Stochastic Modeling, Algorithms and Analysis for Consensus Seeking over Noisy Networks

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Contents

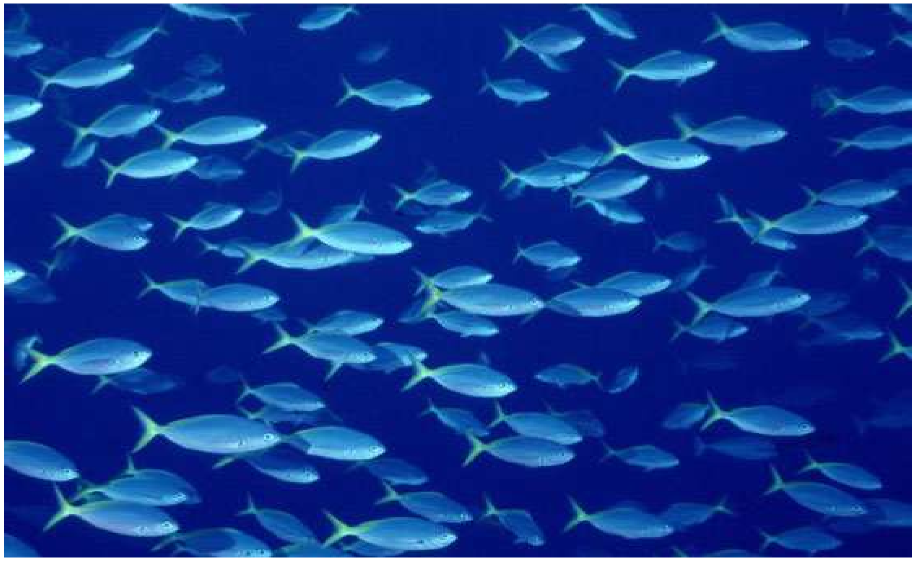
Animal Behavior: Birds

- A group of **birds** fly with coordination in speed and direction (**Flocking**)

Fish

- Huge number of **fish** cooperatively move (**Schooling**)

  -- Important for search for food or for protection from predators



Couzin et.al.
Nature, 2005

Honeybees

- **Honeybees** select a new home from several candidate sites spotted by scout bees

- What is the mechanism for reaching **consensus**? (Visscher, *Nature*, 2003)

-- Important for avoiding population disintegration

From Birds to Bees: from Flocking/Swarming to Consensus

- Each agent has local information about neighboring agents
- and there is a key group objective (e.g., achieve accurate alignment during motion, or agree on a nest site, etc.)

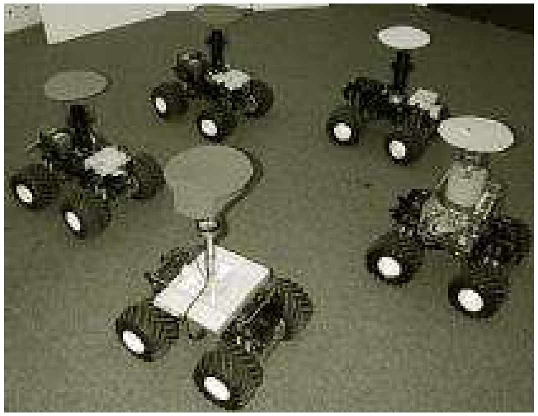
Such coordination amounts to a form of consensus

Math theory for interpretation?

Applications in Technology

- Examples: a group of autonomous vehicles, or robot teams (formation control)

- In such distributed multi-agent control systems – coordination is critical for safety & the performance of tasks (below: simple robots)



Formation of Platoon of Vehicles

- Equalize velocity of different vehicles
- Maintain spacing
- Increase highway capacity and improve safety

The Consensus Issue

- For multi-agent coordination, it is usually important to maintain shared information between agents

- This leads to the key issue of "**Agreeing-on-something**". This agreement may

(1) be the **objective of operation**

(2) or a **condition** for proceeding to further operation

Hence, in this context, **we study consensus problems**.

What Is Consensus?

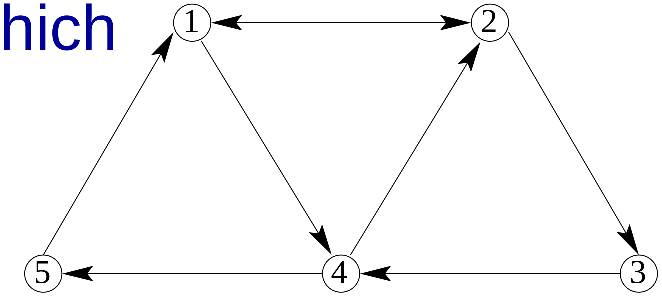
- By consensus seeking, we mean a mechanism whereby the agents adjust their individual values of an **underlying quantity** (e.g., a key state value – angle, velocity, etc.) so as to converge to a **common value**
- In general, **convergence is a primary objective**
- The actually reached limit may be of secondary importance

(small fish schooling)

Background: Models with Exact State Info

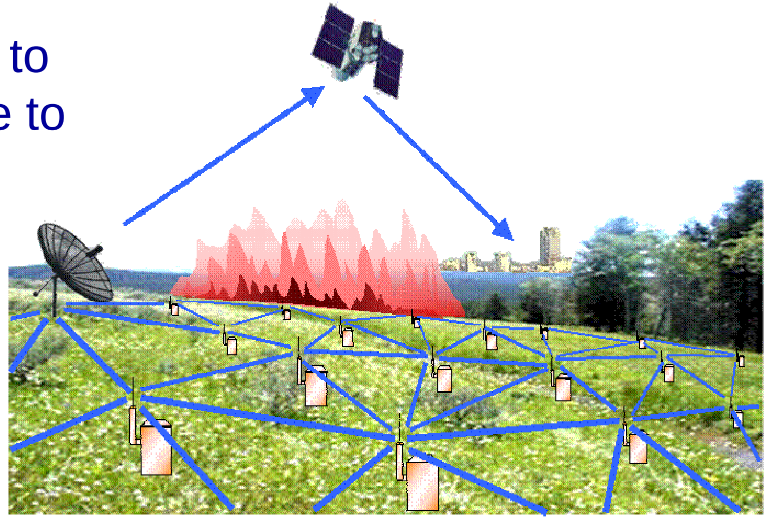
- Most existing research on consensus problems assumes exact state information exchange

- Maintaining certain connectivity (which can be relaxed to different forms) is crucial for achieving consensus



- The most important analytical tools come from the theory of stochastic matrices

Background: Models with Noisy or Inaccurate Measurements



- In a distributed network, it may be impractical to have exact state exchange, for example, due to
  --- receiver noise
  --- quantization, etc. etc.

- Consensus models with additive noises have attracted the interest of many authors
  --- (Ren, Beard and Kingston, ACC'05)
  --- (Xiao, Boyd, and Kim, 2007)
  --- (Huang and Manton, ACC'07, CDC'07, ACC'08,Preprint'06, Preprint'08)
  --- More recent works by various authors …

- Related stochastic models for consensus
  --- (Tsitsiklis, Bertsekas, and Athens, IEEE TAC'86) stochastic gradient based algorithms for distributed function optimization

Definitions

- Definition 1 (weak consensus) The agents are said to reach weak consensus if

$$\lim_{t\to\infty} E|x_t^i - x_t^j|^2 = 0, \quad \forall i, j.$$

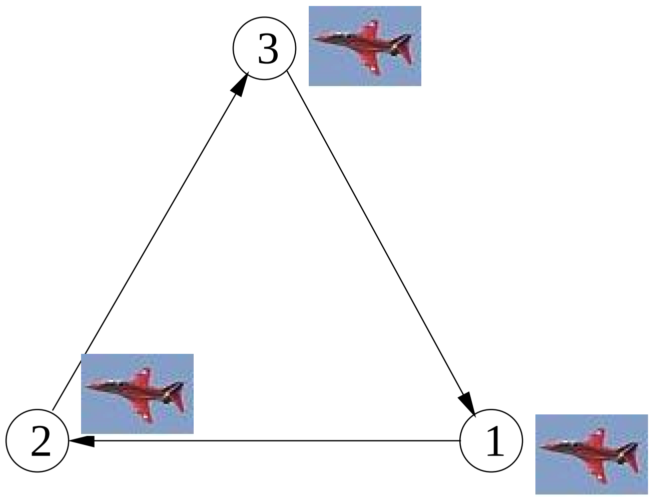
- Definition 2 (mean square consensus) The agents are said to reach m.s. consensus if $E|x_t^i|^2 < \infty, \ \forall i \in \mathcal{N}, t$ and there exists x^* such that

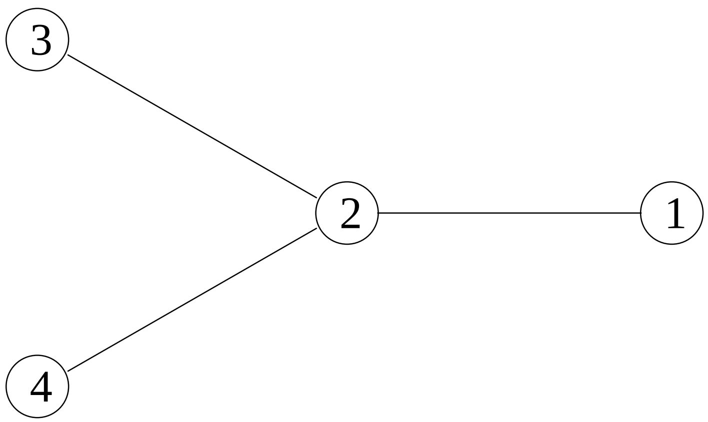
$$\lim_{t\to\infty} E|x_t^i - x^*|^2 = 0, \ \forall i \in \mathcal{N}$$

- Definition 3 (strong consensus) The agents are said to reach strong consensus if there exists x^* such that
$x_t^i \to x^*$ with probability one for all i.

Graph Modeling of Networked Agents

- Consider directed graphs (i.e., digraphs) $G = (\mathcal{N}, \mathcal{E})$
- Each agent is denoted by a node
- In a digraph, arrow indicates neighboring relationship & infor. flow (Example -- right top, agent 1 is a neighbor of agent 2)
- In undirected graph (special case), information is bidirectional





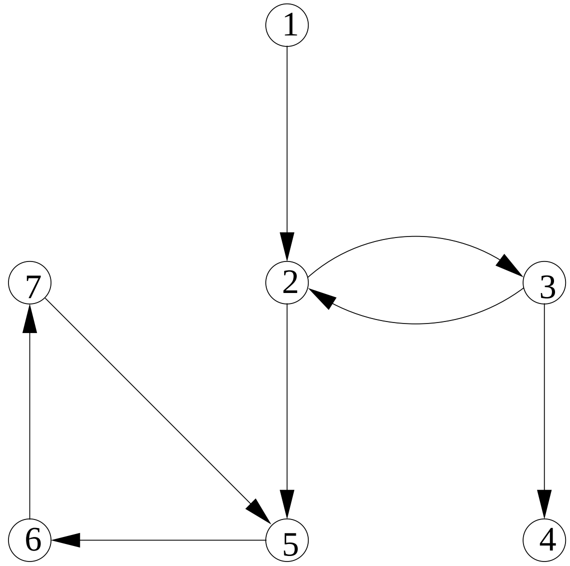
Network Topology Modeling

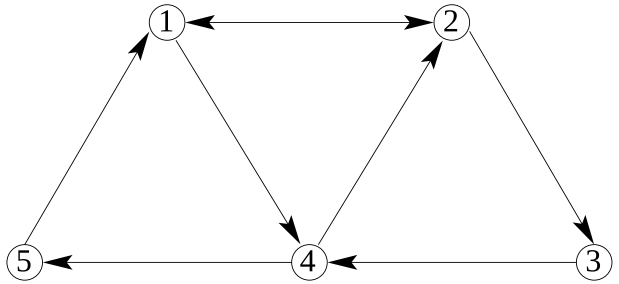
- For our further analysis: we assume---

  The digraph contains a spanning tree (special case: connected undirected graphs)

- Implication: information may propagate across the network from one or more points

- In a deterministic model with fixed topology, Ren et. al. (2005) proved existence of a spanning tree is the weakest connectivity condition for consensus



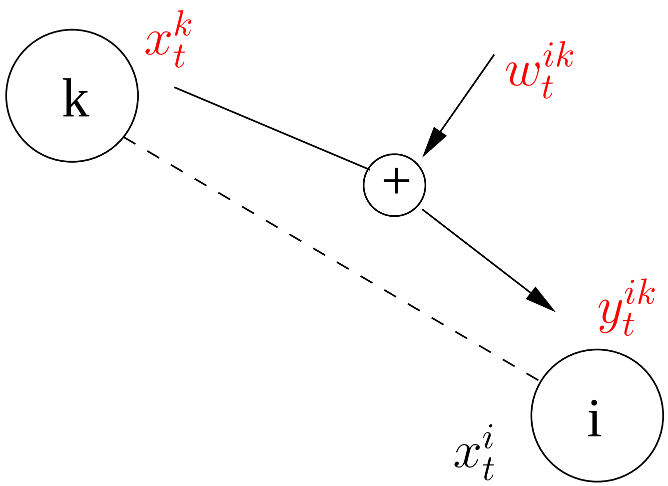


The Measurement Model

- Each agent knows its own state x_t^i exactly,
- and it has noisy observation y_t^{ik} of its neighbors' states, i.e.,

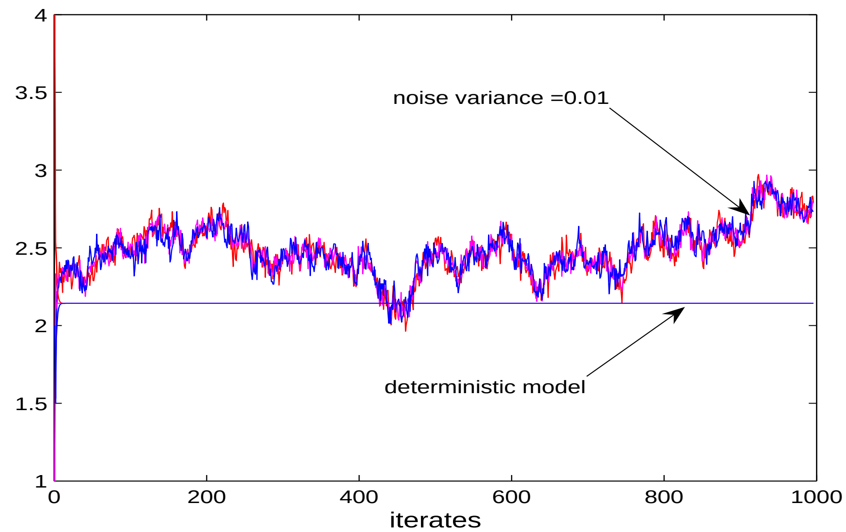
$$y_t^{ik} = x_t^k + w_t^{ik}, \qquad t \in Z^+, \quad k \in \mathcal{N}_i.$$

where w_t^{ik} is additive measurement noise.

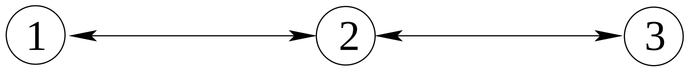


If Fixed Coefficients Are Used in Averaging: What Happens?

$$\begin{cases} x_{t+1}^{1} = \frac{1}{2}(x_t^1 + y_t^{12}) \\ x_{t+1}^{2} = \frac{1}{3}(x_t^2 + y_t^{21} + y_t^{23}) \\ x_{t+1}^{3} = \frac{1}{2}(x_t^3 + y_t^{32}) \end{cases}$$



This algorithm is
essentially a noisy variant
of equal-neighbor
based algorithms
(see related algorithms:
Vicsek et. al. PRL'95
Jadbabaie, Lin, Morse'03, etc.)



Measurement noise causes divergence.

Stochastic Approximation

- We use the averaging rule (convex combination):

$$x_{t+1}^i = (1 - a_t b_{ii}) x_t^i + a_t \sum_{k \in \mathcal{N}_i} b_{ik} y_t^{ik}, \quad t \geq 0$$

$$b_{ik} > 0 \text{ if and only if } k \in \mathcal{N}_i$$

$$b_{ii} = \sum_{k \in \mathcal{N}_i} b_{ik}$$

- The state of a node remains the same if it has no neighbors. (This happens in leader following)

Stochastic Approximation

- The algorithm in vector form:

$$x_{t+1} = x_t + a_t B x_t + a_t \tilde{w}_t$$

where B has zero row sum.

$$B = \begin{bmatrix} -b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & -b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & -b_{nn} \end{bmatrix}$$

where

$$b_{ij} = 0 \text{ if } j \notin \mathcal{N}_i \cup \{i\}.$$

- B is unstable and may be viewed as the generator of a continuous time Markov chain.

Main Assumptions

- (A1) The measurement noises are independent random variables with zero mean, and independent of initial states.
  The noise and initial states have bounded second order moment.

- (A2) The digraph contains a spanning tree.

- (A3) The positive step size sequence $\{a_t, t \geq 0\}$ satisfies:

$$\sum_{i=0}^{\infty} a_i^2 < \infty, \quad \sum_{i=0}^{\infty} a_i = \infty$$

Remark: The independence noise sequence assumption may be relaxed (for instance, a sequence of martingale differences for noise vectors)

Illustration with a Two-agent Model

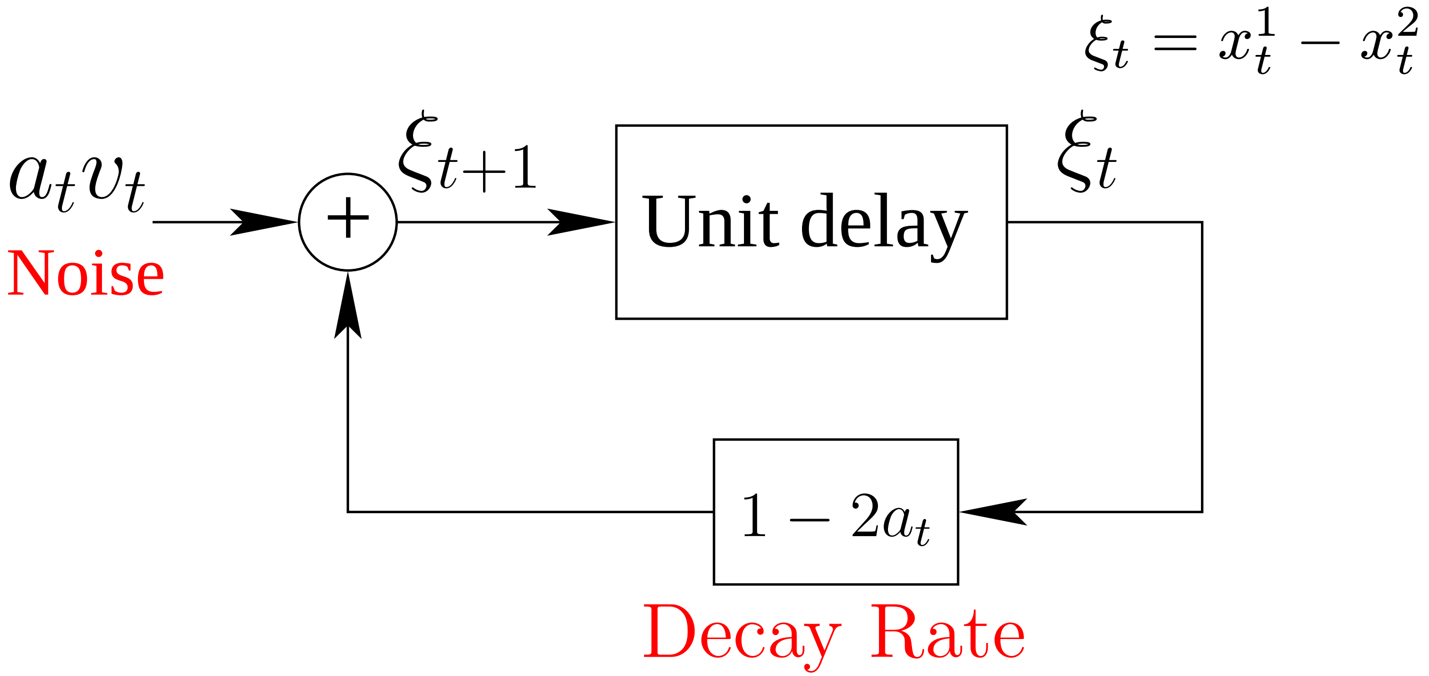
- First, under (A1)-(A2) for noise and step size, it is relatively easy to show (a.s. and m.s.) convergence of the mid-point

$$z_t = \frac{1}{2}(x_t^1 + x_t^2) \to z^*$$

- Next it suffices to show (a.s. and m.s.) convergence of the state gap $\xi_t = x_t^1 - x_t^2$
- We have $\xi_{t+1} = (1 - 2a_t)\xi_t + a_t v_t, \quad t \geq 0$

where $v_t = w_t^{12} - w_t^{21}$

The Diagram for State Gap



- Key idea: show benefits of reducing noise **outweigh** the disadvantage of reducing stability

State Gap as Noise Summation

- Denote $\bar{a}_t = 2a_t$ and

$$\Pi_{l,k} = (1-\bar{a}_l)(1-\bar{a}_{l-1})\cdots(1-\bar{a}_{k+1})a_k$$

for $l > k \geq T_1$. We set $\Pi_{k,k} = a_k$.

- The state gap satisfies

$$\begin{aligned} \xi_{t+1} = \quad & (1-\bar{a}_t)(1-\bar{a}_{t-1})\cdots(1-\bar{a}_{T_1})\xi_{T_1} \\ & +\Pi_{t,T_1}v_{T_1} \\ & \vdots \\ & +\Pi_{t,t-1}v_{t-1} \\ & +\Pi_{t,t}v_t \end{aligned}$$

$\longrightarrow$

- To prove vanishing gap: Show $\Pi_{t,k}$ or related terms sufficiently small

Convergence Analysis

- Mean square convergence

- Sample path convergence

How to Prove M.S. Convergence?

- Use stochastic Lyapunov analysis to show all individual states attract to each other in mean square

- Next, show the individual states actually go to the same limit.

The Lyapunov Function

- Let $S^{n\times n}$ be the set of symmetric matrices and denote

$$\mathcal{D} = \{D \in S^{n\times n} : D \geq 0,\ \mathrm{Null}(D) = \mathrm{span}\{1_n\}\}$$

- Lemma. Under (A2) and given $D \in \mathcal{D}$, the

$$\text{Degenerate Lyapunov Eqn:} \quad QB + B^T Q = -D$$

has a unique solution

$$Q \in \mathcal{D}.$$

- The idea is to show the energy function $V(t) = E x_t^T Q x_t$ will decay to zero.

Energy Decay and Weak Consensus

- Theorem (weak consensus). Under (A1)-(A3),

  (i) There exist $c_1 > 0$, $c_2 > 0$, and a large $T_c > 0$ such that

$$V(t+1) \leq (1 - a_t c_1 + a_t^2 c_2) V(t) + O(a_t^2)$$

  (ii) Consequently $\lim_{t\to\infty} V(t) = 0$, which implies

$$\lim_{t\to\infty} E|x_t^i - x_t^k|^2 = 0, \forall i, k.$$

Stay in $\mathrm{span}\{1_n\}$!

Remark: Here it is not clear yet whether they will converge to the same limit. (so, need an extra step!)

Mean Square Consensus

- Lemma. There is a unique probability measure π such that $\pi^T B = 0$. Further

$$\pi^T x_{t+1} = \pi^T x_t + a_t \pi^T \tilde{w}_t$$

and $\pi^T x_t$ converges in m.s.

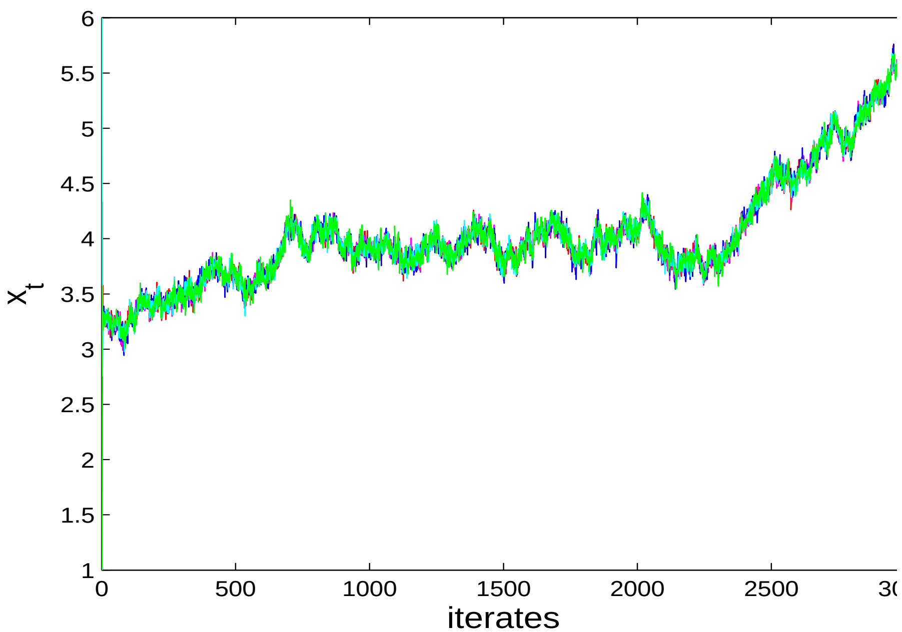
This Lemma combined with $\lim_{t\to\infty} E|x_t^i - x_t^k|^2 = 0, \forall i, k.$

$$\Downarrow$$

Theorem. (A1)-(A3) ensures Mean Square consensus (Huang and Manton, ACC'07,08)

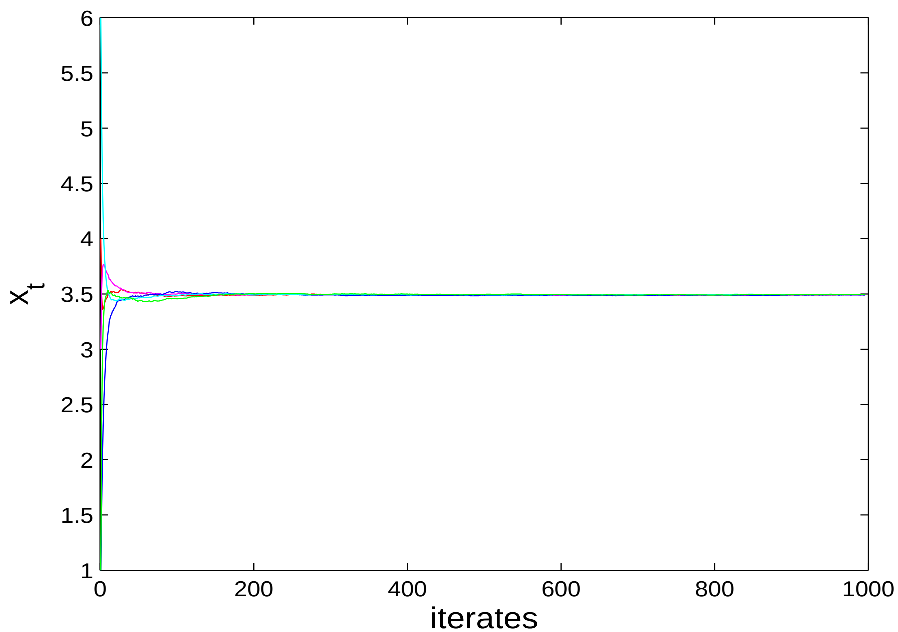
Simulations

- Averaging with fixed weights, noise var=0.01

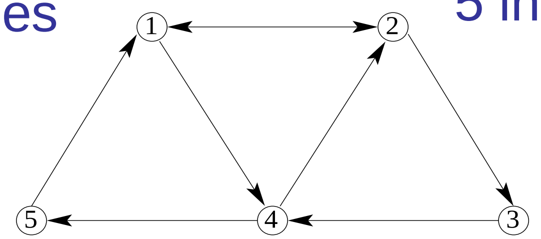


5 individual trajectories

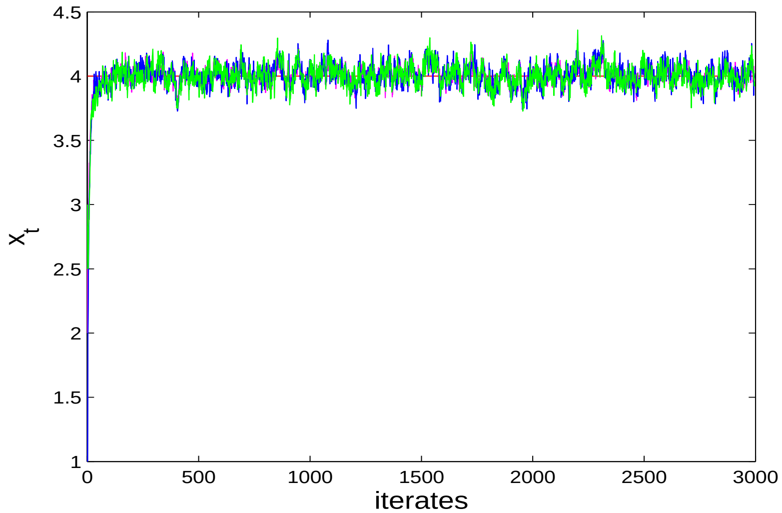
- Stochastic Approx. with decreasing step size

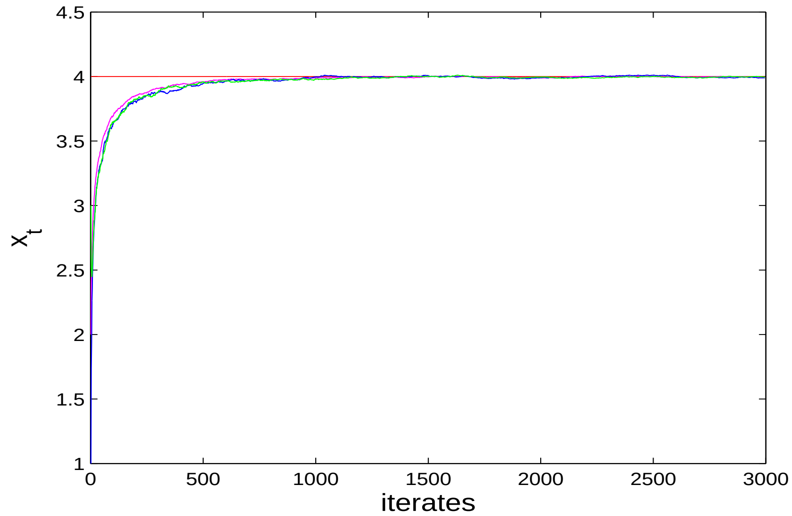


5 individual trajectories

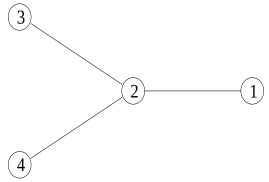


Further Extension to Leader Following

- For leader following, the stochastic Lyapunov analysis is applicable to establish mean square convergence of all other agents’ states to that of the leader (i.e., 4 below).
- 



- Left: use direct averaging   Right: use stochastic approx.



Sample Path Behavior

- What is the group behavior along sample paths?

- In fact, this can be characterized by sample path convergence

Sample Path Convergence

- Theorem 1. Under (A1)-(A3), the Stochastic Approx. (SA) algorithm ensures strong consensus (i.e. sample path convergence).

- Remark: for strong consensus, the second order moment condition for the noise may be relaxed

Sample Path Analysis via Change of Coordinates

- By choosing a suitable change of coordinates $z_t = [z_t^1, z_t^{(n-1)}]^T = \Phi^{-1} x_t,$ the consensus algorithm may be decomposed into the form (Huang & Manton, ACC'08)

$$\begin{cases} z_{t+1}^1 = z_t^1 + a_t v_t^1 \\ z_{t+1}^{(n-1)} = (I + a_t \tilde{B}_{n-1}) z_t^{(n-1)} + a_t v_t^{(n-1)} \end{cases}$$

All eigenvalues of $\tilde{B}_{n-1}$ have negative real parts

Thus, we only need to deal with a random walk and a stable linear SA model

Alternative Proving Tool: Double Array Analysis

- Theorem (Teicher,1985). Let $\{w, w_t, t \geq\}$ be i.i.d. r.v.'s with zero mean and variance Q and $\{a_{ki}, 1 \leq i \leq l_k \uparrow \infty, k \geq 1\}$ a double array of constants. Assume

(i) $\max_{1 \leq i \leq l_k} |a_{ki}| h_i = O(1/\log k)$, where $0 < h_i \uparrow$ $h_i = O(i^{1/\delta})$ for some $\delta \in [1, 2]$

(ii) $\sum_{i=1}^{\infty} P\{|w| > h_i\} < \infty$

(iii) $h_i / i \downarrow$, and $\sum_{i=1}^{l_k} |a_{ki}|^2 h_i^{2-\delta} = o(1/\log k)$, $\sum_{i=1}^{l_k} |a_{ki}|^2 h_i^{2-\delta} = O(1/\log l_k)$

Then

$$\lim_{k \to \infty} \sum_{i=1}^{l_k} a_{ki} w_i = 0, \qquad a.s.$$

Performance?

Performance Analysis

- Due to consensus, denote the limit of the state vector by

$$x_\infty = [x_\infty^1, \cdots, x_\infty^n]^T = x_\infty^1 1_n$$

- Convergence rate --- Roughly, how small is the error term $x_t - x_\infty$ when t is large?

Performance (w/ spanning tree model)

- How fast to reach consensus?(charctrzd by asy. normal.)

- Take step size $a_t = a/t$. Denote $x_t = [x_t^1, \cdots, x_t^n]^T$
- Then under quite standard conditions for noise & coeffic. matrix, we show consensus and furthermore:

$$x_t = x_\infty^1 1_n + x_t^{e,a} + x_t^{e,b}$$

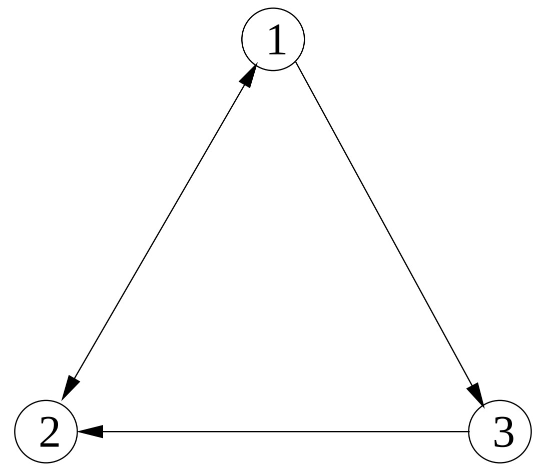
where $x_t^{e,a}$ depends on future noises & $x_t^{e,b}$ is linear in x_t

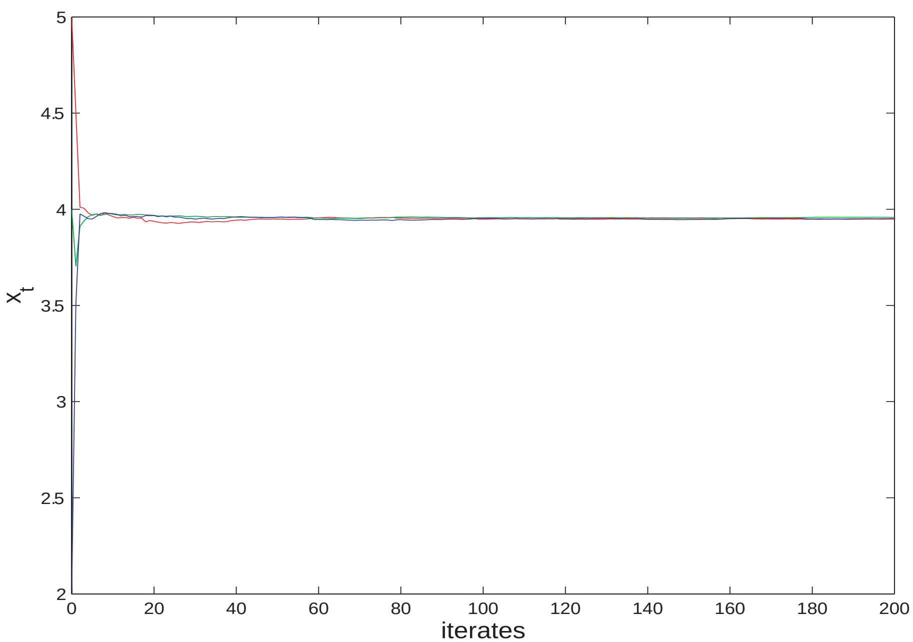
$$\sqrt{t} x_t^{e,a} \xrightarrow{d} N(0, Q_a), \qquad \sqrt{t} x_t^{e,b} \xrightarrow{d} N(0, Q_b)$$

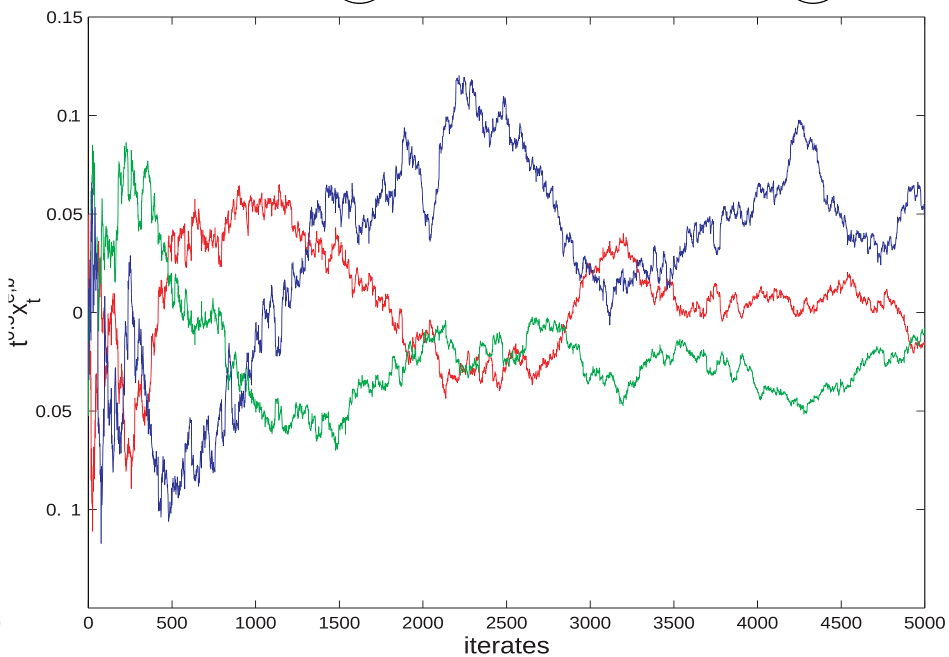
- (H.&M., ACC'08; H. CDC'08 sub) so error decays by rate $\frac{1}{\sqrt{t}}$

Illustration of Asymptotic Normality

- Left bottom --- x_t
- Right bottom --- $t^{1/2} x_t^{e,b}$







Additional Uncertainty Factors

- Random communication link failures

- Quantization effects

Random Link Failures

- The stochastic algorithm may still be applied for the randomly varying topology.
- In this case, the coefficient matrix in the consensus algorithm is given as a sequence of random matrices B_t with mean $\overline{B}$

Random Link Failures (ctn)

- The consensus algorithm

$$x_{t+1} = x_t + a_t B_t x_t + a_t \text{``noise``}$$
$$= x_t + a_t \overline{B} x_t + a_t (B_t - \overline{B}) x_t + a_t \text{``noise``}$$

- This algorithm may be viewed as the standard one (with fixed topology) subject to unbiased perturbations.

- In particular, for i.i.d. link failures with additive measurement noise, a perturbed Lyapunov analysis may be applied to establish convergence (Huang and Manton, ACC'08, and Preprint (submitted to IEEE, June'07))

Quantized Data---How to Achieve Convergence?

Probabilistic Quantization (PQ)

- Suppose the state x_t^i is between two quantization levels $r_k < r_{k+1}$
- The idea of PQ is to produce a randomized output $Q_i(t)$ at the quantizer such that it takes the lower and upper level with probability

$$p_k = (r_{k+1} - x_t^i)/(r_{k+1} - r_k), \quad p_{k+1} = 1 - p_k$$

respectively

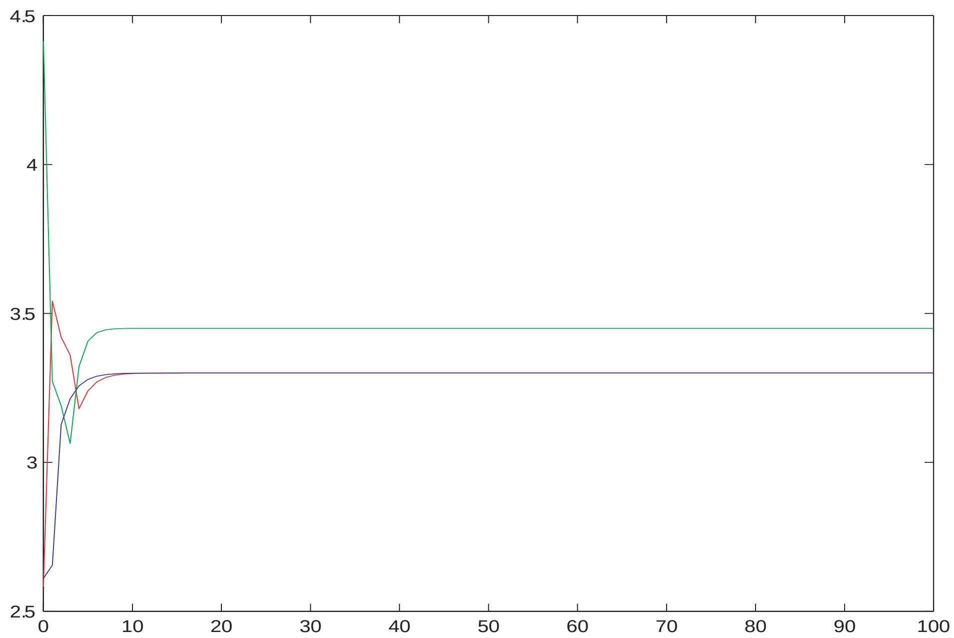
Probabilistic Quantization (PQ)

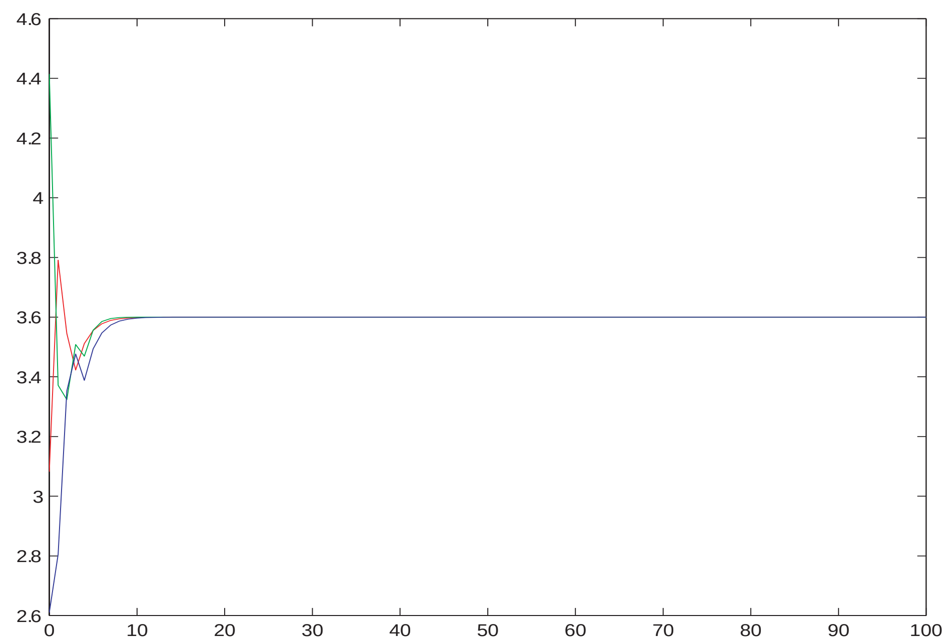
- This approach has been successfully applied for:

- sensor network signal processing (Xiao, Cui, Luo, and Goldsmith, 2006), and
- consensus models (Aysal, Coates and Rabbat, 2007)

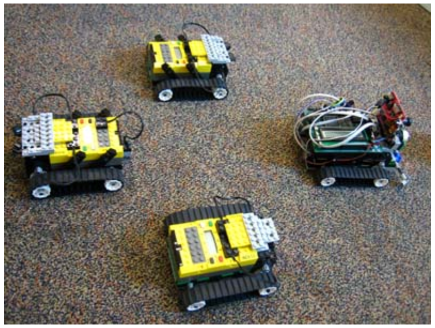
PQ Combined with SA

- In PQ, we may view and quantization error as an additive uncorrelated noise.
- In the consensus algorithm, a decreasing step size may be further used to damp out the noise. Convergence results may be proved. (Huang, Dey, Nair, and Manton, CDC'08 submitted)
- Left: deterministic quantization;   Right: PQ





Concluding Remarks



- Stochastic consensus and convergence
- The key is a decreasing step size for cautious learning
- Stochastic Lyapunov analysis is useful
- Many application opportunities in sensor network signal processing (see, e.g. S. Boyd, J. Hespanha) – networked estim. Prob., sensornet time synchronization, sensornet localization etc. etc. etc.

Many practical modeling scenarios -- high order (inertia) models and asynchronous algorithms, approximate average consensus, etc. etc.