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Mean Field Games for Stochastic Growth with Relative Utility

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Abstract This paper considers continuous time stochastic growth-consumption optimization in a mean field game setting. The individual capital stock evolution is determined by a Cobb–Douglas production function, consumption and stochastic depreciation. The individual utility functional combines an own utility and a relative utility with respect to the population. The use of the relative utility reflects human psychology, leading to a natural pattern of mean field interaction. The fixed point equation of the mean field game is derived with the aid of some ordinary differential equations. Due to the relative utility interaction, our performance analysis depends on some ratio based approximation error estimate.

Keywords Mean field game · Stochastic growth · Relative utility

Mathematics Subject Classification 90C39 · 91A25 · 91B62 · 93E20

1 Introduction

Mean field game theory studies a large population of noncooperative players which are individually insignificant but collectively have a significant impact on a particular player. It provides a powerful methodology for reducing complexity in design and implementation of strategies [22–24, 29]. The solution to the infinite population model

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leads to the construction of a set of decentralized strategies for the original large but finite population with an ε-Nash equilibrium property [22–24]. Another related solution notion in Markov decision models is the oblivious equilibrium [38]. For further literature in the stochastic analysis setting, see [9,10,17,28]. The readers are referred to [6–8,18] for an overview on mean field game theory. Mean field games have found wide applications, and we particularly mention those related to smart grids [14,27,32], economics, finance and operations research [1,10,12,15,19,31].

An application area of interest is capital accumulation with endogenous growth dynamics. Its study in a Nash game setting for multiple producers has existed in the literature [3]. A mean field game approach has been developed in a discrete time model [21] for consumption-accumulation optimization with hyperbolic absolute risk aversion (HARA) utility, where the coupling is due to a congestion effect [5] of the population on the growth dynamics. The recent work [25,26] studies continuous time mean field modeling for growth optimization and takes into account stochastic depreciation for the capital stock of an agent. On the other hand, it has long been observed in the economic literature that the satisfaction of an agent can be affected by the comparison utility with respect to the peers [2,11,20,36]. Relative performance has been introduced into a mean field game model of investment in [15] where an agent, apart from other goals, is concerned with the difference between its own wealth and the average of others at the terminal time. The work [26] considers a different mean field interaction pattern by including within the utility function a multiplicative factor as the ratio of it own consumption to the population average consumption. This modeling feature greatly facilitates the explicit computations of the individual strategies. The fixed point equation for the solution of the mean field game is specified with the aid of a system of ordinary differential equations. However, a remaining complexity issue is that the numerical solution still needs to compute the density evolution of the state process.

In this paper, we adopt a multiplicative coupling similar to [26], but the present relative performance is based on relative utility via a ratio of its own utility to the population average utility. The resulting relative performance is combined with a Cobb–Douglas production function. As it turns out, this modeling framework has a very appealing feature in that the numerical implementation of the strategies no longer needs the density equation of the state process. To characterize the performance of the obtained strategies, a key task is to estimate the concentration of the above ratio around the value one. We further present some error bound on an ε-Nash equilibrium.

It should be noted that except for the linear-quadratic-Gaussian (LQG) [4,22,30] and linear-exponential-quadratic-Gaussian (LEQG) [35] cases of mean field games, it is rare to have closed-form solutions available. For many situations, the implementation of the strategies relies on demanding numerical solutions of partial differential equations. Though not in an LQG setting, our problem formulation is computationally amenable.

The organization of the paper is as follows. Section 2 introduces the dynamics and utility functional of the mean field game. A limiting optimal control problem is analyzed in Sect. 3 to determine the best response. Section 4 introduces the fixed point equation of the mean field game. Section 5 develops error estimates for the mean field approximation, and establishes an ε-Nash equilibrium theorem. Numerical solutions of the fixed point equation are presented in Sect. 6. Section 7 concludes the paper.

2 The Mean Field Model with Finite Population

We start by describing a game of N agents (as economic entities). The capital stock of agent i is denoted by X_t^i which satisfies the stochastic differential equation (SDE)

$$dX_t^i = \left[A(X_t^i)^\alpha - \delta X_t^i - C_t^i\right]dt - \sigma X_t^i dW_t^i, \quad 1 \le i \le N, \ t \ge 0, \tag{1}$$

where the constants $0 < \alpha < 1$, $A > 0$, $X_0^i > 0$, $EX_0^i < \infty$, and $\{W_t^i, 1 \le i \le N\}$ are i.i.d. standard Brownian motions. The N agents have i.i.d. initial states $\{X_0^i, 1 \le i \le N\}$ which are also independent of the N Brownian motions $\{W_t^i, 1 \le i \le N\}$.

The production function $F(x) = Ax^\alpha$ determines the production output contributed by capital stock, and may be regarded as a Cobb–Douglas production function (see e.g. [16,34]) with capital x and a constant labor size. Moreover, $\delta dt + \sigma dW_t^i$ is the stochastic capital depreciation rate and $C_t^i \ge 0$ is the consumption rate. The pioneering work of Merton [33] introduced stochastic differential equations to model economic growth where uncertainty originates from population growth described by a geometric Brownian motion. For existing works examining the effect of stochastic depreciation, see [16,37].

The utility functional of agent i takes the form

$$J_i\left(C^1, \ldots, C^N\right) = E\left[\int_0^T e^{-\rho t} U\left(C_t^i, C_t^{(N,\gamma)}\right) dt + e^{-\rho T} S(X_T)\right], \tag{2}$$

where $C_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^N (C_t^i)^\gamma$ is an average term related to the population and $\gamma \in (0, 1)$. For simplicity, we take S to be only dependent on X_T. We take the utility function

$$U\left(C_t^i, C_t^{(N,\gamma)}\right) = \frac{1}{\gamma}(C_t^i)^{\gamma(1-\lambda)}\left[\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda. \tag{3}$$

The parameter $\lambda \in [0, 1]$. This utility structure has to do with human psychology of comparing with peers. Similar utility functions can be found in [2,20], but they are based on relative consumptions.

Denote

$$U_0(C_t^i) = \frac{1}{\gamma}(C_t^i)^\gamma, \quad U_1\left(C_t^i, C^{(N,\gamma)}\right) = \frac{1}{\gamma}\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}},$$

which will be called the own utility and the relative utility, respectively. Then $U\left(C_t^i, C_t^{(N,\gamma)}\right)$ is a weighted geometric mean of U_0 and U_1, i.e.,

$$U = U_0^{1-\lambda} U_1^{\lambda}.$$

The utility function U reduces to the own utility when $\lambda = 0$, and to the relative utility when $\lambda = 1$.

For a given θ, $U(c, \theta)$ determines a HARA utility since

$$U(c, \theta) = \frac{c^\gamma}{\gamma \theta^\lambda},$$

where $1 - \gamma$ is usually called the relative risk aversion coefficient. It is in fact a constant relative risk aversion (CRRA) utility as a special case of the HARA utility.

We further take

$$S(x) = \frac{\eta x^\gamma}{\gamma}, \tag{4}$$

where $\eta > 0$ is a constant. To develop explicit calculation, we introduce the assumption

$$\gamma = 1 - \alpha.$$

There is economic justification for such a choice of γ while α is an inherent parameter of the growth model. The interpretation is equalizing the coefficient of the relative risk aversion to capital share; see [13,16] for details. For notational simplicity, our further analysis will use the single parameter γ and substitute $\alpha = 1 - \gamma$.

3 The Limiting Model

For sufficiently large N, we may approximate $C_t^{(N,\gamma)}$ by a deterministic function $\bar{C}_t^{(\gamma)}$ defined on $[0, T]$, and this can be heuristically justified by the law of large numbers as long as the individual controls satisfy some mild conditions.

Consider a representative agent. Let its capital stock be denoted by X_t with dynamics

$$dX_t = \left(A X_t^{1-\gamma} - \delta X_t - C_t\right) dt - \sigma X_t dW_t, \quad t \geq 0, \tag{5}$$

where we no longer use the superscript i to label the agent. The initial state $X_0 > 0$, and we have the constraint $X_t \geq 0$, $C_t \geq 0$ for $t \in [0, T]$.

The utility functional is now given as

$$\bar{J}(C(\cdot)) = E\left[\int_0^T e^{-\rho t} U\left(C_t, \bar{C}_t^{(\gamma)}\right) dt + e^{-\rho T} S(X_T)\right], \tag{6}$$

where $U(C_t, \bar{C}_t^{(\gamma)})$ and $S(X_T)$ are given as in (3)–(4),

$$U(C_t, \bar{C}_t^{(\gamma)}) = \frac{1}{\gamma}\Big[C_t^\gamma\Big]^{1-\lambda}\left[\frac{C_t^\gamma}{\bar{C}_t^{(\gamma)}}\right]^\lambda, \quad S(X_T) = \frac{\eta X_T^\gamma}{\gamma}.$$

Let $C([0, T]; \mathbb{R}^+)$ denote the set of continuous functions which are strictly positive on $[0, T]$. To avoid a zero division problem, we will consider $\bar{C}^{(\gamma)}(\cdot) \in C([0, T]; \mathbb{R}^+)$.

The admissible control set consists of all consumption processes C_t adapted to the filtration generated by $X_0, W_s, s \le t$ such that $X_t \ge 0$ for all $t \in [0, T]$. A natural problem is to choose a consumption plan to maximize the functional $\bar{J}$ for the agent in question.

Before further analysis, we make a note on notation. We use t in X_t, C_t, W_t, etc. to indicate the value of the process or function at time t. Only for $V_t(t, x)$ appearing in various Hamilton–Jacobi–Bellman (HJB) equations, it means the partial derivative with respect to t. The interpretation should be clear from the context. Sometimes we use $C(\cdot)$, $\bar{C}^{(\gamma)}(\cdot)$, etc. to indicate a process or function on $[0, T]$. We use D to denote a generic constant that may change from place to place.

3.1 2-Step Solution

The solution of this infinite population model consists of two steps:

Step 1. Find the optimal strategy $\hat{C}_t$ when the function $\bar{C}^{(\gamma)}(\cdot)$ is fixed.

Step 2. Write the closed-loop state equation

$$dX_t = \left(AX_t^{1-\gamma} - \delta X_t - \hat{C}_t\right) dt - \sigma X_t dW_t,$$

and, following the standard approach in mean field games, further impose the consistency condition

$$\bar{C}_t^{(\gamma)} = E\hat{C}_t^\gamma, \quad t \in [0, T], \tag{7}$$

which is due to the fact that $\bar{C}_t^{(\gamma)}$ is used to approximate $\frac{1}{N}\sum_{i=1}^N (\hat{C}_t^i)^\gamma$.

The remaining part of this section will carry out Step 1.

3.2 The Best Response and HJB Equation

For the optimal control problem (5)–(6), we consider a general function $\bar{C}_t^{(\gamma)}$ without imposing the consistency condition (7). For $0 \le t \le T$ and $x > 0$, further define the utility functional associated with the initial pair (t, x) as

$$\bar{J}(t, x, C(\cdot)) = E_{t,x}\left[\int_t^T e^{-\rho(s-t)} U\left(C_s, \bar{C}_s^{(\gamma)}\right) ds + e^{-\rho(T-t)} S(X_T)\right],$$

where $E_{t,x}$ denotes the expectation given $X_t = x$. Define the value function

$$V(t, x) = \sup_{C(\cdot)} \bar{J}(t, x, C(\cdot)).$$

We write the HJB equation

$$\rho V(t, x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + \sup_c \Big[U(c, \bar{C}_t^{(\gamma)}) + \Big(A x^{1-\gamma} - \delta x - c \Big) V_x \Big], \; x > 0, \tag{8}$$
$$V(T, x) = S(x).$$

3.3 More Explicit Form of the HJB Equation

Let $\bar{C}_t^{(\gamma)}$ be fixed. Denote

$$B_t = \big(\bar{C}_t^{(\gamma)}\big)^{\lambda}.$$

Equation (8) reduces to

$$\rho V(t, x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + \sup_c \left\{ \frac{c^\gamma}{\gamma B_t} + \Big(A x^{1-\gamma} - \delta x - c \Big) V_x \right\}. \tag{9}$$

If the condition

$$V_x > 0 \tag{10}$$

holds, $\sup_c \left\{ \frac{c^\gamma}{\gamma B_t} - c V_x \right\}$ is attained at

$$c = \big(B_t V_x \big)^{\frac{1}{\gamma - 1}} \tag{11}$$

and accordingly (9) is equivalent to

$$\rho V(t, x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + \Big(A x^{1-\gamma} - \delta x \Big) V_x + \frac{1-\gamma}{\gamma} B_t^{\frac{1}{\gamma-1}} V_x^{\frac{\gamma}{\gamma-1}}. \tag{12}$$

The terminal condition is $V(T, x) = \frac{\eta x^\gamma}{\gamma}$ due to (4).

To solve (12), we try the ansatz

$$V(t, x) = \frac{1}{\gamma} \Big[p(t) x^\gamma + h(t) \Big], \quad x > 0, \; t \geq 0.$$

Then we have

$$V_t = \frac{1}{\gamma}\Big[\dot{p}(t)x^{\gamma} + \dot{h}(t)\Big],$$

and

$$V_x = p(t)x^{\gamma-1}, \quad V_{xx} = (\gamma-1)p(t)x^{\gamma-2}.$$

Substituting these expressions into (12) yields

$$\frac{\rho}{\gamma}\Big(px^{\gamma}+h\Big) = \frac{1}{\gamma}\Big[\dot{p}x^{\gamma}+\dot{h}\Big] + \frac{\sigma^2}{2}(\gamma-1)px^{\gamma} + Ap - \delta px^{\gamma} + \frac{1-\gamma}{\gamma}B_t^{\frac{1}{\gamma-1}}p^{\frac{\gamma}{\gamma-1}}x^{\gamma}. \tag{13}$$

By (13), we obtain two ordinary differential equations (ODEs)

$$\dot{p}(t) = \left[\rho + \frac{\sigma^2\gamma(1-\gamma)}{2} + \delta\gamma\right]p(t) - (1-\gamma)B_t^{\frac{1}{\gamma-1}}p^{\frac{\gamma}{\gamma-1}}(t), \tag{14}$$

$$p(T) = \eta,$$

$$\dot{h}(t) = \rho h(t) - \gamma A p(t), \tag{15}$$

$$h(T) = 0.$$

Theorem 1 *For given $\bar{C}^{(\gamma)} \in C([0,T];\mathbb{R}^+)$, the system (14)–(15) has a unique solution (p,h), where $p \in C([0,T];\mathbb{R}^+)$, and the optimal control in (5)–(6) is given in the feedback form*

$$\hat{C}_t = \big(B_t p(t)\big)^{\frac{1}{\gamma-1}}X_t.$$

Proof Define

$$a = \frac{1}{1-\gamma}\left[\rho + \frac{\sigma^2\gamma(1-\gamma)}{2} + \delta\gamma\right], \quad b_t = B_t^{\frac{1}{\gamma-1}}.$$

Define the new function φ via $p = \varphi^{1-\gamma}$. Then (14) reduces to

$$(1-\gamma)\varphi^{-\gamma}\dot{\varphi} = (1-\gamma)a\varphi^{1-\gamma} - (1-\gamma)b_t\varphi^{-\gamma},$$

which gives $\dot{\varphi} = a\varphi - b_t$, and $\varphi(T) = \eta^{\frac{1}{1-\gamma}}$. Solving this ODE we obtain a unique solution

$$\varphi(t) = e^{a(t-T)}\eta^{\frac{1}{1-\gamma}} + e^{at}\int_t^T e^{-as}b_s ds > 0.$$

Consequently, we obtain the unique solution

$$p(t) = \left[e^{a(t-T)}\eta^{\frac{1}{1-\gamma}} + e^{at}\int_t^T e^{-as} b_s ds\right]^{1-\gamma} > 0.$$

It is clear that $p \in C([0, T]; \mathbb{R}^+)$. We continue to solve (15), and the unique solution of h can be obtained accordingly. The optimal control follows from the relation (11). □

It is seen that the solution of (p, h) ensures condition (10).

Theorem 2 *The closed-loop system of* (5) *with the control* $\hat{C}_t$ *has a unique strong solution* X_t*,* $t \in [0, T]$.

Proof The closed-loop dynamics are

$$dX_t = \left[AX_t^{1-\gamma} - \delta X_t - \left(B_t p(t)\right)^{\frac{1}{\gamma-1}} X_t\right] dt - \sigma X_t dW_t, \quad X_0 > 0.$$

Denote $\tau = \inf\{t | X_t = 0, t \le T\}$. Following the method in [34], define $Z_t = X_t^{\gamma}$ for $t < \tau$. According to Itô's formula, Z_t satisfies the following linear SDE

$$dZ_t = \left\{\gamma A - \gamma\left[\delta + (B_t p(t))^{\frac{1}{\gamma-1}} + \frac{\sigma^2(1-\gamma)}{2}\right] Z_t\right\} dt - \gamma\sigma Z_t dW_t, \quad Z_0 = X_0^{\gamma}.$$

Note that from this equation we can solve a unique solution $Z_t > 0$ on $[0, T]$. This determines a unique solution for X_t on $[0, T]$ and so $P(\tau \le T) = 0$. □

4 The Fixed Point Equation

This section carries out Step 2 outlined in Sect. 3.1. Recall that

$$B_t = \left(\bar{C}_t^{(\gamma)}\right)^{\lambda}, \quad b_t = B_t^{\frac{1}{\gamma-1}}. \tag{16}$$

Although we may formalize the fixed point condition in terms of $\bar{C}_t^{(\gamma)}$, it turns out to be more convenient to deal with b_t. Let b_t be given and $b \in C([0, T]; \mathbb{R}^+)$. For $0 \le t \le T$, denote

$$\Gamma_0(b)_t = p^{\frac{1}{\gamma-1}}(t) = \left[e^{a(t-T)}\eta^{\frac{1}{1-\gamma}} + e^{at}\int_t^T e^{-as} b_s ds\right]^{-1}, \qquad \Gamma_1(b)_t = b_t\Gamma_0(b)_t. \tag{17}$$

We use $\Gamma_k(b)_t$ to denote the value of the function $\Gamma_k(b)$ at t, $k = 0, 1$. Thus, the best response is given in the form $\hat{C}_t = \Gamma_1(b)_t X_t$, which gives the closed-loop state equation

$$dX_t = \Big[AX_t^{1-\gamma} - \delta X_t - \Gamma_1(b)_t X_t\Big]dt - \sigma X_t dW_t. \tag{18}$$

Based on (18), define the operator Λ by

$$\Lambda(b)_t = \big(EX_t^\gamma\big)^{\frac{1}{\gamma}}, \quad 0 \le t \le T. \tag{19}$$

According to (16), $b_t = (\bar{C}_t^{(\gamma)})^{\frac{\lambda}{\gamma-1}}$. The equation of X_t further gives

$$(E\hat{C}_t^\gamma)^{\frac{\lambda}{\gamma-1}} = \big[\big(\Gamma_1(b)_t\big)^\gamma EX_t^\gamma\big]^{\frac{\lambda}{\gamma-1}} = [\Gamma_1(b)_t\Lambda(b)_t]^{\frac{\lambda\gamma}{\gamma-1}} =: \Gamma(b)_t, \tag{20}$$

which together with the consistency condition (7) leads to the fixed point equation

$$b_t = \Gamma(b)_t, \quad t \in [0, T]. \tag{21}$$

We summarize the following theorem.

Theorem 3 *Suppose that $b \in C([0, T]; \mathbb{R}^+)$ is a solution of (21). Denote by X_t^* the solution of (18) and set the continuous function*

$$\bar{C}_t^{(\gamma)} = \big(\Gamma_1(b)_t\big)^\gamma E(X_t^*)^\gamma, \quad 0 \le t \le T.$$

Then the control law $\hat{C}_t = \Gamma_1(b)_t X_t$ is optimal for the control problem (5)–(6) with $\bar{C}_t^{(\gamma)}$ selected as above and furthermore, the closed-loop system gives $E\hat{C}_t^\gamma = \bar{C}_t^{(\gamma)}$.

Next, we consider the fixed point problem (21). For simplicity, we further assume that the i.i.d. initial conditions $\{X_0^i, i \ge 1\}$ satisfy

$$d_1 \le X_0^i \le d_2, \quad i \ge 1,$$

for some positive constants d_1, d_2. Denote $d_0 = \big[E(X_0^i)^\gamma\big]^{\frac{1}{\gamma}}$. For positive numbers $D_1 < D_2$, let $C([0, T]; [D_1, D_2])$ denote the subset of $C([0, T]; \mathbb{R})$ which contains all continuous functions from $[0, T]$ to $[D_1, D_2]$. For $b_1, b_2 \in C([0, T]; [D_1, D_2])$, denote $d(b_1, b_2) = \|b_1 - b_2\|_\infty$. Then $\big(C([0, T]; [D_1, D_2]), d(\cdot, \cdot)\big)$ is a complete metric space. We have the following lemma.

Lemma 4 (i) *There exist constants D_4 and D_6 such that for any $b \in C([0, T]; \mathbb{R}^+)$,*

$$\Gamma_0(b)_t \le D_4, \quad \Lambda(b)_t \le D_6, \quad 0 \le t \le T. \tag{22}$$

(ii) *If $b \in C([0, T]; \mathbb{R}^+)$ and $\|b\|_\infty \le D_2$ for some constant D_2, then there exist constants $D_3 > 0$, $D_5 > 0$ such that*

$$\Gamma_0(b)_t \ge D_3, \quad \Lambda(b)_t \ge D_5, \quad 0 \le t \le T. \tag{23}$$

Proof (i) It follows from (17) that

$$\Gamma_0(b)_t \le e^{aT}\eta^{-\frac{1}{1-\gamma}} =: D_4. \tag{24}$$

Next, let X_t be the solution to (18) and X_t^u the solution of the SDE

$$dX_t^u = A(X_t^u)^{1-\gamma}dt - \sigma X_t^u dW_t, \quad X_0^u = X_0. \tag{25}$$

Note that $AX^{1-\gamma} - \big(\delta + \Gamma_1(b)_t\big)X \le AX^{1-\gamma}$ for any $b \in C([0,T];\mathbb{R}^+)$, $0 \le t \le T$ and $X > 0$. Hence, according to the comparison theorem to the solutions to (18) and (25), we have $X_t \le X_t^u$ for $0 \le t \le T$. Denote $Z_t^u = (X_t^u)^\gamma$. By Itô's formula,

$$dZ_t^u = \gamma\Big[A - \frac{(1-\gamma)\sigma^2}{2}Z_t^u\Big]dt - \gamma\sigma Z_t^u dW_t, \quad Z_0^u = X_0^\gamma.$$

This linear SDE admits the explicit solution

$$Z_t^u = \exp\Big\{-\frac{\gamma\sigma^2}{2}t - \gamma\sigma W_t\Big\}\Big[X_0^\gamma + \gamma A\int_0^t \exp\Big\{\frac{\gamma\sigma^2}{2}s + \gamma\sigma W_s\Big\}ds\Big]. \tag{26}$$

Since $\gamma \in (0,1)$, by taking expectations on both sides of (26) and using the identity $E[\exp(\sigma W_t)] = \exp\Big(\frac{\sigma^2}{2}t\Big)$, we arrive at

$$\begin{aligned} EZ_t^u &= E(X_0)^\gamma E\exp\Big\{-\frac{\gamma\sigma^2}{2}t - \gamma\sigma W_t\Big\} \\ &\quad + \gamma A\int_0^t E\exp\Big\{-\frac{\gamma\sigma^2}{2}(t-s) - \gamma\sigma(W_t - W_s)\Big\}ds \\ &= d_0^\gamma \exp\Big(\frac{\gamma(\gamma-1)\sigma^2 t}{2}\Big) + \gamma A\int_0^t \exp\Big(\frac{\gamma(\gamma-1)\sigma^2(t-s)}{2}\Big)ds \\ &\le d_0^\gamma + \gamma AT. \end{aligned} \tag{27}$$

Thus, it follows from (19) that

$$\Lambda(b)_t = \big[E\big(X_t\big)^\gamma\big]^{\frac{1}{\gamma}} \le \big[E\big(X_t^u\big)^\gamma\big]^{\frac{1}{\gamma}} = \big[EZ_t^u\big]^{\frac{1}{\gamma}} \le \big[d_0^\gamma + \gamma AT\big]^{\frac{1}{\gamma}} =: D_6 \tag{28}$$

for any $0 \le t \le T$.

(ii) Since $0 < b_t \le D_2$ for $0 \le t \le T$, it follows from (17) that

$$\Gamma_0(b)_t \ge \Big[\eta^{\frac{1}{1-\gamma}} + TD_2\Big]^{-1} =: D_3. \tag{29}$$

To proceed, let X_t^l be the solution to the SDE

$$dX_t^l = \Big[A(X_t^l)^{1-\gamma} - (\delta + D_2 D_4)X_t^l\Big]dt - \sigma X_t^l dW_t, \quad X_0^l = X_0. \tag{30}$$

Since $b_t \le D_2$ and $\Gamma_0(b)_t \le D_4$, $AX^{1-\gamma} - (\delta + D_2 D_4)X \le AX^{1-\gamma} - (\delta + \Gamma_1(b)_t)X$ for $0 \le t \le T$, $X > 0$. By the comparison theorem for (18) and (30), we have $X_t^l \le X_t$ for $0 \le t \le T$. Denote $Z_t^l = (X_t^l)^\gamma$. Then Itô's formula yields

$$dZ_t^l = \gamma\Big[A - \Big(\delta + D_2 D_4 + \frac{(1-\gamma)\sigma^2}{2}\Big)Z_t^l\Big]dt - \gamma\sigma Z_t^l dW_t, \quad Z_0^l = X_0^\gamma. \tag{31}$$

This linear SDE admits the explicit solution

$$\begin{aligned} Z_t^l =& \exp\Big\{-\gamma\Big(\delta + D_2 D_4 + \frac{\sigma^2}{2}\Big)t - \gamma\sigma W_t\Big\} \\ &\times\Big[X_0^\gamma + \gamma A\int_0^t \exp\Big\{\gamma\Big(\delta + D_2 D_4 + \frac{\sigma^2}{2}\Big)s + \gamma\sigma W_s\Big\}ds\Big] \\ \ge& X_0^\gamma \exp\Big\{-\gamma\Big(\delta + D_2 D_4 + \frac{\sigma^2}{2}\Big)t - \gamma\sigma W_t\Big\}. \end{aligned} \tag{32}$$

Again, using the identity $E[\exp(-\sigma W_t)] = \exp\Big(\frac{\sigma^2 t}{2}\Big)$ and taking expectations in the above equation and inequality yield

$$\begin{aligned} EZ_t^l &\ge d_0^\gamma E\exp\Big\{-\gamma\Big(\delta + D_2 D_4 + \frac{\sigma^2}{2}\Big)t - \gamma\sigma W_t\Big\} \\ &\ge d_0^\gamma \exp\Big\{-\gamma\Big(\delta + D_2 D_4\Big)T\Big\}. \end{aligned}$$

Therefore, for any $0 \le t \le T$,

$$\begin{aligned} \Lambda(b)_t = \Big[E(X_t)^\gamma\Big]^{\frac{1}{\gamma}} \ge \Big[E(X_t^l)^\gamma\Big]^{\frac{1}{\gamma}} &= \Big[EZ_t^l\Big]^{\frac{1}{\gamma}} \\ &\ge d_0 \exp\Big\{-\Big(\delta + D_2 D_4\Big)T\Big\} =: D_5. \end{aligned} \tag{33}$$

This completes the proof. □

Lemma 5 *There exist constants K_0, K_1, K_2 such that for any $b^1, b^2 \in C([0,T];\mathbb{R}^+)$,*

$$d\Big(\Gamma_0(b^1), \Gamma_0(b^2)\Big) \le K_0 d(b^1, b^2), \tag{34}$$

$$d\Big(\Gamma_1(b^1), \Gamma_1(b^2)\Big) \le K_1 d(b^1, b^2), \tag{35}$$

$$d\Big(\Lambda(b^1), \Lambda(b^2)\Big) \le K_2 d(b^1, b^2). \tag{36}$$

Proof Denote

$$K_0 = TD_4^2, \quad K_1 = D_2 K_0 + D_4. \tag{37}$$

Then (34) and (35) are obtained from (17) by direct calculations. It remains to prove (36). To this end, let $X_t^{b^1}$ and $X_t^{b^2}$ be solutions to the following SDEs

$$\begin{aligned} dX_t^{b^1} &= \Big[A(X_t^{b^1})^{1-\gamma} - \big(\delta + \Gamma_1(b^1)_t\big)X_t^{b^1}\Big]dt - \sigma X_t^{b^1} dW_t, \quad X_0^{b^1} = X_0, \\ dX_t^{b^2} &= \Big[A(X_t^{b^2})^{1-\gamma} - \big(\delta + \Gamma_1(b^2)_t\big)X_t^{b^2}\Big]dt - \sigma X_t^{b^2} dW_t, \quad X_0^{b^2} = X_0. \end{aligned}$$

Denote $Z_t^1 = \left(X_t^{b^1}\right)^\gamma$ and $Z_t^2 = \left(X_t^{b^2}\right)^\gamma$. Again, by Itô's formula, Z_t^1 and Z_t^2 satisfy

$$\begin{aligned} dZ_t^1 &= \gamma\Big[A - \Big(\delta + \Gamma_1(b^1)_t + \frac{(1-\gamma)\sigma^2}{2}\Big)Z_t^1\Big]dt - \gamma\sigma Z_t^1 dW_t, \quad Z_0^1 = X_0^\gamma, \\ dZ_t^2 &= \gamma\Big[A - \Big(\delta + \Gamma_1(b^2)_t + \frac{(1-\gamma)\sigma^2}{2}\Big)Z_t^2\Big]dt - \gamma\sigma Z_t^2 dW_t, \quad Z_0^2 = X_0^\gamma. \end{aligned}$$

These linear SDEs admit the explicit solutions

$$\begin{aligned} Z_t^i &= \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)t - \gamma\int_0^t \Gamma_1(b^i)_s ds - \gamma\sigma W_t\Big\} \\ &\quad \times \Big[X_0^\gamma + \gamma A\int_0^t \exp\Big\{\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)s + \gamma\int_0^s \Gamma_1(b^i)_u du + \gamma\sigma W_s\Big\}ds\Big] \\ &= X_0^\gamma \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)t - \gamma\int_0^t \Gamma_1(b^i)_s ds - \gamma\sigma W_t\Big\} \\ &\quad + \gamma A\int_0^t \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)(t-s) - \gamma\int_s^t \Gamma_1(b^i)_u du - \gamma\sigma(W_t - W_s)\Big\}ds, \end{aligned} \tag{38}$$

for $i = 1, 2$. Again, by a comparison theorem, we have $Z_t^i \le Z_t^u$ for $i = 1, 2$, where Z_t^u is defined in (26). We have

$$d(\Lambda(b^1), \Lambda(b^2)) = \sup_{0\le t\le T}\left|\Lambda(b^1)_t - \Lambda(b^2)_t\right| = \sup_{0\le t\le T}\left|\left[EZ_t^1\right]^{\frac{1}{\gamma}} - \left[EZ_t^2\right]^{\frac{1}{\gamma}}\right|. \tag{39}$$

By the inequality

$$\left|a^{\frac{1}{\gamma}} - b^{\frac{1}{\gamma}}\right| \le \frac{|a-b|}{\gamma}\max\left\{a^{\frac{1-\gamma}{\gamma}}, b^{\frac{1-\gamma}{\gamma}}\right\}, \quad a, b > 0$$

and the fact that $\max\{EZ_t^1, EZ_t^2\} \le EZ_t^u$, we obtain

$$\Big|\big[EZ_t^1\big]^{\frac{1}{\gamma}} - \big[EZ_t^2\big]^{\frac{1}{\gamma}}\Big| \le \frac{1}{\gamma}\big|EZ_t^1 - EZ_t^2\big|\big(EZ_t^u\big)^{\frac{1-\gamma}{\gamma}}. \tag{40}$$

Next, we estimate $\big|EZ_t^1 - EZ_t^2\big|$. Using the inequality $|e^{-a} - e^{-b}| \le |a - b|$ for $a, b \ge 0$ and (35), we have

$$\begin{aligned}
&\Big|\exp\Big\{-\gamma\int_s^t \Gamma_1(b^1)_u du\Big\} - \exp\Big\{-\gamma\int_s^t \Gamma_1(b^2)_u du\Big\}\Big| \\
&\le \gamma\int_s^t \big|\Gamma_1(b^1)_u - \Gamma_1(b^2)_u\big| du \\
&\le \gamma T K_1 d(b^1, b^2),
\end{aligned} \tag{41}$$

for any $0 \le s \le t \le T$. It follows from (38) and (41) that

$$\begin{aligned}
&\big|EZ_t^1 - EZ_t^2\big| \\
&\le \big[EX_0^\gamma\big]\Big[E\exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)t - \gamma\sigma W_t\Big\}\Big] \\
&\quad\times\Big|\exp\Big\{-\gamma\int_0^t \Gamma_1(b^1)_s ds\Big\} - \exp\Big\{-\gamma\int_0^t \Gamma_1(b^2)_s ds\Big\}\Big| \\
&\quad+\gamma A\int_0^t \Big[E\exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)(t-s) - \gamma\sigma(W_t - W_s)\Big\}\Big] \\
&\quad\times\Big|\exp\Big\{-\gamma\int_s^t \Gamma_1(b^1)_u du\Big\} - \exp\Big\{-\gamma\int_s^t \Gamma_1(b^2)_u du\Big\}\Big| ds \\
&\le \gamma T K_1 d(b^1, b^2)\Big[d_0^\gamma \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)t + \frac{\gamma^2\sigma^2 t}{2}\Big\} \\
&\quad+\gamma A\int_0^t \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)(t-s) + \frac{\gamma^2\sigma^2(t-s)}{2}\Big\} ds\Big] \\
&\le \gamma T\big[d_0^\gamma + \gamma AT\big] K_1 d(b^1, b^2).
\end{aligned} \tag{42}$$

We have used the fact that $\gamma \in (0, 1)$ in the last inequality. Combining (39), (40), (27) and (42), we have

$$d(\Lambda(b^1), \Lambda(b^2)) \le T\big[d_0^\gamma + \gamma AT\big]^{\frac{1}{\gamma}} K_1 d(b^1, b^2) = K_1 D_6 T d(b^1, b^2).$$

This implies (36) with

$$K_2 = K_1 D_6 T. \tag{43}$$

This completes the proof. □

Theorem 6 (i) *There exist positive constants* D_1, D_2 *such that*

$$\Gamma : C([0,T];[D_1,D_2]) \to C([0,T];[D_1,D_2]).$$

(ii) *There exists a constant* K *such that*

$$d(\Gamma(b^1), \Gamma(b^2)) \leq K d(b^1, b^2) \tag{44}$$

for any pair b^1, b^2 *in* $C([0,T];[D_1,D_2])$.

Proof (i) First we take

$$D_2 = (D_4 D_6)^{\frac{\theta}{1-\theta}}, \tag{45}$$

where $\theta = \frac{\lambda\gamma}{1-\gamma}$, D_4 and D_6 are respectively given in (24) and (28). Thus, by Lemma 4 (i), for any $b \in C([0,T];\mathbb{R}^+)$ satisfying $b_t \leq D_2$ for any $0 \leq t \leq T$, we have

$$\Gamma(b)_t = \left[b_t \Gamma_0(b)_t \Lambda(b)_t\right]^{\frac{\lambda\gamma}{1-\gamma}} \leq (D_2 D_4 D_6)^{\theta} = D_2. \tag{46}$$

Next, we apply Lemma 4 (ii) with D_2 given in (45) and define D_3 and D_5 respectively as in (29) and (33). Denote

$$D_1 = (D_3 D_5)^{\frac{\theta}{1-\theta}}. \tag{47}$$

It follows from Lemma 4 (ii) that for any $b \in C([0,T];\mathbb{R}^+)$ satisfying $b_t \geq D_1$ for any $0 \leq t \leq T$, we have

$$\Gamma(b)_t = \left[b_t \Gamma_0(b)_t \Lambda(b)_t\right]^{\frac{\lambda\gamma}{1-\gamma}} \geq (D_1 D_3 D_5)^{\theta} = D_1. \tag{48}$$

Combining (46) and (48) implies $\Gamma : C([0,T];[D_1,D_2]) \to C([0,T];[D_1,D_2])$.

(ii) For any $b \in C([0,T];[D_1,D_2])$, we have

$$\left(\Gamma_1(b)_t \Lambda(b)_t\right)^{\frac{\lambda\gamma}{1-\gamma}-1} = \frac{\Gamma(b)_t}{b_t \Gamma_0(b)_t \Lambda(b)_t} \leq \frac{D_2}{D_1 D_3 D_5}. \tag{49}$$

Therefore, using the elementary inequality $|a^\theta - b^\theta| \leq \theta|a-b| \max(a^{\theta-1}, b^{\theta-1})$ for $\theta = \frac{\lambda\gamma}{1-\gamma}$, $a, b > 0$, Lemmas 4 and 5, we have

$$\begin{aligned} |\Gamma(b^1)_t - \Gamma(b^2)_t| &= \left|\left(\Gamma_1(b^1)_t \Lambda(b^1)_t\right)^{\frac{\lambda\gamma}{1-\gamma}} - \left(\Gamma_1(b^2)_t \Lambda(b^2)_t\right)^{\frac{\lambda\gamma}{1-\gamma}}\right| \\ &\leq \frac{\lambda\gamma}{1-\gamma}\Big|\Gamma_1(b^1)_t \Lambda(b^1)_t \end{aligned}$$

$$
\begin{aligned}
&- \Gamma_1(b^2)_t\Lambda(b^2)_t\Big| \max_{i=1,2}\left\{\left(\Gamma_1(b^i)_t\Lambda(b^i)_t\right)^{\frac{\lambda\gamma}{1-\gamma}-1}\right\} \\
&\leq \frac{\lambda\gamma}{1-\gamma}\frac{D_2}{D_1D_3D_5}\Big|\Gamma_1(b^1)_t\Lambda(b^1)_t - \Gamma_1(b^2)_t\Lambda(b^2)_t\Big| \\
&\leq \frac{\lambda\gamma}{1-\gamma}\frac{D_2}{D_1D_3D_5}\Big[\Gamma_1(b^1)_t\Big|\Lambda(b^1)_t - \Lambda(b^2)_t\Big| \\
&+ \Big|\Gamma_1(b^1)_t - \Gamma_1(b^2)_t\Big|\Lambda(b^2)_t\Big] \\
&\leq \frac{\lambda\gamma}{1-\gamma}\frac{D_2}{D_1D_3D_5}\big(D_2D_4K_1 + D_6K_2\big)d(b^1,b^2). \qquad (50)
\end{aligned}
$$

This leads to

$$
d\left(\Gamma(b^1),\Gamma(b^2)\right) = \sup_{0\leq t\leq T}\Big|\Gamma(b^1)_t - \Gamma(b^2)_t\Big| \leq Kd(b^1,b^2),
$$

where

$$
K = \frac{\lambda\gamma}{1-\gamma}\frac{D_2}{D_1D_3D_5}\big(D_2D_4K_1 + D_6K_2\big). \qquad (51)
$$

□

The following corollary is a direct consequence of Theorem 6 and the contraction mapping theorem.

Corollary 7 *If $K < 1$ in (51), then (21) has a unique solution $b \in C([0,T],[D_1,D_2])$, where D_1 and D_2 are respectively defined in (47) and (45).*

Remark 1 A potentially useful approach to prove the existence of a solution to (21) is to apply Schauder's theorem. This would rely on analyzing equicontinuity properties of functions defined on $[0,T]$ under the operator Γ.

5 Mean Field Approximation and ε-Nash Equilibrium

So far our analysis in Sects. 3 and 4 focusses on the infinite population model where the utility functional involves $\bar{C}^{(\gamma)}$. The question now is how to justify such an approximation in a finite population model.

The capital stock of agent i, $1 \leq i \leq N$, satisfies the following equation

$$
dX_t^i = \left[A(X_t^i)^{1-\gamma} - \delta X_t^i - C_t^i\right]dt - \sigma X_t^i dW_t^i, \quad t \geq 0,
$$

and the utility functional of agent i has the form

$$
J_i(C^1,\ldots,C^N) = E\left[\int_0^T \frac{e^{-\rho t}}{\gamma}\left[\frac{\left(C_t^i\right)^\gamma}{\left(C_t^{(N,\gamma)}\right)^\lambda}\right]dt + e^{-\rho T}\eta\frac{(X_T^i)^\gamma}{\gamma}\right].
$$

Below we consider the case $\lambda > 0$. Once b_t is determined from (21), we further obtain

$$\bar{C}_t^{(\gamma)} = b_t^{\frac{\gamma-1}{\lambda}}. \tag{52}$$

Let all the agents apply the decentralized strategies

$$\hat{C}_t^i = \Gamma_1(b)_t X_t^i, \quad 1 \le i \le N,$$

which correspond to the following closed-loop state equations:

$$d\hat{X}_t^i = \Big[A(\hat{X}_t^i)^{1-\gamma} - \delta\hat{X}_t^i - \Gamma_1(b)_t\hat{X}_t^i\Big]dt - \sigma\hat{X}_t^i dW_t^i, \quad 1 \le i \le N,\ t \ge 0. \tag{53}$$

Denote

$$\hat{C}_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^{N}(\hat{C}_t^i)^\gamma.$$

The error estimate for the mean field approximation is given in the following theorem.

Theorem 8 *Suppose that* $b \in C([0,T];\mathbb{R}^+)$ *is a solution of* (21) *with* $\lambda > 0$ *and the i.i.d. initial conditions* X_0^i *satisfy* $E|X_0^i|^{2\gamma} < \infty$. *Then*

$$E|\hat{C}_t^{(N,\gamma)} - \bar{C}_t^{(\gamma)}|^2 = O\Big(\frac{1}{N}\Big).$$

Proof Denote $\hat{Z}_t^i = (\hat{X}_t^i)^\gamma$. Itô's formula yields the following linear SDE

$$d\hat{Z}_t^i = \gamma\left\{A - \left[\delta + \Gamma_1(b)_t + \frac{\sigma^2(1-\gamma)}{2}\right]\hat{Z}_t^i\right\}dt - \gamma\sigma\hat{Z}_t^i dW_t^i. \tag{54}$$

Similar to (38), we have

$$\begin{aligned}\hat{Z}_t^i &= (X_0^i)^\gamma \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)t - \gamma\int_0^t \Gamma_1(b)_s ds - \gamma\sigma W_t^i\Big\}\\ &\quad + \gamma A\int_0^t \exp\Big\{-\gamma\Big(\delta + \frac{\sigma^2}{2}\Big)(t-s) - \gamma\int_s^t \Gamma_1(b)_u du - \gamma\sigma(W_t^i - W_s^i)\Big\}ds.\end{aligned} \tag{55}$$

Since $E|\hat{Z}_0^i|^2 = E|X_0^i|^{2\gamma} < \infty$, it follows that $\sup_{0\le t\le T} E|\hat{Z}_t^i|^2 < \infty$ and $\sup_{0\le t\le T} E|\hat{X}_t^i|^{2\gamma} < \infty$. Note that $(\hat{C}_t^i)^\gamma = (\Gamma_1(b)_t\hat{X}_t^i)^\gamma = (\Gamma_1(b)_t)^\gamma\hat{Z}_t^i$ and $D_1D_3 \le$

$\Gamma_1(b)_t \leq D_2 D_4$ for $0 \leq t \leq T$. Hence, $\left\{(\hat{C}_t^i)^\gamma, 1 \leq i \leq N\right\}$ is a sequence of i.i.d. random variables with bounded second moments for each fixed t, $0 \leq t \leq T$. Since b_t is the solution to the fixed point Eq. (21), Theorem 3 implies $\bar{C}_t^{(\gamma)} = E\big(\hat{C}_t^i\big)^\gamma$ for $i = 1, \ldots, N$. We have

$$E|\hat{C}_t^{(N,\gamma)} - \bar{C}_t^{(\gamma)}|^2 = E\left|\frac{1}{N}\sum_{i=1}^N \Big((\hat{C}_t^i)^\gamma - E(\hat{C}_t^i)^\gamma\Big)\right|^2$$
$$= \frac{1}{N^2}\sum_{i=1}^N E\Big((\hat{C}_t^i)^\gamma - E(\hat{C}_t^i)^\gamma\Big)^2$$
$$= \frac{1}{N}E\Big((\hat{C}_t^1)^\gamma - E(\hat{C}_t^1)^\gamma\Big)^2 = O\Big(\frac{1}{N}\Big).$$

This completes the proof. □

Let the consumption of all agents other than agent i be $\hat{C}^{-i} = (\hat{C}^1, \ldots, \hat{C}^{i-1}, \hat{C}^{i+1}, \ldots, \hat{C}^N)$. Recall that $\bar{J}(\cdot)$ is the utility functional of the limiting problem defined in (6). For simplicity of further performance estimates, we consider the case that all initial states are bounded, i.e.,

$$d_1 \leq X_0^i \leq d_2, \quad 1 \leq i \leq N,$$

for some positive constants d_1 and d_2. For the performance estimate, some special analysis is required to deal with the non-Lipschitz form of the growth dynamics and the ratio type coupling term in the utility functional. We have the following estimate on the approximation of utility functionals.

Theorem 9 *Suppose that $b \in C([0, T]; \mathbb{R}^+)$ is a solution of (21) with $\lambda > 0$ and bounded i.i.d. initial conditions X_0^i. Then*

$$\left|J_i\left(\hat{C}^i, \hat{C}^{-i}\right) - \bar{J}\left(\hat{C}^i\right)\right| = O\left(N^{-\frac{1}{2}}\right).$$

Denote by $\mathcal{U}_i$ the set of all admissible consumption processes $C_t^i \geq 0$ which are adapted to the filtration generated by X_0^j, W_s^j, $j = 1, \ldots, N$, $s \leq t$ such that the corresponding state $X_t^i > 0$ for all $t \in [0, T]$. We are now in a position to state the main result on the ε-Nash equilibrium.

Theorem 10 *Under the conditions of Theorem 9, we have*

$$J_i(\hat{C}^i, \hat{C}^{-i}) \leq \sup_{C^i(\cdot)\in\mathcal{U}_i} J_i(C^i, \hat{C}^{-i}) \leq J_i(\hat{C}^i, \hat{C}^{-i}) + \varepsilon_N,$$

where $\varepsilon_N = O(N^{-\frac{1}{2}})$.

Note that C^i in Theorem 10 is allowed to use sample path information of all agents.

5.1 Proof of Theorems 9 and 10

We have the following lemma.

Lemma 11 *There is a fixed constant D such that $E\int_0^T C_t^i dt \leq D$ for all admissible consumption processes $C^i \in \mathcal{U}_i$.*

Proof Let C_t^i be a fixed. Then the equation

$$dX_t^i = \left[A(X_t^i)^{1-\gamma} - \delta X_t^i - C_t^i\right]dt - \sigma X_t^i dW_t^i, \quad 0 \leq t \leq T$$

has a unique solution X_t^i and $X_t^i > 0$ on $[0, T]$. Let Y_t^i be the unique solution to the linear stochastic differential equation

$$dY_t^i = \left[A(Y_t^i)^{1-\gamma} - \delta Y_t^i\right]dt - \sigma Y_t^i dW_t^i,\ 0 \leq t \leq T, \quad Y_0^i = X_0^i.$$

Denote $Z_t = (X_t^i)^\gamma$, $\tilde{Z}_t = (Y_t^i)^\gamma$, $B = \delta + \frac{(1-\gamma)\sigma^2}{2}$, $f_t = \gamma C_t^i (X_t^i)^{\gamma-1}$ and $z_t = \tilde{Z}_t - Z_t$. Then Itô's formula gives

$$dz_t = (-\gamma B z_t + f_t)dt - \gamma\delta z_t dW_t^i,\ 0 \leq t \leq T, \quad z_0 = 0.$$

We can show this equation has a unique solution. For each positive integer k, denote by z_t^k the unique solution to the following equation

$$dz_t^k = \left(-\gamma B z_t^k + f_t\right)dt - \gamma\delta z_t^k dW_t^i,\ 0 \leq t \leq T, \quad z_0^k = \frac{1}{k}.$$

It is clear that $z_t^k = z_t + y_t^k$ where y_t^k is represented by the linear equation

$$dy_t^k = -\gamma B y_t^k dt - \gamma\delta y_t^k dW_t^i,\ 0 \leq t \leq T, \quad y_0^k = \frac{1}{k}$$

which admits the explicit solution

$$dy_t^k = \frac{1}{k}\exp\left\{-\left(\gamma B + \frac{\gamma^2\sigma^2}{2}\right)t - \gamma\sigma W_t^i\right\}, \quad 0 \leq t \leq T. \tag{56}$$

To proceed, we shall prove that $z_t^k > 0$ for $t \in [0, T]$ and any positive integer k. For $n = 1, 2, \ldots$, let $\tau_n = \inf\{t > 0 : z_t^k = \frac{1}{2kn}\}$ then $\tau_1 < \tau_2 < \ldots$. Denote $x_t^k = \log z_t^k$ if $0 \leq t \leq \tau_n \wedge T$ for some n. Then we can show that on $[0, \tau_n \wedge T]$, x_t^k has the following representation

$$\begin{aligned} x_t^k &= -\log k + \int_0^t \left[-\left(\gamma B + \frac{\gamma^2\sigma^2}{2}\right) + \frac{f_s}{z_s^k}\right]ds - \int_0^t \gamma\sigma dW_s^i \\ &= -\log k - \left(\gamma B + \frac{\gamma^2\sigma^2}{2}\right)t - \gamma\sigma W^i(t) + \int_0^t \frac{f_s}{z_s^k}ds. \end{aligned}$$

Note that on the set $\{\lim_{n\to\infty}\tau_n \le T\}$ we must have $\lim_{n\to\infty} x^k_{\tau_n\wedge T} = -\infty$. Since $\frac{f_s}{z^k_s} > 0$ for $0 \le s \le T$, the above equation implies that $P(\lim_{n\to\infty} x^k_{\tau_n\wedge T} = -\infty) = 0$. Thus, $\lim_{n\to\infty}\tau_n > T$ with probability 1 and we have $z^k_t > 0$ for $t \in [0, T]$.

Since $y^k_t \to 0$ almost surely as $k \to \infty$ by virtue of (56), it follows that $z_t = z^k_t - y^k_t \ge 0$ for any $0 \le t \le T$. This gives $X^i_t \le Y^i_t$ for $t \in [0, T]$. Note that $\tilde{Z}_t = (Y^i_t)^\gamma$ is a solution to a linear stochastic differential equation with constant coefficients and bounded initial condition, it has bounded moment of any order. In particular, $E(Y^i_t)^{1-\gamma} = E(\tilde{Z}_t)^{\frac{1-\gamma}{\gamma}} \le D$ for some constant D for all $t \in [0, T]$.

Next, taking the expectation in both sides of the equation

$$X^i_T = X^i_0 + \int_0^T \left[A(X^i_t)^{1-\gamma} - \delta X^i_t - C^i_t\right]dt - \int_0^T \sigma X^i_t dW^i_t$$

and using the fact that $X^i_T \ge 0$, we obtain

$$EX^i_T = EX^i_0 + \int_0^T \left[AE(X^i_t)^{1-\gamma} - \delta EX^i_t - EC^i_t\right]dt \ge 0.$$

Therefore,

$$\int_0^T EC^i_t dt \le EX^i_0 + \int_0^T AE(X^i_t)^{1-\gamma}dt \le D := EX^i_0 + \int_0^T AE(Y^i_t)^{1-\gamma}dt < \infty.$$

□

Note that as a consequence of the above lemma and Hölder's inequality, if $\gamma p < 1$ there is a fixed constant D such that

$$\int_0^T E(C^i_t)^{\gamma p}dt \le T^{1-\gamma p}\Big(\int_0^T E(C^i_t)dt\Big)^{\gamma p} \le D \tag{57}$$

for all admissible consumption processes $C^i \in \mathcal{U}_i$.

Next, we have following estimate.

Proposition 12 *Under the conditions of Theorem 9, for any $q > 1$ there exists a constant D such that*

$$E\int_0^T \left| \left(\frac{\bar{C}^{(\gamma)}_t}{\hat{C}^{(N,\gamma)}_t}\right)^\lambda - 1 \right|^q dt \le DN^{-\frac{q}{2}}.$$

Proof Let $\hat{Z}^i_t$ be defined as in (54). By (55), under the assumption $d_1 \le X^i_0 \le d_2$ for $i = 1, \dots, N$, we can prove that $E(\hat{Z}^i_t)^p < \infty$ for any real number $p \in \mathbb{R}$. Since $\hat{X}^i_t = (\hat{Z}^i_t)^{1/\gamma}$, $\hat{C}^i_t = b_t\Gamma_0(b)_t\hat{X}^i_t$ and $D_1 \le b_t \le D_2$, it follows that $E(\hat{C}^i_t)^p < \infty$ for any real number p.

For $1 \le i \le N$, denote $Y_i = (\hat{C}_t^i)^\gamma / \bar{C}_t^{(\gamma)}$ and $S_N = \sum_{i=1}^N Y_i$. Then $Y_i \ge 0$, $EY_i = 1$ and $E|Y_i|^p < D_p < \infty$ for any p where D_p is a constant that does not depend on t. In addition,

$$\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}} = \frac{N}{S_N}. \tag{58}$$

Using the inequality $|a^\lambda - 1| \le \lambda |a-1| \max\{a^{\lambda-1}, 1\}$ for $a = N/S_N > 0$, we have

$$\begin{aligned} \left| \left(\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}} \right)^\lambda - 1 \right|^q &= \left| \left(\frac{N}{S_N} \right)^\lambda - 1 \right|^q \\ &\le \lambda^q \left| \frac{N}{S_N} - 1 \right|^q \max\left\{ \left(\frac{N}{S_N} \right)^{(\lambda-1)q}, 1 \right\} \\ &= \lambda^q \left(\frac{N}{S_N} \right)^q \left| \frac{S_N}{N} - 1 \right|^q \max\left\{ \left(\frac{N}{S_N} \right)^{(\lambda-1)q}, 1 \right\}. \end{aligned} \tag{59}$$

Let p_1, p_2, p_3 be positive numbers such that $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, $qp_2 > 2$ and $(1-\lambda)qp_3 > 1$. By (59) and Hölder's inequality, we have

$$\begin{aligned} &E \left| \left(\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}} \right)^\lambda - 1 \right|^q \\ &\le \lambda^q \left[E \left(\frac{N}{S_N} \right)^{qp_1} \right]^{\frac{1}{p_1}} \left[E \left| \frac{S_N}{N} - 1 \right|^{qp_2} \right]^{\frac{1}{p_2}} \left[E \max\left\{ \left(\frac{N}{S_N} \right)^{(\lambda-1)qp_3}, 1 \right\} \right]^{\frac{1}{p_3}}. \end{aligned} \tag{60}$$

By the convexity of the function $x \mapsto x^{-qp_1}$, $x > 0$, and Jensen's inequality we have

$$E \left(\frac{N}{S_N} \right)^{qp_1} = E \left(\frac{S_N}{N} \right)^{-qp_1} \le \frac{1}{N} \sum_{i=1}^N E(Y_i)^{-qp_1} \le D. \tag{61}$$

By a similar way with the convexity of the function $x \mapsto x^{(1-\lambda)qp_3}$, $x > 0$, there is a constant D independent of t such that

$$\begin{aligned} &E \max\left\{ \left(\frac{N}{S_N} \right)^{(\lambda-1)qp_3}, 1 \right\} \\ &\le E \left[\left(\frac{S_N}{N} \right)^{(1-\lambda)qp_3} + 1 \right] \le 1 + \frac{1}{N} \sum_{i=1}^N E(Y_i)^{(1-\lambda)qp_3} \le D. \end{aligned} \tag{62}$$

Next, since $Y_1, Y_2, \ldots, Y_N$ are independent identically distributed random variables with $EY_i = 1$ for $1 \le i \le N$, $M_n = \sum_{i=1}^n (Y_i - 1)$, $1 \le n \le N$ is a martingale. By Burkholder-Davis-Gundy inequality and Jensen's inequality we have

$$\begin{aligned} E\left|\frac{S_N}{N} - 1\right|^{qp_2} &= N^{-qp_2} E|M_N|^{qp_2} \\ &\le N^{-\frac{qp_2}{2}} E\left[\frac{1}{N}\sum_{i=1}^N (Y_i - 1)^2\right]^{\frac{qp_2}{2}} \\ &\le N^{-\frac{qp_2}{2}} \frac{1}{N}\sum_{i=1}^N E(Y_i - 1)^{qp_2} \\ &\le DN^{-\frac{qp_2}{2}}, \end{aligned} \tag{63}$$

where D is a constant independent of t. Combining (60)–(63), we obtain

$$E\left|\left(\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}}\right)^\lambda - 1\right|^q \le DN^{-\frac{q}{2}},$$

where the constant D is independent of t. By taking integration on both sides, this completes the proof. □

Proof of Theorem 9 Let p, q be positive numbers such that $p^{-1} + q^{-1} = 1$. Since the initial condition X_0^i is bounded, it follows from (55) that $\sup_{0\le t\le T} E|\hat{Z}_t^i|^p < \infty$ for any positive number p. This leads to $\sup_{0\le t\le T} E|\hat{X}_t^i|^p < \infty$ and $\sup_{0\le t\le T} E|\hat{C}_t^i|^p < \infty$ for any positive number p. Therefore, by the boundedness of $\bar{C}_t$, Hölder's inequality and Proposition 12, we have

$$\begin{aligned} &\left|J_i(\hat{C}^i, \hat{C}^{-i}) - \bar{J}_i(\hat{C}^i)\right| \\ &= \left|E\int_0^T \frac{e^{-\rho t}}{\gamma}\left(\left[\frac{(\hat{C}_t^i)^\gamma}{(\hat{C}_t^{(N,\gamma)})^\lambda}\right] - \left[\frac{(\hat{C}_t^i)^\gamma}{(\bar{C}_t^{(\gamma)})^\lambda}\right]\right)dt\right| \\ &\le DE\int_0^T |\hat{C}_t^i|^\gamma\left|\left(\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}}\right)^\lambda - 1\right|dt \\ &\le D\left[E\int_0^T |\hat{C}_t^i|^{p\gamma}dt\right]^{1/p}\left[E\int_0^T \left|\left(\frac{\bar{C}_t^{(\gamma)}}{\hat{C}_t^{(N,\gamma)}}\right)^\lambda - 1\right|^q dt\right]^{1/q} \\ &\le DN^{-\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 10 The first inequality is trivial. Thus, it suffices to prove the second one. Let $1 \le i \le N$ and $C_t^i \in \mathcal{U}_i$ be fixed. Let X_t^i be the state of agent i corresponding to the consumption C_t^i. For $1 \le j \le N$, let $\hat{C}_t^j = \Gamma_1(b)_t \hat{X}_t^j$ be the decentralized strategy given in Theorem 3 where $\hat{X}_t^j$ is the corresponding state and b is the solution to the fixed point equation (21). Note that

$$\hat{C}_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^{N}(\hat{C}_t^j)^\gamma, \quad C_t^{(N,\gamma)} = \frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^\gamma + \frac{1}{N}(C_t^i)^\gamma.$$

We write

$$\begin{aligned} J_i(C^i, \hat{C}^{-i}) &= E\left[\int_0^T \frac{e^{-\rho t}}{\gamma}\left[\frac{(C_t^i)^\gamma}{\left(\bar{C}_t^{(\gamma)}\right)^\lambda}\right]dt + e^{-\rho T}\eta\frac{(X_T^i)^\gamma}{\gamma}\right] \\ &\quad + E\int_0^T \frac{e^{-\rho t}}{\gamma}\left(\left[\frac{(C_t^i)^\gamma}{\left(C_t^{(N,\gamma)}\right)^\lambda}\right] - \left[\frac{(C_t^i)^\gamma}{(\hat{C}_t^{(N,\gamma)})^\lambda}\right]\right)dt \\ &\quad + E\int_0^T \frac{e^{-\rho t}}{\gamma}\left(\left[\frac{(C_t^i)^\gamma}{\left(\hat{C}_t^{(N,\gamma)}\right)^\lambda}\right] - \left[\frac{(C_t^i)^\gamma}{\left(\bar{C}_t^{(\gamma)}\right)^\lambda}\right]\right)dt \\ &=: \bar{J}(C^i) + I_1^i + I_2^i, \end{aligned} \tag{64}$$

where $\bar{J}$ is calculated using the dynamics of X_t^i,

To proceed, we observe that

$$\left(\hat{C}_t^{(N,\gamma)}\right)^\lambda - \left(C_t^{(N,\gamma)}\right)^\lambda \le \frac{\lambda}{N}(\hat{C}_t^i)^\gamma\Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^\gamma\Big]^{\lambda-1}. \tag{65}$$

Indeed, this inequality holds true if $(\hat{C}_t^i)^\gamma \le (C_t^i)^\gamma$ as in this case $\left(\hat{C}_t^{(N,\gamma)}\right)^\lambda - \left(C_t^{(N,\gamma)}\right)^\lambda \le 0$. Next, given $\lambda \in (0,1)$, if $(\hat{C}_t^i)^\gamma > (C_t^i)^\gamma$, using the inequality $|a^\lambda - b^\lambda| \le \lambda|a-b|\max\{a^{\lambda-1}, b^{\lambda-1}\}$ for $a, b > 0$, we get

$$\begin{aligned} \left(\hat{C}_t^{(N,\gamma)}\right)^\lambda - \left(C_t^{(N,\gamma)}\right)^\lambda &\le \lambda\Big[\hat{C}_t^{(N,\gamma)} - C_t^{(N,\gamma)}\Big]\big(C_t^{(N,\gamma)}\big)^{\lambda-1} \\ &\le \frac{\lambda}{N}\Big[(\hat{C}_t^i)^\gamma - (C_t^i)^\gamma\Big]\Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^\gamma\Big]^{\lambda-1} \\ &\le \frac{\lambda}{N}(\hat{C}_t^i)^\gamma\Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^\gamma\Big]^{\lambda-1} \end{aligned} \tag{66}$$

which implies (65). Note that in the second inequality we have used the fact that $\lambda - 1 < 0$ and the inequality $\min\left\{\hat{C}_t^{(N,\gamma)}, C_t^{(N,\gamma)})\right\} \ge \frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^\gamma$. Using this inequality again we obtain

$$\left(\hat{C}_t^{(N,\gamma)} C_t^{(N,\gamma)}\right)^{-\lambda} \le \Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^{\gamma}\Big]^{-2\lambda}. \tag{67}$$

Next, by Jensen's inequality for the convex function $f(x) = x^{-\lambda-1}$, $x > 0$,

$$\Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^{\gamma}\Big]^{-\lambda-1} \le \frac{D}{N-1}\sum_{j\neq i}(\hat{C}_t^j)^{-\gamma(\lambda+1)}. \tag{68}$$

Combining (65)–(68) yields

$$\begin{aligned} I_1^i &= E\int_0^T \frac{e^{-\rho t}}{\gamma}(C_t^i)^{\gamma}\Big[\big(\hat{C}_t^{(N,\gamma)}\big)^{\lambda} - \big(C_t^{(N,\gamma)}\big)^{\lambda}\Big]\left(\hat{C}_t^{(N,\gamma)} C_t^{(N,\gamma)}\right)^{-\lambda} dt \\ &\le \frac{D}{N} E\int_0^T (C_t^i)^{\gamma}(\hat{C}_t^i)^{\gamma}\Big[\frac{1}{N}\sum_{j\neq i}(\hat{C}_t^j)^{\gamma}\Big]^{-\lambda-1} dt \\ &\le \frac{D}{N(N-1)} E\int_0^T (C_t^i)^{\gamma}(\hat{C}_t^i)^{\gamma}\Big[\sum_{j\neq i}(\hat{C}_t^j)^{-\gamma(\lambda+1)}\Big] dt \\ &= \frac{D}{N(N-1)}\sum_{j\neq i} E\int_0^T (C_t^i)^{\gamma}(\hat{C}_t^i)^{\gamma}(\hat{C}_t^j)^{-\gamma(\lambda+1)} dt. \end{aligned}$$

Note that $\gamma < 1$, $E(C_t^i) < \infty$ and $E(\hat{C}_t^i)^p < \infty$ for any real number p. Let (p, q, r) be positive numbers such that $\gamma p < 1$ and $p^{-1} + q^{-1} + r^{-1} = 1$. By Hölder's inequality, we obtain

$$\begin{aligned} &E\int_0^T (C_t^i)^{\gamma}(\hat{C}_t^i)^{\gamma}(\hat{C}_t^j)^{-\gamma(\lambda+1)} dt \\ &\le \Big[\int_0^T E\big(C_t^i\big)^{\gamma p} dt\Big]^{\frac{1}{p}}\Big[\int_0^T E\big(\hat{C}_t^i\big)^{\gamma q} dt\Big]^{\frac{1}{q}}\Big[\int_0^T E\big(\hat{C}_t^j\big)^{-\gamma(\lambda+1)r} dt\Big]^{\frac{1}{r}} \\ &\le D \end{aligned}$$

for some constant D. Note that we have used (57) in the last inequality. This implies

$$I_1^i \le O(N^{-1}). \tag{69}$$

Next, similar to Theorem 9, we have $I_2^i = O(N^{-\frac{1}{2}})$. Thus, it follows from (64), (69) and Theorem 9 that

$$\begin{aligned} \sup_{C^i\in\mathcal{U}_i} J_i(C^i, \hat{C}^{-i}) &\le \sup_{C^i} \bar{J}_i(C^i) + O(N^{-\frac{1}{2}} + N^{-1}) \\ &= \bar{J}_i(\hat{C}^i) + O(N^{-\frac{1}{2}}) \\ &= J_i(\hat{C}^i, \hat{C}^{-i}) + O(N^{-\frac{1}{2}}). \end{aligned}$$

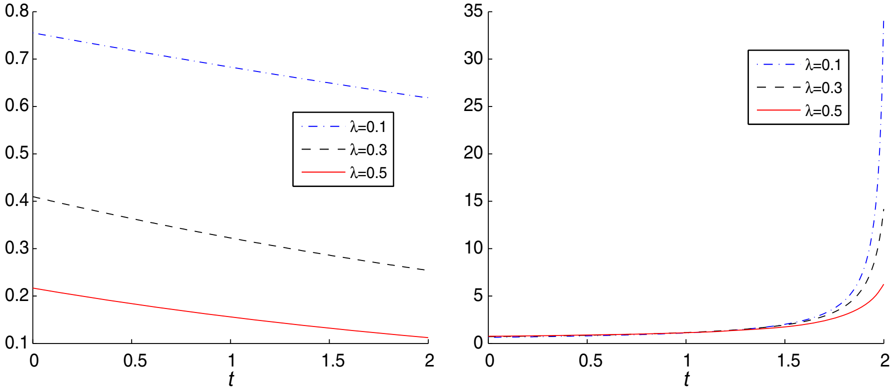

Fig. 1 *Left* b_t solved from the fixed point equation (21); *right* $b_t \Gamma_0(b)_t$

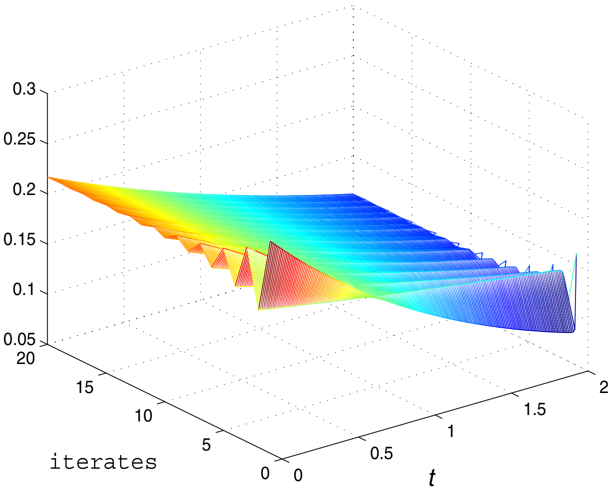

Fig. 2 The computation of b_t in the first 20 iterates by operator Γ

This completes the proof. □

6 Numerical Examples

We solve the fixed equation $b = \Gamma(b)$ in (21) with the following parameters

$$T = 2,\ A = 1,\ \delta = 0.05,\ \gamma = 0.6,\ \eta = 0.2,\ \rho = 0.04,\ \sigma = 0.08,$$

where λ will take three different values 0.1, 0.3, 0.5 for comparisons. The reader is referred to [16] for typical parameter values in capital growth models with stochastic depreciation. Time is discretized with step size 0.01. Fig. 1 (left) solves b by 100 iterates of Γ, and Fig. 1 (right) displays $b_t \Gamma_0(b)_t$ which is the gain of the state feedback policy

$\hat{C}_t^i$. It suggests that when the agent is more concerned with the relative utility (i.e., taking larger λ), it tends to consume with more caution during the late stage. Fig. 2 shows the iteration of b when $\lambda = 0.5$.

Note that Corollary 7 identifies a sufficient condition for Γ to be a contraction mapping. The method there only intends to provide a qualitative result and can be restrictive since various bound estimates obtained may be loose. Our numerical examples show satisfactory convergence to fixed points even when λ is relatively large, indicating strong interaction of the agents. On the other hand, when we replace η by a much smaller value (such as 0.05), it will be easier to encounter non-convergence of the iteration with a moderate value of λ. This is expectable since a very small η causes inadequate regularizing effect near the terminal time and consequently the agents can behave more aggressively, making it unlikely to produce a stable interaction between an individual and the mean field.

7 Conclusion

This paper considers continuous time stochastic growth-consumption optimization in a mean field game setting. The individual performance is based on combining the own utility and the relative utility with respect to the population. Our approach is to apply mean field approximations of the population average utility to determine the best response of a representative agent. An ε-Nash equilibrium property is proved for the resulting set of decentralized strategies.

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