze a scalar case of the Riccati ODEs (19) and (20), i.e., $n=1$, and suppose $B\neq 0$ for the model to be nontrivial. Consider

$$\begin{cases} \dot\Lambda_1 = \Lambda_1^2 - 2A\Lambda_1 - Q, & (48)\\ \dot\Lambda_2 = 2\Lambda_1\Lambda_2 + \Lambda_2^2 - \Lambda_2(2A+G) - \Lambda_1 G + Q\Gamma, & (49)\end{cases}$$

where $\Lambda_1(T) = Q_f$ and $\Lambda_2(T) = -Q_f\Gamma_f$. Without loss of generality we only deal with the case $M=1$ since otherwise a change of variable may be used to convert (19)–(20) to the above form with appropriately modified parameters Q and Q_f. Although $\Lambda_1(t)$ can be explicitly solved for a general Q_f, one usually cannot further solve $\Lambda_2(t)$ in a closed form. To overcome this difficulty, we will further take particular choices of the terminal conditions to obtain explicit solutions. Our method is to choose Q_f appropriately to solve $\Lambda_1(t)$ as a constant so that (49) becomes a Riccati equation with constant coefficients.

In this section we further suppose the pair $(A,\sqrt{Q})$ is detectable. Denote the algebraic Riccati equation

$$\Lambda_{1\infty}^2 - 2A\Lambda_{1\infty} - Q = 0,$$

which gives the stabilizing solution

$$\Lambda_{1\infty} = A + \sqrt{A^2+Q} \ge 0, \tag{50}$$

such that $A - M\Lambda_{1\infty} = A - \Lambda_{1\infty} < 0$. Below we take

$$M=1, \quad Q_f = \Lambda_{1\infty}, \quad \Gamma_f = 0. \tag{51}$$

Then (48) has a constant solution $\Lambda_1(t)\equiv\Lambda_{1\infty}$, and (49) becomes

$$\dot\Lambda_2 = 2\hat a\Lambda_2 + \Lambda_2^2 + \hat Q, \quad \Lambda_2(T) = 0, \tag{52}$$

where

$$\hat a = \sqrt{A^2+Q} - \frac{G}{2}, \quad \hat Q = Q\Gamma - \big(A + \sqrt{A^2+Q}\big)G. \tag{53}$$

To solve (52), let $\Lambda_2 = -\frac{u'}{u}$. Then (52) leads to

$$u'' - 2\hat a u' + \hat Q u = 0. \tag{54}$$

Denote

$$\hat\Delta = \hat a^2 - \hat Q = \frac14(2A+G)^2 + Q(1-\Gamma).$$

Proposition 13: The Riccati ODE (52) has a unique solution on $[0,T]$ for all $T>0$ under either of the two conditions: i) $\hat Q\le 0$; ii) $0<\hat Q\le \hat a^2$ and $\hat a>0$.

Proof: See Appendix D. □

Proposition 14: i) If $0<\hat Q\le\hat a^2$ and $\hat a<0$, the solution of (52) is given by

$$\Lambda_2(t) = \begin{cases} \frac{\hat Q\left(e^{\alpha(T-t)} - e^{-\alpha(T-t)}\right)}{\hat\lambda_2 e^{-\alpha(T-t)} - \hat\lambda_1 e^{\alpha(T-t)}}, & \text{if } \hat Q < \hat a^2,\\ \frac{\hat a^2(T-t)}{\hat a(t-T)-1}, & \text{if } \hat Q = \hat a^2,\end{cases}$$

where $\alpha = \sqrt{\hat\Delta}$ and $\hat\lambda_1 = \hat a+\alpha$, $\hat\lambda_2 = \hat a - \alpha$ are solutions to the characteristic equation of (54).

ii) If $\hat Q > \hat a^2$, then

$$\Lambda_2(t) = \frac{\sqrt{\hat Q}\sin\beta(t-T)}{\sin\big(\beta(T-t)+\theta\big)}, \tag{55}$$

where $\beta = \sqrt{\hat Q - \hat a^2} > 0$ and $\hat a + \beta i = \sqrt{\hat Q}e^{i\theta}$ for $\theta\in(0,\pi)$.

Proof: See Appendix D. □

Remark 6: The assumptions in the four cases in Propositions 13 and 14 are categorized according to the distribution of the two eigenvalues of the characteristic equation of (54).

Remark 7: Depending on the value of T, the solutions in both i) and ii) of Proposition 14 may have a maximal existence internal as a proper subset of $[0,T]$.

Remark 8: If

$$0<\hat Q<\hat a^2, \quad \hat a < 0, \tag{56}$$

by Proposition 14, Λ_2 has a finite escape time $\hat t\in[0,T)$ satisfying $T - \hat t = \check T := \frac{\ln(\hat\lambda_2/\hat\lambda_1)}{2\alpha}$ if $\check T\in(0,T]$.

Example 1: Consider the system with

$$A = -\frac14, \quad G = \frac45, \quad Q = \frac{1}{16}, \quad \Gamma = \frac43.$$

It can be verified that the system satisfies (56).

The parameters in Example 1 are constructed by first fixing A and Q, and next searching for (G,Γ) subject to the two constraints in (56).

B. The TPBV Problem and Non-uniqueness

For the scalar case $n=1$, we take $M=1$ and $Q_f=\Lambda_{1\infty}$ so that $\Lambda_1=\Lambda_{1\infty}$. Then (35) reduces to the form

$$\mathbb{A}_\infty=\begin{bmatrix} G-\sqrt{A^2+Q} & -1 \\ Q\Gamma-(A+\sqrt{A^2+Q})G & \sqrt{A^2+Q}\end{bmatrix}\in\mathbb{R}^{2\times 2}, \quad (57)$$

which has the characteristic polynomial

$$|\lambda I-\mathbb{A}_\infty|=\lambda^2-G\lambda+Q\Gamma-(A^2+Q+AG).$$

Note that for the TPBV problem (34) in the fixed point approach to have multiple solutions, a necessary condition is that asymptotic solvability fails by Theorem 11. For constructing non-uniqueness results, below we largely impose conditions in Proposition 14 i). If

$$\hat{\Delta}>0, \quad (58)$$

$|\lambda I-\mathbb{A}_\infty|=0$ has the real-valued solutions

$$\lambda_1=\frac{G}{2}+\sqrt{\hat{\Delta}},\quad \lambda_2=\frac{G}{2}-\sqrt{\hat{\Delta}}. \quad (59)$$

Restricting our attention to two distinct real roots will streamline the presentation in constructing non-uniqueness examples. Under (58), denote

$$c_1=-\hat{a}-\sqrt{\hat{\Delta}},\quad c_2=-\hat{a}+\sqrt{\hat{\Delta}}.$$

$\mathbb{A}_\infty$ has two eigenvectors

$$v_k=[1,c_k]^T,\qquad k=1,2.$$

corresponding to the eigenvalues λ_1 and λ_2, respectively. Now for (36), we have

$$\Phi(t,\tau)=[v_1,v_2]\begin{bmatrix} e^{\lambda_1(t-\tau)} & \\ & e^{\lambda_2(t-\tau)}\end{bmatrix}[v_1,v_2]^{-1}$$

as a 2×2 matrix function. We further calculate

$$\Phi_{21}(t,\tau)=\frac{c_1c_2(e^{\lambda_1(t-\tau)}-e^{\lambda_2(t-\tau)})}{2\sqrt{\hat{\Delta}}}, \quad (60)$$

$$\Phi_{22}(t,\tau)=\frac{c_2e^{\lambda_2(t-\tau)}-c_1e^{\lambda_1(t-\tau)}}{2\sqrt{\hat{\Delta}}}.$$

Given the parameters in (51), (34) becomes

$$\begin{cases}\frac{d\overline{X}}{dt}=(A-\Lambda_{1\infty}+G)\overline{X}-s,\\ \dot{s}=-(A^T-\Lambda_{1\infty})s-\Lambda_{1\infty}G\overline{X}+Q(\Gamma\overline{X}+\eta),\end{cases} \quad (61)$$

where $\overline{X}(0)=x_0$ and $s(T)=-\Lambda_{1\infty}\eta_f$.

In order to construct models with non-uniqueness results, here we treat T and x_0 in (61) as adjustable parameters.

Proposition 15: Assume $Q_f=\Lambda_{1\infty}$. If (56) holds, then $\Phi_{21}(T,0)\neq 0$ for all $T>0$ and there exists a unique $\hat{T}>0$ such that $\Phi_{22}(\hat{T},0)=0$.

Proof: It can be shown that (56) holds if and only if

$$\hat{\Delta}>0,\quad c_1>0, \quad (62)$$

which implies that

$$0<\lambda_2<\lambda_1,\quad 0<c_1<c_2,$$

where λ_1 and λ_2 are given by (59). It is clear that $\Phi_{21}(T,0)\neq 0$. Note that $\Phi_{22}(T,0)=0$ if and only if $c_2e^{\lambda_2 T}=c_1e^{\lambda_1 T}$, for which we uniquely solve $T=\hat{T}:=\frac{\ln(c_2/c_1)}{2\sqrt{\hat{\Delta}}}=\check{T}>0$; see Remark 8 for $\check{T}$. □

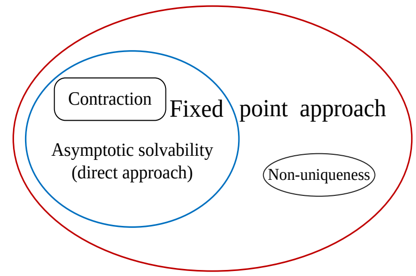

Fig. 2. Comparison of the two approaches

For constructing the TPBV problem below, we suppose the assumptions of Proposition 15 hold, and uniquely solve $\hat{x}_0$ from

$$\Phi_{21}(\hat{T},0)\hat{x}_0+\int_0^{\hat{T}}\Phi_{22}(\hat{T},\tau)Q\eta\, d\tau=-\Lambda_{1\infty}\eta_f \quad (63)$$

since $\Phi_{21}(\hat{T},0)\neq 0$. We calculate

$$\int_0^{\hat{T}}\Phi_{22}(\hat{T},\tau)d\tau=\frac{c_2\lambda_1(e^{\lambda_2\hat{T}}-1)-c_1\lambda_2(e^{\lambda_1\hat{T}}-1)}{2\sqrt{\hat{\Delta}}\lambda_1\lambda_2}.$$

Now for the scalar case with $M=1$, $Q_f=\Lambda_{1\infty}$, $\Gamma_f=0$ and $T=\hat{T}$, (61) specializes to the TPBV problem

$$\begin{cases}\frac{d\overline{X}}{dt}=(A-\Lambda_{1\infty}+G)\overline{X}-s,\\ \dot{s}(t)=-(A^T-\Lambda_{1\infty})s-\Lambda_{1\infty}G\overline{X}+Q(\Gamma\overline{X}+\eta),\end{cases} \quad (64)$$

where $\overline{X}(0)=\hat{x}_0$ and $s(\hat{T})=-\Lambda_{1\infty}\eta_f$.

Proposition 16: Assume (51) and (56) hold. Then a solution $(\overline{X},s)$ of (64) can be obtained by taking any initial condition $s(0)$. Therefore, (64) has an infinite number of solutions.

Proof: Recalling (39), (64) is solvable if and only if one can find $s(0)$ to satisfy

$$\Phi_{21}(\hat{T},0)\hat{x}_0+\Phi_{22}(\hat{T},0)s(0)+\int_0^{\hat{T}}\Phi_{22}(\hat{T},\tau)Q\eta\, d\tau$$
$$=-\Lambda_{1\infty}\eta_f. \quad (65)$$

By (63) and $\Phi_{22}(\hat{T},0)=0$, (65) holds for any choice of $s(0)$. □

C. Comparison of Two Approaches

Consider the system given by Example 1 with time horizon $[0,T]$. It satisfies (56). Then $\check{T}=\hat{T}$.

If we take $T\in(0,\hat{T})$, then asymptotic solvability holds and the TPBV problem (61) has a unique solution by Theorem 11.

If $T=\hat{T}$, then Λ_2 has a finite escape time at $t=0$ implying no asymptotic solvability. However, in this case the TPBV problem (64) has an infinite number of solutions, which in turn can be used to construct an infinite number of ε-Nash equilibria for the N-player game.

If $T>\hat{T}$, asymptotic solvability fails but (61) has a unique solution since $\Phi_{22}(T,0)\neq 0$ by Proposition 15.

Based on Theorems 11 and 12, and the comparison above, the relation between the two approaches is illustrated in Fig. 2. The rectangle region represents models satisfying the contraction condition $\kappa_0<1$ in Theorem 12.

VI. Long Time Behavior

For this section, we make the following assumption:

(H1) The pair (A,B) is stabilizable, and the pair $(A,Q^{\frac{1}{2}})$ is detectable.

Within the setup of continuous time dynamical systems, a matrix $Z\in\mathbb{R}^{k\times k}$ is called stable or Hurwitz if all its eigenvalues have a strictly negative real part.

A. Steady State Form of Riccati ODEs

For (19), we introduce the ARE

$$\Lambda_{1\infty}M\Lambda_{1\infty}-(\Lambda_{1\infty}A+A^T\Lambda_{1\infty})-Q=0.$$

Note that under (H1) there exists a unique solution $\Lambda_{1\infty}\geq 0$ from the class of positive semi-definite matrices. Corresponding to (20), we introduce the algebraic equation

$$\begin{aligned}0=\;&\Lambda_{1\infty}M\Lambda_{2\infty}+\Lambda_{2\infty}M\Lambda_{1\infty}+\Lambda_{2\infty}M\Lambda_{2\infty}\\&-(\Lambda_{1\infty}G+\Lambda_{2\infty}(A+G)+A^T\Lambda_{2\infty})+Q\Gamma,\end{aligned}\tag{66}$$

which is a non-symmetric algebraic Riccati equation (NARE). When (66) has a solution in $\mathbb{R}^{n\times n}$, it is possible that multiple such solutions exist. The question is how to determine a solution of interest, and this amounts to imposing appropriate constraints on the solution. For related methods on choosing a desirable solution of NAREs by fulfilling some stability conditions, see [33].

B. Stabilizing Solution

Suppose $\Lambda_{2\infty}\in\mathbb{R}^{n\times n}$ is a solution to (66). Denote

$$A_G=A-M(\Lambda_{1\infty}+\Lambda_{2\infty})+G,\tag{67}$$

$$A_M=A-M(\Lambda_{1\infty}+\Lambda_{2\infty}^T).\tag{68}$$

To motivate the restrictions to be imposed on $\Lambda_{2\infty}$, we examine the two ODEs (24) and (30), where the latter is the closed-loop mean field dynamics. We start by checking the stability of the solution of (24) when t is simply allowed to tend to $-\infty$. If $\Lambda_2(t)$ can converge to a limit $\Lambda_{2\infty}$ at all, it is well justified to study the stability of the limiting ODE

$$\begin{aligned}\dot\chi_1&=[(\Lambda_{1\infty}+\Lambda_{2\infty})M-A^T]\chi_1+Q\eta\\&=-A_M^T\chi_1+Q\eta,\qquad t\in(-\infty,T],\end{aligned}\tag{69}$$

which is constructed by replacing $(\Lambda_1(t),\Lambda_2(t))$ by $(\Lambda_{1\infty},\Lambda_{2\infty})$ in (24). The solution of (69) converges to a constant vector $\chi_{1\infty}$ as $t\to-\infty$ if A_M is Hurwitz. Thus the generation of stable long time behavior suggests we impose a stability condition on A_M. For (30) we similarly introduce a limiting ODE of the form

$$\begin{aligned}\frac{d\bar X}{dt}&=(A-M(\Lambda_{1\infty}+\Lambda_{2\infty})+G)\bar X-M\chi_{1\infty}\\&=A_G\bar X-M\chi_{1\infty},\qquad t\in[0,\infty),\end{aligned}\tag{70}$$

and further introduce a stability condition on A_G in order to have a stable solution.

Definition 17: $\Lambda_{2\infty}\in\mathbb{R}^{n\times n}$ is called a stabilizing solution of (66) if it satisfies (66) and both A_G and A_M are Hurwitz.

If $\Lambda_{2\infty}$ is a stabilizing solution, it has the interpretation as a locally stable equilibrium point of the Riccati ODE (20). We take a limiting form of (20) by replacing Λ_1 by $\Lambda_{1\infty}$ and for convenience of analysis next reverse time to obtain the new equation

$$\begin{aligned}\dot Y(t)=&-\Lambda_{1\infty}MY-YM\Lambda_{1\infty}-YMY\\&+(\Lambda_{1\infty}G+Y(A+G)+A^TY)-Q\Gamma,\quad t\geq 0,\end{aligned}\tag{71}$$

for which we take a general initial condition $Y(0)$. The linearized ODE for (71) around $\Lambda_{2\infty}$ is

$$\dot Z(t)=A_M^TZ+ZA_G,\qquad t\geq 0,$$

which is a Sylvester ODE with solution $Z(t)=e^{A_M^Tt}Z(0)e^{A_Gt}$. This ODE is asymptotically stable given any $Z(0)$ if the matrices A_G and A_M are Hurwitz.

We proceed to determine conditions for existence of a stabilizing solution. Denote

$$\mathbb{A}_\infty=\begin{bmatrix}A-M\Lambda_{1\infty}+G & -M\\ Q\Gamma-\Lambda_{1\infty}G & -A^T+\Lambda_{1\infty}M\end{bmatrix}\in\mathbb{R}^{2n\times 2n},\tag{72}$$

which may be viewed as a steady state form of $\mathbb{A}(t)$ in (35).

Let $\mathbb{A}_o\in\mathbb{R}^{k\times k}$ be any matrix. An l-dimensional subspace $\mathscr{V}$ of $\mathbb{R}^k$ is called an invariant subspace of $\mathbb{A}_o$ if $\mathbb{A}_o\mathscr{V}\subset\mathscr{V}$; in this case $\mathbb{A}_oV=VA_o$ for some $A_o\in\mathbb{R}^{l\times l}$ where $V\in\mathbb{R}^{k\times l}$ and $\mathrm{span}\{V\}=\mathscr{V}$. If A_o is Hurwitz, $\mathscr{V}$ is called a stable invariant subspace. Below we give some standard definitions related to structural properties of an invariant subspace (see e.g. [6], [35]). For $1\leq l<k$, an l-dimensional invariant subspace $\mathscr{V}_g$ of $\mathbb{A}_o\in\mathbb{R}^{k\times k}$ is called a graph subspace if $\mathscr{V}_g$ is spanned by the columns of a $k\times l$ matrix whose leading $l\times l$ submatrix (i.e., its first l rows) is invertible. The k eigenvalues of $\mathbb{A}_o$ have a strong (k_1,k_2) c-splitting if the open left half plane and the open right half plane contain k_1 and k_2 eigenvalues, respectively, for $k_1\geq 1$, $k_2\geq 1$, $k_1+k_2=k$.

We introduce the following condition on $\mathbb{A}_\infty$:

(H_g) The eigenvalues of $\mathbb{A}_\infty$ are strong (n,n) c-splitting and the associated n-dimensional stable invariant subspace is a graph subspace.

Theorem 18: i) The NARE (66) has a stabilizing solution $\Lambda_{2\infty}$ if and only if (H_g) holds.

ii) If (H_g) holds, (66) has a unique stabilizing solution.

Proof: i) Step 1. To show necessity, suppose that $\Lambda_{2\infty}$ is a stabilizing solution. Denote

$$K=\begin{bmatrix}I_n & 0\\ \Lambda_{2\infty} & I_n\end{bmatrix}.$$

Since (66) holds, it can be checked that

$$K^{-1}\mathbb{A}_\infty K=\begin{bmatrix}A_G & -M\\ 0 & -A_M^T\end{bmatrix}.\tag{73}$$

By the definition of a stabilizing solution, A_G and A_M are Hurwitz. So $-A_M^T$ has all its eigenvalues in the open right half plane. Therefore, the eigenvalues of $\mathbb{A}_\infty$ have a strong (n,n) c-splitting. Now the columns of

$$\begin{bmatrix}I_n\\ \Lambda_{2\infty}\end{bmatrix}$$

span the n-dimensional stable invariant subspace of $\mathbb{A}_\infty$ as a graph subspace.

Step 2. We continue to show sufficiency. Suppose the columns of the matrix

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \in \mathbb{R}^{2n\times n} \tag{74}$$

spans the n-dimensional stable invariant subspace of $\mathbb{A}_\infty$, where U_1 is invertible. We take

$$\Lambda_{2\infty} = U_2 U_1^{-1}. \tag{75}$$

Then one can directly verify that $\Lambda_{2\infty}$ solves (66) (see e.g. [6, Corollary 2.2, pp. 34]), and (73) holds where A_M and A_G in (67)–(68) are determined by use of (75). Since A_G is associated with the stable invariant subspace, it is necessarily a Hurwitz matrix. Since the eigenvalues of $\mathbb{A}_\infty$ are (n,n) c-splitting, $-A_M^T$ has n eigenvalues in the open right half plane, which implies that A_M is Hurwitz. Hence, (66) has a stabilizing solution.

ii) Suppose $\Lambda_{2\infty}$ and $\bar{\Lambda}_{2\infty}$ are two stabilizing solutions. Denote

$$Y = \begin{bmatrix} I_n \\ \Lambda_{2\infty} \end{bmatrix}, \qquad \bar{Y} = \begin{bmatrix} I_n \\ \bar{\Lambda}_{2\infty} \end{bmatrix}.$$

By Step 1, $\mathrm{span}\{Y\} = \mathrm{span}\{\bar{Y}\}$ since they both are equal to the n-dimensional stable invariant subspace of $\mathbb{A}_\infty$. Now for each $1 \le i \le n$, the ith column Y_i of Y is in $\mathrm{span}\{\bar{Y}\}$, which further implies that Y_i is equal to the ith column of $\bar{Y}$. Therefore $\Lambda_{2\infty} = \bar{\Lambda}_{2\infty}$, and uniqueness follows. □

Theorem 18 presents a qualitative criterion on the existence of a stabilizing solution to the NARE (66). Step 2 in the proof further provides a computational procedure. When (H_g) holds, one may choose any n basis vectors of the n-dimensional stable invariant subspace to form the matrix in (74) and the resulting matrix $U_1 \in \mathbb{R}^{n\times n}$ is necessarily invertible. Subsequently one uses (75) to find the stabilizing solution. In fact, there is a simple means to test whether (H_g) holds. If the eigenvalues of $\mathbb{A}_\infty$ are strong (n,n) c-splitting, one takes any n basis vectors of the stable invariant subspace to form a matrix as in (74) with U_1 to be further checked. Finally, if U_1 is invertible, (H_g) holds; and (H_g) fails otherwise.

VII. Numerical Examples

A. Asymptotic Solvability

Consider the Riccati ODEs (19) and (20) with $n = 1$.

Example 2: Take the parameters $A = 0.2$, $B = G = Q = R = 1$, $Q_f = \Lambda_{1\infty}$, $\Gamma = 1.2$, $\Gamma_f = 0$. Then (19) gives $\Lambda_1(t) \equiv \Lambda_{1\infty}$ and (20) becomes

$$\dot{\Lambda}_2 = 2\Lambda_{1\infty}\Lambda_2 + \Lambda_2^2 - (\Lambda_{1\infty} + 1.4\Lambda_2) + 1.2, \quad \Lambda_2(T) = 0.$$

By verifying condition i) in Proposition 13, we see that Λ_2 has a solution on $[0,T]$ for any $T > 0$. So asymptotic solvability holds.

Example 3: Take $Q_f = 0$ and $T = 3$. All other parameters are the same as in Example 2. Now (19) and (20) reduce to

$$\dot{\Lambda}_1 = \Lambda_1^2 - 0.4\Lambda_1 - 1, \quad \Lambda_1(T) = 0,$$
$$\dot{\Lambda}_2 = 2\Lambda_1\Lambda_2 + \Lambda_2^2 - (\Lambda_1 + 1.4\Lambda_2) + 1.2, \quad \Lambda_2(T) = 0.$$

Λ_1 can be solved explicitly on $[0,T]$. Fig. 3 shows that Λ_2 does not have a solution on the whole interval $[0,T]$ implying no asymptotic solvability.

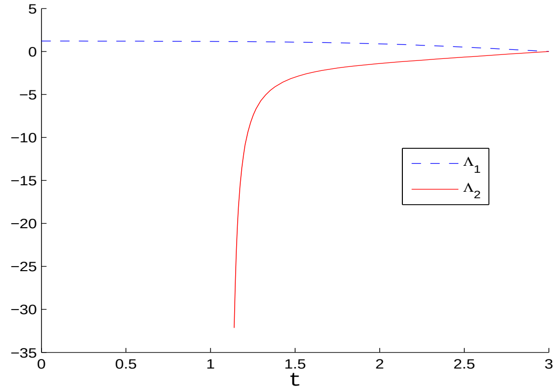

Fig. 3. Λ_2 has a maximal existence interval small than $[0,T]$

Examples 2 and 3 reveal a significant role of Λ_1 in affecting the existence interval of Λ_2.

Example 4: Consider a system with parameters in Example 1 and $T = 35$. Following the notation in Proposition 14, then

$$\hat{a} = -0.046447, \quad \hat{Q} = 4.906209 \times 10^{-4}.$$

So

$$0 < \hat{Q} < \hat{a}^2 = 0.002157,$$

and $\frac{1}{2\alpha}\ln(\hat{\lambda}_2/\hat{\lambda}_1) = 33.587095$. By Proposition 14, $\Lambda_2(t)$ has a finite escape time at $\hat{t} \approx 1.4129$. The TPBV problem (61) has a unique solution since $\Phi_{22}(35,0) \neq 0$.

B. Non-uniqueness

Consider a system with parameters in Example 1 and $\eta = \eta_f = 1$. Following the notation in subsection V-B,

$$\hat{\Delta} = 0.001667, \quad c_1 = 0.005622, \quad c_2 = 0.087271,$$

which satisfy the conditions in Proposition 15, and further determine

$$\hat{T} = 33.587095, \quad \hat{x}_0 = -0.394732.$$

Fig. 4 displays $\Phi_{21}(T,0)$ and $\Phi_{22}(T,0)$, where T is treated as a variable. It shows that $\Phi_{22}(T,0) = 0$ when $T = \hat{T}$.

Now consider the model (1)–(2) with time horizon $[0,\hat{T}]$. In this case, we have no asymptotic solvability since Λ_2 has the maximal existence interval $(0,\hat{T}]$. However, the TPBV problem (64) has an infinite number of solutions.

C. Stabilizing Solution for the NARE (66)

Example 5: We take

$$A = \begin{bmatrix} 1 & 1 \\ -0.5 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \ \eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and $G = Q = I_2$, $R = 1$. Then (66) has a stabilizing solution

$$\Lambda_{2\infty} = \begin{bmatrix} 16.238985 & 4.099679 \\ 4.132523 & 1.570208 \end{bmatrix}.$$

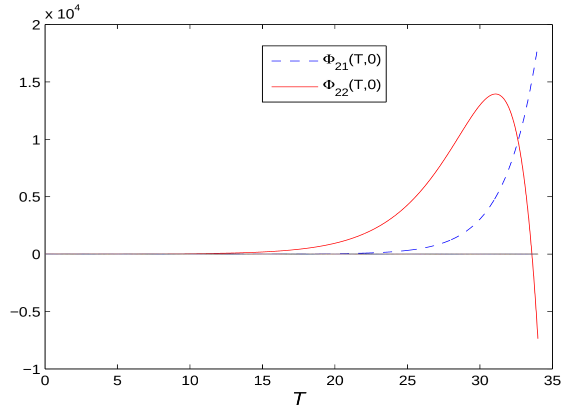

Fig. 4. $\Phi_{21}(T,0)$ and $\Phi_{22}(T,0)$ as a function of T.

In fact, the columns of the matrix

$$\begin{bmatrix} -0.167388 & -0.161703 \\ 0.448957 & 0.742511 \\ -0.877636 & 0.418170 \\ 0.013220 & 0.497657 \end{bmatrix}$$

span the stable invariant subspace of $\mathbb{A}_\infty$ as a graph subspace. $\mathbb{A}_\infty$ has the eigenvalues

$$-1.022350 \pm 0.730733i, \quad 2.022350 \pm 0.707903i.$$

Example 6: We take $G = -1.2I_2$ and all other parameters are the same as in Example 5. Then there exists no stabilizing solution $\Lambda_{2\infty}$ since in this case $\mathbb{A}_\infty$ has the eigenvalues

$$-1.090328 \pm 0.762501i, \quad -0.109672 \pm 0.692413i.$$

VIII. Conclusion

This paper investigates an asymptotic solvability problem in LQ mean field games, and studies its connection with the fixed point approach which involves a TPBV problem. For asymptotic solvability we derive a necessary and sufficient condition via a non-symmetric Riccati ODE. It is shown that asymptotic solvability provides a sufficient condition for the TPBV problem in the fixed point approach to have a unique solution. We identify situations for the TPBV problem to be solvable or have multiple solutions when asymptotic solvability does not hold. The long time behavior of the non-symmetric Riccati ODE in the asymptotic solvability problem is addressed by studying the stabilizing solution to a non-symmetric algebraic Riccati equation.

The re-scaling technique used in studying asymptotic solvability can be extended to more general models in terms of dynamics, interaction and information patterns [8], [25], [27], [30]. This will be reported in our future work.

Appendix A: Proof of Theorem 3

Lemma A.1: We assume that (10) has a solution $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ on $[0,T]$. Then the following holds.

i) $\boldsymbol{P}_1(t)$ has the representation

$$\boldsymbol{P}_1(t) = \begin{bmatrix} \Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_4(t) & \cdots & \Pi_4(t) \\ \Pi_2^T(t) & \Pi_4(t) & \Pi_3(t) & \cdots & \Pi_4(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^T(t) & \Pi_4(t) & \Pi_4(t) & \cdots & \Pi_3(t) \end{bmatrix}, \quad \text{(A.1)}$$

where Π_1, Π_3 and Π_4 are $n \times n$ symmetric matrices.

ii) For $i > 1$, $\boldsymbol{P}_i(t) = J_{1i}^T \boldsymbol{P}_1(t) J_{1i}$.

Proof: Step 1. It is straightforward to show

$$J_{23}^T \Psi J_{23} = \Psi,$$

where Ψ is any matrix from $\widehat{\boldsymbol{A}}$, $\boldsymbol{Q}_i$, $\boldsymbol{Q}_{if}$ and $\boldsymbol{B}_i R^{-1} \boldsymbol{B}_i^T$, $i \neq 2,3$. And moreover,

$$\begin{aligned} &J_{23}^T \boldsymbol{Q}_2 J_{23} = \boldsymbol{Q}_3, \qquad J_{23}^T \boldsymbol{Q}_3 J_{23} = \boldsymbol{Q}_2 \\ &J_{23}^T \boldsymbol{Q}_{2f} J_{23} = \boldsymbol{Q}_{3f} \qquad J_{23}^T \boldsymbol{Q}_{3f} J_{23} = \boldsymbol{Q}_{2f} \\ &J_{23}^T \boldsymbol{B}_2 R^{-1} \boldsymbol{B}_2^T J_{23} = \boldsymbol{B}_3 R^{-1} \boldsymbol{B}_3^T, \\ &J_{23}^T \boldsymbol{B}_3 R^{-1} \boldsymbol{B}_3^T J_{23} = \boldsymbol{B}_2 R^{-1} \boldsymbol{B}_2^T. \end{aligned}$$

Denote $J_{23}^T \boldsymbol{P}_i J_{23} = \boldsymbol{P}_i^\dagger$ for $1 \le i \le N$. Multiplying both sides of (10) from the left by J_{23}^T and next from the right by J_{23}, we obtain

$$\begin{aligned} \dot{\boldsymbol{P}}_i^\dagger = &- \boldsymbol{P}_i^\dagger J_{23}^T \widehat{\boldsymbol{A}} J_{23} - J_{23}^T \widehat{\boldsymbol{A}}^T J_{23} \boldsymbol{P}_i^\dagger \\ &+ \boldsymbol{P}_i^\dagger \sum_{k=1}^N J_{23}^T \boldsymbol{B}_k R^{-1} \boldsymbol{B}_k^T J_{23} \boldsymbol{P}_k^\dagger \\ &+ \sum_{k=1}^N \boldsymbol{P}_k^\dagger J_{23}^T \boldsymbol{B}_k R^{-1} \boldsymbol{B}_k^T J_{23} \boldsymbol{P}_i^\dagger \\ &- \boldsymbol{P}_i^\dagger J_{23}^T \boldsymbol{B}_i R^{-1} \boldsymbol{B}_i^T J_{23} \boldsymbol{P}_i^\dagger - J_{23}^T \boldsymbol{Q}_i J_{23}. \end{aligned}$$

Hence for $i \neq 2,3$,

$$\begin{aligned} \dot{\boldsymbol{P}}_i^\dagger = &- \boldsymbol{P}_i^\dagger \widehat{\boldsymbol{A}} - \widehat{\boldsymbol{A}}^T \boldsymbol{P}_i^\dagger + \boldsymbol{P}_i^\dagger \sum_{k \notin \{2,3\}}^N \boldsymbol{B}_k R^{-1} \boldsymbol{B}_k^T \boldsymbol{P}_k^\dagger \\ &+ \boldsymbol{P}_i^\dagger (\boldsymbol{B}_3 R^{-1} \boldsymbol{B}_3^T \boldsymbol{P}_2^\dagger + \boldsymbol{B}_2 R^{-1} \boldsymbol{B}_2^T \boldsymbol{P}_3^\dagger) \\ &+ \sum_{k \notin \{2,3\}}^N \boldsymbol{P}_k^\dagger \boldsymbol{B}_k R^{-1} \boldsymbol{B}_k^T \boldsymbol{P}_i^\dagger \\ &+ (\boldsymbol{P}_2^\dagger \boldsymbol{B}_3 R^{-1} \boldsymbol{B}_3^T + \boldsymbol{P}_3^\dagger \boldsymbol{B}_2 R^{-1} \boldsymbol{B}_2^T) \boldsymbol{P}_i^\dagger \\ &- \boldsymbol{P}_i^\dagger \boldsymbol{B}_i R^{-1} \boldsymbol{B}_i^T \boldsymbol{P}_i^\dagger - \boldsymbol{Q}_i, \end{aligned}$$

where $\boldsymbol{P}_i^\dagger(T) = \boldsymbol{Q}_{if}$. Similarly, we can write the equations for $\dot{\boldsymbol{P}}_2^\dagger$ and $\dot{\boldsymbol{P}}_3^\dagger$ for which we omit the details. Note that $\boldsymbol{P}_2^\dagger(T) = \boldsymbol{Q}_{3f}$ and $\boldsymbol{P}_3^\dagger(T) = \boldsymbol{Q}_{2f}$. Subsequently, we list the N equations by the order of $\dot{\boldsymbol{P}}_1^\dagger$, $\dot{\boldsymbol{P}}_3^\dagger$, $\dot{\boldsymbol{P}}_2^\dagger$, $\dot{\boldsymbol{P}}_4^\dagger$, $\cdots$, $\dot{\boldsymbol{P}}_N^\dagger$, and it turns out that

$$(J_{23}^T \boldsymbol{P}_1 J_{23},\ J_{23}^T \boldsymbol{P}_3 J_{23},\ J_{23}^T \boldsymbol{P}_2 J_{23},\ J_{23}^T \boldsymbol{P}_4 J_{23}, \cdots, J_{23}^T \boldsymbol{P}_N J_{23})$$

satisfies (10) as $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ does.

Step 2. For $1 \le i \le N$, denote $\boldsymbol{P}_i = (\boldsymbol{P}_i^{jk})_{1 \le j,k \le N}$, where each $\boldsymbol{P}_i^{jk}$ is an $n \times n$ matrix. By Step 1, $\boldsymbol{P}_1 = J_{23}^T \boldsymbol{P}_1 J_{23}$, which implies

$$\boldsymbol{P}_1^{12} = \boldsymbol{P}_1^{13}, \quad \boldsymbol{P}_1^{22} = \boldsymbol{P}_1^{33}, \quad \boldsymbol{P}_1^{23} = \boldsymbol{P}_1^{32}. \quad \text{(A.2)}$$

Repeating the above procedure by using J_{2k}, $k \geq 4$, in place of J_{23}, we obtain

$$\boldsymbol{P}_1^{12} = \boldsymbol{P}_1^{13} = \cdots = \boldsymbol{P}_1^{1N}, \qquad \boldsymbol{P}_1^{22} = \boldsymbol{P}_1^{33} = \cdots = \boldsymbol{P}_1^{NN}.$$

We similarly obtain $\boldsymbol{P}_1 = J_{34}^T \boldsymbol{P}_1 J_{34}$, and this gives

$$\boldsymbol{P}_1^{23} = \boldsymbol{P}_1^{24}.$$

Repeating a similar argument, we can check all other remaining off-diagonal submatrices. Since $\boldsymbol{P}_1$ is symmetric (also see Remark 1), $(\boldsymbol{P}_1^{23})^T = \boldsymbol{P}_1^{32}$, which implies that $\boldsymbol{P}_1^{23}$ is symmetric by (A.2). By the above method we can show that the off-diagonal submatrices P_1^{ij}, where $i \neq j$ and $2 \leq i \leq N$, $2 \leq j \leq N$, are equal and symmetric. Therefore we obtain the representation of $\boldsymbol{P}_1$.

Step 3. We can verify that

$$(J_{12}^T \boldsymbol{P}_2 J_{12},\ J_{12}^T \boldsymbol{P}_1 J_{12},\ J_{12}^T \boldsymbol{P}_3 J_{12}, \cdots, J_{12}^T \boldsymbol{P}_N J_{12})$$

satisfies (10) as $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ does. Hence $\boldsymbol{P}_2 = J_{12}^T \boldsymbol{P}_1 J_{12}$. All other cases can be similarly checked. □

Proof of Theorem 3:

By Lemma A.1, we have

$$\begin{aligned}
\dot{\Pi}_1(t) &= \Pi_1 M \Pi_1 + (N-1)(\Pi_2 M \Pi_2 + \Pi_2^T M \Pi_2^T) \\
&\quad - \Big(\Pi_1 (A + \frac{G}{N}) + (A^T + \frac{G^T}{N})\Pi_1\Big) \\
&\quad - (1 - \frac{1}{N})(\Pi_2 G + G^T \Pi_2^T) \\
&\quad - (I - \frac{\Gamma^T}{N}) Q (I - \frac{\Gamma}{N}), \qquad\qquad (A.3)\\
\Pi_1(T) &= (I - \frac{\Gamma_f^T}{N}) Q_f (I - \frac{\Gamma_f}{N}),
\end{aligned}$$

and

$$\begin{aligned}
\dot{\Pi}_2(t) &= \Pi_1 M \Pi_2 + \Pi_2 M \Pi_1 + \Pi_2^T M \Pi_3 \\
&\quad + (N-2)\Pi_2 M \Pi_2 + (N-2)\Pi_2^T M \Pi_4 \\
&\quad - \Big(\Pi_1 \frac{G}{N} + \frac{G^T}{N}\Pi_3 + \frac{N-2}{N} G^T \Pi_4 \\
&\quad + \Pi_2 (A + \frac{N-1}{N} G) + (A^T + \frac{G^T}{N})\Pi_2\Big) \\
&\quad + (I - \frac{\Gamma^T}{N}) Q \frac{\Gamma}{N}, \qquad\qquad (A.4)\\
\Pi_2(T) &= -(I - \frac{\Gamma_f^T}{N}) Q_f \frac{\Gamma_f}{N},
\end{aligned}$$

and

$$\begin{aligned}
\dot{\Pi}_3(t) &= \Pi_2^T M \Pi_2 + \Pi_3 M \Pi_1 + \Pi_1 M \Pi_3 \\
&\quad + (N-2)(\Pi_4 M \Pi_2 + \Pi_2^T M \Pi_4) \\
&\quad - \Big(\frac{1}{N}(\Pi_2^T G + G^T \Pi_2) \\
&\quad + \Pi_3 (A + \frac{G}{N}) + (A^T + \frac{G^T}{N})\Pi_3 \\
&\quad + \frac{N-2}{N}(\Pi_4 G + G^T \Pi_4)\Big) - \frac{\Gamma^T}{N} Q \frac{\Gamma}{N}, \qquad\qquad (A.5)\\
\Pi_3(T) &= \frac{\Gamma_f^T}{N} Q_f \frac{\Gamma_f}{N},
\end{aligned}$$

and

$$\begin{aligned}
\dot{\Pi}_4(t) &= \Pi_2^T M \Pi_2 + \Pi_4 M \Pi_1 + \Pi_1 M \Pi_4 + \Pi_3 M \Pi_2 \\
&\quad + \Pi_2^T M \Pi_3 + (N-3)(\Pi_4 M \Pi_2 + \Pi_2^T M \Pi_4) \\
&\quad - \Big(\frac{1}{N}(\Pi_2^T G + G^T \Pi_2 + \Pi_3 G + G^T \Pi_3) \\
&\quad + \Pi_4 (A + \frac{N-2}{N} G) + (A^T + \frac{N-2}{N} G^T)\Pi_4\Big) \\
&\quad - \frac{\Gamma^T}{N} Q \frac{\Gamma}{N}, \qquad\qquad (A.6)\\
\Pi_4(T) &= \frac{\Gamma_f^T}{N} Q_f \frac{\Gamma_f}{N}.
\end{aligned}$$

The last two ODEs lead to

$$\begin{aligned}
\frac{d}{dt}(\Pi_3 - \Pi_4) &= (\Pi_3 - \Pi_4)(M\Pi_1 - M\Pi_2 - A) \\
&\quad + (\Pi_1 M - \Pi_2^T M - A^T)(\Pi_3 - \Pi_4),
\end{aligned}$$

where $\Pi_3(T) - \Pi_4(T) = 0$. This can be viewed as a linear ODE once Π_1 and Π_2 are fixed. Therefore $\Pi_3 \equiv \Pi_4$ on $[0,T]$. This completes the proof. □

Appendix B: Proof of Theorem 4

Proof: i) We can find a constant C_z such that $\sup_{0 \leq t \leq T} |x^z(t)| \leq C_z$, and $\sup_{0 < \varepsilon \leq 1} |z_\varepsilon| \leq C_z$. Fix the open ball $B_{2C_z}(0)$. For $x, y \in B_{2C_z}(0)$ and $t \in [0,T]$, we have

$$|\phi(t,x) - \phi(t,y)| \leq \mathrm{Lip}(2C_z)|x - y|.$$

For each $\varepsilon \leq 1$, by (A1)–(A3), (17) has a solution $y^\varepsilon(t)$ defined either (a) for all $t \in [0,T]$ or (b) on a maximal interval $[0, t_{\max})$ for some $0 < t_{\max} \leq T$.

Below we show that for all small ε, (b) does not occur. We prove by contradiction. Suppose for any small $\varepsilon_0 > 0$, there exists $0 < \varepsilon < \varepsilon_0$ such that (b) occurs with the corresponding $0 < t_{\max} \leq T$. Since $[0, t_{\max})$ is the maximal existence interval, we have $\lim_{t \uparrow t_{\max}} |y^\varepsilon(t)| = \infty$ [22]. Therefore for some $0 < t_m < t_{\max}$,

$$y^\varepsilon(t_m) \in \partial B_{2C_z}(0), \qquad (B.1)$$

and

$$y^\varepsilon(t) \in B_{2C_z}(0), \quad \forall\ 0 \leq t < t_m. \qquad (B.2)$$

For $t < t_{\max}$, we have

$$y^\varepsilon(t) - x^z(t) = z_\varepsilon - z + \int_0^t \zeta(\tau) d\tau,$$

where $\zeta(\tau) = f(\tau, y^\varepsilon(\tau)) + g(\varepsilon, \tau, y^\varepsilon(\tau)) - f(\tau, x^z(\tau))$. It follows from (A3) that

$$\begin{aligned}
|\zeta(\tau)| &= |\zeta(\tau) - g(\varepsilon, \tau, x^z(\tau)) + g(\varepsilon, \tau, x^z(\tau))| \\
&\leq \mathrm{Lip}(2C_z)|y^\varepsilon(\tau) - x^z(\tau)| + |g(\varepsilon, \tau, x^z(\tau))|.
\end{aligned}$$

Now for $0 \leq t < t_m$,

$$\begin{aligned}
|y^\varepsilon(t) - x^z(t)| &\leq |z_\varepsilon - z| + \delta_\varepsilon \\
&\quad + \int_0^t \mathrm{Lip}(2C_z)|y^\varepsilon(\tau) - x^z(\tau)| d\tau.
\end{aligned}$$

Note that $\delta_\varepsilon = \int_0^T |g(\varepsilon,\tau,x^z(\tau))|d\tau \to 0$ as $\varepsilon \to 0$. By Gronwall's lemma,

$$|y^\varepsilon(t) - x^z(t)| \le (\delta_\varepsilon + |z_\varepsilon - z|)e^{\mathrm{Lip}(2C_z)t}$$

for all $t \le t_m$. We can find $\bar\varepsilon > 0$ such that for all $\varepsilon \le \bar\varepsilon$,

$$(\delta_\varepsilon + |z_\varepsilon - z|)e^{\mathrm{Lip}(2C_z)T} < \frac{C_z}{3}.$$

Then for all $0 \le t \le t_m$, $y^\varepsilon(t) \in B_{3C_z/2}(0)$, which is a contradiction to (B.1). We conclude for all $0 < \varepsilon \le \bar\varepsilon$, y^ε is defined on $[0,T]$. Next, (18) follows readily.

ii) We have

$$y^{\varepsilon_i}(t) = z_{\varepsilon_i} + \int_0^t \Big[f\big(\tau, y^{\varepsilon_i}(\tau)\big) + g\big(\varepsilon,\tau,y^{\varepsilon_i}(\tau)\big)\Big]d\tau, \qquad \text{(B.3)}$$

and

$$\begin{aligned}
&|f\big(\tau,y^{\varepsilon_i}(\tau)\big) + g\big(\varepsilon,\tau,y^{\varepsilon_i}(\tau)\big)| \\
&\le \mathrm{Lip}(C_2)|y^{\varepsilon_i}(\tau)| + |f(\tau,0) + g(\varepsilon,\tau,0)| \\
&\le \mathrm{Lip}(C_2)|y^{\varepsilon_i}(\tau)| + C_1 \\
&\le \mathrm{Lip}(C_2)C_2 + C_1, \qquad \text{(B.4)}
\end{aligned}$$

where C_1 is given in (A1).

By (B.3)–(B.4), the functions $\{y^{\varepsilon_i}(\cdot), i \ge 1\}$ are uniformly bounded and equicontinuous. By Arzelà-Ascoli theorem [53], there exists a subsequence $\{y^{\varepsilon_{i_j}}(\cdot), j \ge 1\}$ such that $y^{\varepsilon_{i_j}}$ converges to $y^* \in C\big([0,T],\mathbb{R}^K\big)$ uniformly on $[0,T]$, as $j \to \infty$. Hence,

$$y^*(t) = z + \int_0^t f\big(\tau,y^*(\tau)\big) + g\big(\varepsilon,\tau,y^*(\tau)\big)d\tau$$

for all $t \in [0,T]$. So (16) has a solution. □

The proof in part i) follows the method in [46, sec. 2.4] and [49, pp. 486].

Appendix C

Proof of Theorem 5:

Taking $\Pi_3 = \Pi_4$ into account, we rewrite the system of (A.3), (A.4) and (A.5) by use of a set of new variables

$$\Lambda_1^N = \Pi_1(t),\ \Lambda_2^N = N\Pi_2(t),\ \Lambda_3^N = N^2\Pi_3(t).$$

Here and hereafter N is used as a superscript in various places. This should be clear from the context. We can determine functions g_k, $1 \le k \le 3$, and obtain

$$\begin{aligned}
\dot\Lambda_1^N &= \Lambda_1^N M \Lambda_1^N - (\Lambda_1^N A + A^T \Lambda_1^N) - Q \\
&\quad + g_1(1/N, \Lambda_1^N, \Lambda_2^N), \qquad \text{(C.1)}\\
\Lambda_1^N(T) &= (I - \frac{\Gamma_f^T}{N})Q_f(I - \frac{\Gamma_f}{N}),
\end{aligned}$$

$$\begin{aligned}
\dot\Lambda_2^N &= \Lambda_1^N M \Lambda_2^N + \Lambda_2^N M \Lambda_1^N + \Lambda_2^N M \Lambda_2^N \\
&\quad - (\Lambda_1^N G + \Lambda_2^N(A+G) + A^T\Lambda_2^N) + Q\Gamma \\
&\quad + g_2(1/N, \Lambda_2^N, \Lambda_3^N), \qquad \text{(C.2)}\\
\Lambda_2^N(T) &= -(I - \frac{\Gamma_f^T}{N})Q_f\Gamma_f,
\end{aligned}$$

$$\begin{aligned}
\dot\Lambda_3^N &= (\Lambda_2^N)^T M \Lambda_2^N + \Lambda_3^N M \Lambda_1^N + \Lambda_1^N M \Lambda_3^N \\
&\quad + \Lambda_3^N M \Lambda_2^N + (\Lambda_2^N)^T M \Lambda_3^N \\
&\quad - \big((\Lambda_2^N)^T G + G^T\Lambda_2^N + \Lambda_3^N(A+G) + (A^T + G^T)\Lambda_3^N\big) \\
&\quad - \Gamma^T Q \Gamma + g_3(1/N, \Lambda_2^N, \Lambda_3^N), \qquad \text{(C.3)}\\
\Lambda_3^N(T) &= \Gamma_f^T Q_f \Gamma_f.
\end{aligned}$$

In particular, we can determine

$$\begin{aligned}
g_1 =& \frac{1}{N}(1 - \frac{1}{N})(\Lambda_2^N M \Lambda_2^N + (\Lambda_2^N)^T M (\Lambda_2^N)^T) \\
&- \frac{1}{N}(\Lambda_1^N G + G^T \Lambda_1^N) - \frac{1}{N}(1 - \frac{1}{N})(\Lambda_2^N G + G^T(\Lambda_2^N)^T) \\
&+ \frac{1}{N}(\Gamma^T Q + Q\Gamma) - \frac{1}{N^2}\Gamma^T Q \Gamma.
\end{aligned}$$

The expressions of g_2 and g_3 can be determined in a similar way and the detail is omitted here.

Letting $N \to \infty$ in (C.1)–(C.3), this gives a limiting ODE system consisting of (19), (20) and (22).

If (20) has a unique solution on $[0,T]$, we can uniquely solve Λ_3 from a linear ODE (22). In view of g_1, g_2, g_3 and the terminal conditions in (C.1)–(C.3), by Theorem 4 i) and Remark 3, there exists N_0 such that for all $N \ge N_0$, the system (C.1)–(C.3) has a solution on $[0,T]$ and

$$\sup_{N \ge N_0, 0 \le t \le T} (|\Lambda_1^N| + |\Lambda_2^N| + |\Lambda_3^N|) < \infty, \qquad \text{(C.4)}$$

which implies (15) and so (13). Consequently, asymptotic solvability follows.

Conversely, if asymptotic solvability holds, there exists N_0 such that the system (C.1)–(C.3) has a solution on $[0,T]$ for all $N \ge N_0$ and (C.4) holds. By Theorem 4 ii), (20) has a unique solution on $[0,T]$. This completes the proof of Theorem 5. □

Proof of Proposition 7:

We can check that

$$(J_{23}^T \boldsymbol{S}_1,\ J_{23}^T \boldsymbol{S}_3,\ J_{23}^T \boldsymbol{S}_2,\ J_{23}^T \boldsymbol{S}_4, \cdots, J_{23}^T \boldsymbol{S}_N)$$

satisfies (11). Hence $\boldsymbol{S}_1 = J_{23}^T \boldsymbol{S}_1$. We can further show $\boldsymbol{S}_1 = J_{12}^T \boldsymbol{S}_2$. By the method in the proof of Lemma A.1, we obtain the representation for $\boldsymbol{S}_i$. Next, for each i we have

$$\begin{cases}
\dot{\boldsymbol{r}}_i(t) = \theta_1^T M \theta_1 + 2(N-1)\theta_2^T M \theta_1 - \mathrm{Tr}(D^T \Pi_1 D) \\
\qquad -(N-1)\mathrm{Tr}(D^T \Pi_3 D) - \eta^T Q \eta, \\
\boldsymbol{r}_i(T) = \eta_f^T Q_f \eta_f,
\end{cases}$$

and therefore $\boldsymbol{r}_1 = \cdots = \boldsymbol{r}_N$. □

Proof of Proposition 8:

Recalling $M = BR^{-1}B^T$, by Proposition 7 we derive

$$\begin{cases}
\dot\theta_1(t) = \Pi_1 M \theta_1 + (N-1)(\Pi_2 M \theta_1 + \Pi_2^T M \theta_2) \\
\qquad -\big(A^T + \frac{G^T}{N}\big)\theta_1 - \frac{N-1}{N} G^T \theta_2 \\
\qquad +\big(I - \frac{\Gamma^T}{N}\big)Q\eta, \\
\theta_1(T) = -(I - \frac{\Gamma_f^T}{N})Q_f \eta_f,
\end{cases} \qquad \text{(C.5)}$$

and

$$\begin{cases}
\dot\theta_2(t) = \big(\Pi_2^T + (N-1)\Pi_3\big)M\theta_1 \\
\qquad +\big(\Pi_1 + (N-2)\Pi_2^T\big)M\theta_2 - \frac{1}{N}G^T\theta_1 \\
\qquad -\big(A^T + \frac{N-1}{N}G^T\big)\theta_2 - \frac{1}{N}\Gamma^T Q \eta, \\
\theta_2(T) = \frac{1}{N}\Gamma_f^T Q_f \eta_f.
\end{cases} \qquad \text{(C.6)}$$

Based on (C.5)–(C.6), we may write the ODEs of $\chi_1^N(t)$ and $\chi_2^N(t)$. Under asymptotic solvability, we uniquely solve $(\Lambda_1,\Lambda_2,\Lambda_3,\chi_1,\chi_2)$ on $[0,T]$. We obtain (27) by writing the ODE system of $(\Lambda_1^N,\Lambda_2^N,\Lambda_3^N,\chi_1^N,\chi_2^N)$ and next applying Theorem 4. The proposition follows. $\square$

APPENDIX D

Proof of Proposition 13:

i) If $\hat{Q}\le 0$, (52) is the Riccati ODE in a standard optimal control problem [49], and so has a unique solution on $[0,T]$.

ii) The characteristic equation of (54) has solutions $\hat{\lambda}_1=\hat{a}+\alpha$, $\hat{\lambda}_2=\hat{a}-\alpha$, where $\alpha=\sqrt{\Delta}$.

If $\alpha>0$, we write $u=C_1e^{\hat{\lambda}_1 t}+e^{\hat{\lambda}_2 t}$. Then $u'=C_1\hat{\lambda}_1e^{\hat{\lambda}_1 t}+\hat{\lambda}_2e^{\hat{\lambda}_2 t}$. By $\Lambda_2(T)=0$, we obtain $C_1=-(\hat{\lambda}_2/\hat{\lambda}_1)e^{-2\alpha T}$ and

$$\begin{aligned}\Lambda_2(t)&=\frac{\hat{\lambda}_1\hat{\lambda}_2\left(e^{\alpha(T-t)}-e^{-\alpha(T-t)}\right)}{\hat{\lambda}_2e^{-\alpha(T-t)}-\hat{\lambda}_1e^{\alpha(T-t)}}\\&=\frac{\hat{Q}\left(e^{\alpha(T-t)}-e^{-\alpha(T-t)}\right)}{\hat{\lambda}_2e^{-\alpha(T-t)}-\hat{\lambda}_1e^{\alpha(T-t)}},\end{aligned}$$

which exists on $[0,T]$.

If $\alpha=0$, we write the solution of (54) as $u=C_1e^{\hat{a}t}+te^{\hat{a}t}$. This gives $u'=C_1\hat{a}e^{\hat{a}t}+e^{\hat{a}t}+t\hat{a}e^{\hat{a}t}$. Since $\Lambda_2(T)=0$, $C_1\hat{a}+1+T\hat{a}=0$. Therefore,

$$\Lambda_2(t)=-\frac{u'}{u}=-\frac{C_1\hat{a}+1+t\hat{a}}{C_1+t}=\frac{\hat{a}^2(T-t)}{\hat{a}(t-T)-1},$$

which exists on $[0,T]$. $\square$

Proof of Proposition 14:

i) The computation is similar to the case in Proposition 13 and we omit the details.

ii) The characteristic equation of (54) has solutions $\hat{\lambda}_{1,2}=\hat{a}\pm\beta i$. To solve (54), we take $u=C_1e^{\hat{a}t}\cos\beta t+e^{\hat{a}t}\sin\beta t$. Now

$$u'=\hat{a}e^{\hat{a}t}(C_1\cos\beta t+\sin\beta t)+\beta e^{\hat{a}t}(-C_1\sin\beta t+\cos\beta t).$$

Since $\Lambda_2(T)=0$, we determine

$$\begin{aligned}C_1=&\frac{\hat{a}\sin\beta T+\beta\cos\beta T}{-\hat{a}\cos\beta T+\beta\sin\beta T}\\=&\frac{\cos\theta\sin\beta T+\sin\theta\cos\beta T}{-\cos\theta\cos\beta T+\sin\theta\sin\beta T}\\=&-\frac{\sin(\beta T+\theta)}{\cos(\beta T+\theta)}.\end{aligned}$$

For this moment we suppose $\cos(\beta T+\theta)\neq 0$ so that C_1 above is well defined. Subsequently,

$$\begin{aligned}C_1\cos\beta t+\sin\beta t=&\frac{\sin(\beta t-\beta T-\theta)}{\cos(\beta T+\theta)},\\-C_1\sin\beta t+\cos\beta t=&\frac{\cos(\beta t-\beta T-\theta)}{\cos(\beta T+\theta)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Lambda_2(t)=&-\frac{\hat{a}\sin(\beta t-\beta T-\theta)+\beta\cos(\beta t-\beta T-\theta)}{\sin(\beta t-\beta T-\theta)}\\=&\frac{\sqrt{\hat{Q}}\sin\beta(t-T)}{\sin\left(\beta(T-t)+\theta\right)}.\end{aligned}\tag{D.1}$$

If $\cos(\beta T+\theta)=0$ occurs, we start by taking $u=e^{\hat{a}t}\cos\beta t+C_2e^{\hat{a}t}\sin\beta t$. We may determine $C_2=0$ and still obtain the same form of Λ_2 as in (D.1). $\square$

IX. ACKNOWLEDGMENT

We would like to thank the anonymous reviewers and the Associate Editor for very helpful suggestions.

REFERENCES

[1] R. Aumann. Markets with a continuum of traders. *Econometrica*, 32:39–50, 1964.

[2] M. Bardi and M. Fischer. On non-uniqueness and uniqueness of solutions in finite-horizon mean field games. arXiv:1707.00628, 2017.

[3] T. Basar and G. J. Olsder. *Dynamic Noncooperative Game Theory*. SIAM, Philadelphia, PA, USA, 2nd edition, 1999.

[4] A. Bensoussan, J. Frehse, and P. Yam. *Mean Field Games and Mean Field Type Control Theory*. Springer-Verlag, New York, USA, 2013.

[5] A. Bensoussan, K. C. J. Sung, S. C. P. Yam, and S. P. Yung. Linear-quadratic mean field games. *J. Optim. Theory Appl.*, 169(2):496–529, 2016.

[6] D. A. Bini, B. Iannazzo, and B. Meini. *Numerical Solution of Algebraic Riccati Equations*. SIAM, Philadelphia, PA, USA, 2012.

[7] P. E. Caines, M. Huang, and R. P. Malhamé. Mean field games. In T. Basar and G. Zaccour, editors, *Handbook of Dynamic Game Theory*, pages 345–372. Springer, Berlin, Germany, 2017.

[8] P. E. Caines and A. C. Kizilkale. ε-Nash equilibria for partially observed LQG mean field games with a major player. *IEEE Trans. Autom. Control*, 62(7):3225–3234, July 2017.

[9] P. Cardaliaguet. Notes on mean field games. University of Paris, Dauphine, 2012.

[10] P. Cardaliaguet, F. Delarue, J. M. Lasry, and P. L. Lions. The master equation and the convergence problem in mean field games. arXiv:1509.02505, 2015.

[11] G. Carmona and K. Podczeck. Approximation and characterization of Nash equilibria of large games. Working paper, Univ. Cambridge and Universitat Wien, 2011.

[12] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, Cham, Switzerland, 2018.

[13] R. Carmona and X. Zhu. A probabilistic approach to mean field games with major and minor players. *Ann. Appl. Probab.*, 26(3):1535–1580, 2016.

[14] M. Cirant and D. Tonon. Time-dependent focusing mean-field games: the sub-critical case. *J. Dyn. Diff. Equat.*, 31(1):49–79, 2019.

[15] T. Eisele. Nonexistence and nonuniqueness of open-loop equilibria in linear-quadratic differential games. *J. Optim. Theory Appl.*, 37(4):443–468, 1982.

[16] J. Engwerda. Feedback Nash equilibria in the scalar infinite horizon LQ-game. *Automatica*, 36(1):135–139, 2000.

[17] M. Fischer. On the connection between symmetric *N*-player games and mean field games. *Ann. Appl. Probab.*, 27(2):757–810, 2017.

[18] G. Freiling. A survey of nonsymmetric Riccati equations. *Linear Algebra Appl.*, 351:243–270, 2002.

[19] D. A. Gomes, L. Nurbekyan, and M. Prazeres. One-dimensional stationary mean-field games with local coupling. *Dyn. Games Appl.*, 8(2):315–351, 2018.

[20] D. A. Gomes and J. Saude. Mean field games models: A brief survey. *Dyn. Games Appl.*, 4(2):110–154, 2014.

[21] E. J. Green. Continuum and finite-player noncooperative models of competition. *Econometrica*, 52(4):975–993, 1984.

[22] J. K. Hale. *Ordinary Differential Equations*. Wiley, New York, USA, 1969.

[23] A. Haurie and P. Marcotte. On the relationship between Nash-Cournot and Wardrop equilibria. *Networks*, 15(3):295–308, 1985.

[24] M. Herty, L. Pareschi, and S. Steffensen. Mean-field control and Riccati equations. *Networks and Heterogeneous Media*, 10(3):699–715, 2015.

[25] J. Huang, S. Wang, and Z. Wu. Backward mean-field linear-quadratic-gaussian (LQG) games: Full and partial information. *IEEE Trans. Autom. Control*, 61(12):3784–3796, December 2016.

[26] M. Huang. *Stochastic control for distributed systems with applications to wireless communications*. PhD thesis, McGill Univ., Montreal, Canada, 2003.

[27] M. Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM J. Control Optim.*, 48(5):3318–3353, 2010.

[28] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Trans. Autom. Control*, 52(9):1560–1571, September 2007.

[29] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: Closed-loop Mckean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Info. Syst.*, 6(3):221–252, 2006.

[30] M. Huang and S. L. Nguyen. Stochastic control of mean field models with mixed players. In *Proc. 18th IFAC World Congress*, pages 4465–4470, Milan, Italy, 2011.

[31] M. Huang and M. Zhou. Linear quadratic mean field games–Part I: The asymptotic solvability problem. In *Proc. 23rd Internat. Symp. Math. Theory Networks and Systems (MTNS)*, pages 489–495, Hong Kong, China, 2018.

[32] V. N. Kolokoltsov, M. Troeva, and W. Yang. On the rate of convergence for the mean-field approximation of controlled diffusions with large number of players. *Dyn. Games Appl.*, 4(2):208–230, 2014.

[33] D. Kremer and R. Stefan. Non-symmetric Riccati theory and linear quadratic Nash games. In *Proc. 15th Internat. Symp. Math. Theory Networks and Systems (MTNS)*, Univ. Notre Dame, USA, 2002.

[34] D. Lacker. A general characterization of the mean field limit for stochastic differential games. *Probability Theory and Related Fields*, 165(3-4):581–648, 2016.

[35] P. Lancaster and L. Rodman. *Algebraic Riccati Equations*. Clarendon press, Oxford, UK, 1995.

[36] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn J. Math.*, 2(1):229–260, 2007.

[37] T. Li and J.-F. Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Trans. Autom. Control*, 53(7):1643–1660, August 2008.

[38] E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So, and T. M. Antonsen. Exact results for the Kuramoto model with a bimodal frequency distribution. *Physical Review E*, 79(2):026204, 2009.

[39] A. Mas-Colell. Walrasian equilibria as limits of noncooperative equilibria. Part I: mixed strategies. *J. Econ. Theory*, 30(1):153–170, 1983.

[40] J. Moon and T. Basar. Linear-quadratic stochastic differential Stackelberg games with a high population of followers. In *Proc. 54th IEEE Conf. Decis. Control*, pages 2270–2275, Osaka, Japan, 2015.

[41] M. Nourian and P. E. Caines. ε-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM J. Control Optim.*, 51(4):3302–3331, 2013.

[42] M. Nourian, P. E. Caines, R. P. Malhamé, and M. Huang. Nash, social and centralized solutions to consensus problems via mean field control theory. *IEEE Trans Autom. Control*, 58(3):639–653, March 2013.

[43] E. Ott and T. M. Antonsen. Low dimensional behavior of large systems of globally coupled oscillators. *Chaos*, 18(3):037113, 2008.

[44] G. P. Papavassilopoulos. Limiting behavior of LQ deterministic infinite horizon Nash games with symmetric players as the number of players goes to infinity. arXiv:1403.3300, 2014.

[45] D. Pazó and E. Montbrió. Low-dimensional dynamics of populations of pulse-coupled oscillators. *Physical Review X*, 4(1):011009, 2014.

[46] L. Perko. *Differential Equations and Dynamical Systems*. Springer, New York, USA, 2nd edition, 1996.

[47] F. S. Priuli. Linear-quadratic N-person and mean-field games: infinite horizon games with discounted cost and singular limits. *Dyn. Games Appl.*, 5(3):397–419, 2015.

[48] R. Salhab, R. P. Malhamé, and J. Le Ny. A dynamic game model of collective choice in multiagent systems. *IEEE Trans. Autom. Control*, 63(3):768–782, March 2018.

[49] E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, USA, 2nd edition, 1998.

[50] R. F. Tchuendom. Uniqueness for linear-quadratic mean field games with common noise. *Dyn. Games Appl.*, 8(1):199–210, March 2018.

[51] H. Tembine, Q. Zhu, and T. Basar. Risk-sensitive mean-field games. *IEEE Trans. Autom. Control*, 59(4):835–850, April 2014.

[52] B. C. Wang and J. F. Zhang. Mean field games for large-population multiagent systems with Markov jump parameters. *SIAM J. Control Optim.*, 50(4):2308–2334, 2012.

[53] K. Yosida. *Functional Analysis*. Springer, Berlin, Germany, 6th edition, 1980.

PLACE PHOTO HERE

Minyi Huang (S'01-M'04) received the B.Sc. degree from Shandong University, Jinan, Shandong, China, in 1995, the M.Sc. degree from the Institute of Systems Science, Chinese Academy of Sciences, Beijing, in 1998, and the Ph.D. degree from the Department of Electrical and Computer Engineering, McGill University, Montreal, QC, Canada, in 2003, all in systems and control.

He was a Research Fellow first in the Department of Electrical and Electronic Engineering, the University of Melbourne, Melbourne, Australia, from February 2004 to March 2006, and then in the Department of Information Engineering, Research School of Information Sciences and Engineering, the Australian National University, Canberra, from April 2006 to June 2007. He joined the School of Mathematics and Statistics, Carleton University, Ottawa, ON, Canada as an Assistant Professor in July 2007, where he is now a Professor. His research interests include mean field stochastic control and dynamic games, multi-agent control and computation in distributed networks with applications.

PLACE PHOTO HERE

Mengjie Zhou received the B.Sc. degree in applied mathematics from Shanghai Jiao Tong University, Shanghai, China, in 2014, the M.Sc. degree in applied mathematics from Western University, London, ON, Canada, in 2015, and the M.Sc. degree in probability and statistics from Carleton University, Ottawa, ON, Canada, in 2017. She is currently pursuing the Ph.D. degree in probability and statistics under the supervision of Dr. Minyi Huang at Carleton University.

Her research interests include mean field stochastic control and Markov decision processes.