Chapter 1

NASH EQUILIBRIA FOR LARGE-POPULATION LINEAR STOCHASTIC SYSTEMS OF WEAKLY COUPLED AGENTS

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Abstract We consider dynamic games in large population conditions where the agents evolve according to non-uniform dynamics and are weakly coupled via their dynamics and the individual costs. A state aggregation technique is developed to obtain a set of decentralized control laws for the individuals which possesses an ε-Nash equilibrium property. An attraction property of the mass behaviour is established. The methodology and the results contained in this paper reveal novel behavioural properties of the relationship of any given individual with respect to the mass of individuals in large-scale noncooperative systems of weakly coupled agents.

1. Introduction

The control and optimization of large-scale complex systems is evidently of importance due to their ubiquitous appearance in engineering, industrial, social and economic settings. These systems are usually characterized by features such as high dimensionality and uncertainty, and the system evolution is associated with complex interactions among its constituent parts or sub-systems.

In the past decades considerable research effort has been devoted to a significant variety of large-scale systems, and a range of techniques has been developed for their analysis and optimization, including model reduction, aggregation, and hierarchical optimization, etc.

To date, although considerable progress has been made in different directions concerning the optimization of large-scale dynamical systems, general theoretical principles and methodologies are still lacking; this may well be an inherent problem in this domain given the great diversity in the nature of the systems under consideration and their associated optimization problems. So far, most work on optimization of large-scale dynamical systems is based upon centralized performance measures. However, in many social, economic, and engineering models, the individuals or agents involved have conflicting objectives and it is more appropriate to consider optimization based upon individual payoffs or costs. This gives rise to noncooperative game theoretic approaches partly based upon the vast corpus of relevant work within economics and the social sciences. In particular, game theoretic methods have been used in the engineering context in the study of wireless and wired networks optimization, as in Altman, Basar and Srikant (2002), Dziong and Mason (1996).

Game theoretic approaches are intended to capture the individual interest seeking nature of agents in many social, economic and manmade systems; however, in a large-scale dynamic model this approach results in an analytic complexity which is in general prohibitively high, and correspondingly leads to few implementable results on dynamic optimization. We note that the so-called evolutionary games which have been used to treat large population dynamic models at reduced complexity (see Fudenberg and Levine (1998)) are useful mainly for analyzing the asymptotic behaviour of the overall system, and do not lead to a satisfactory framework for the dynamic quantitative optimization of individual performance since the revision of agents' strategies is specified a priori via heuristic rules.

In this paper, we investigate the optimization of large-scale linear control systems wherein many agents (also to be called players) are each coupled with others via the individual dynamics and the costs in a particular form. We view this to be the characteristic property of a class of situations which we term (*distributed*) control problems with weak coupling. The study of such large-scale weakly coupled systems is motivated by a variety of scenarios, for instance, dynamic economic models involving agents linked via a market, and power control in mobile wireless communications. In the latter case, different users have independent power control mechanisms and statistically independent communication channels, but they interact with each other via mutual interference as reflected by the resulting signal-to-interference ratio (SIR) performance indices (cf. Huang, Caines and Malhamé (2003), Huang, Caines and Malhamé (2004b)). Indeed, the model studied in this paper is also related to the research on swarming, flocking, behaviour of human crowds, and formation control of autonomous mobile agents, where each agent has its individual dynamics in which an average effect by all others or the surrounding agents acts as a nominal driving term. For relevant literature, see, e.g., Helbing, Farkas and Vicsek (2000), Tanner, Jadbabaie and Pappas (2003), Liu and Passino (2004), Low (2000). Also, see the large-scale electric load model in Malhamé and Chong (1985).

In the literature, within the optimal control context weakly interconnected systems were studied by Bensoussan (1988). Dynamic LQG games were considered by Papavassilopoulos (1982), and Petrovic and Gajic (1988) proposed an iterative computing procedure with small coupling coefficients for two players assuming existence of a solution. In a two player noncooperative nonlinear dynamic game setting, the Nash equilibria were analyzed in Srikant and Basar (1991) where the coefficients for the coupling terms in the dynamics and costs were restricted to be sufficiently small. In contrast to existing work, our concentration is on games with large populations. We analyze the ε-Nash equilibrium properties for a control law by which each individual optimizes using *local information* its cost function depending upon the state of the individual agent and the average effect of all agents taken together, hereon referred to as "the mass". In preceding work (see Huang, Caines and Malhamé (2003)) we considered the LQG game for a population of uniform agents and introduced a state aggregation procedure for the design of decentralized control with an ε-Nash equilibrium property. In the non-uniform case studied in Huang, Caines and Malhamé (2004a) a given agent only has exact information on its own dynamics, and the information concerning other agents is available in a statistical sense as described by a randomized parametrization for agents' dynamics across the population. Building upon our previous results, in this paper we consider the more general model where the aggregated population effect is incorporated into the individual dynamics. Due to the particular structure of the individual dynamics and costs, the mass formed by all agents impacts any given agent as a nearly deterministic quantity. In response to any known mass influence, a given individual will select its localized control strategy to minimize its own cost. In a practical situation the mass influence cannot be assumed known *a priori*. It turns out, however, that this does not present any difficulty for applying the individual-mass interplay methodology as described below.

In the noncooperative game setup studied here, an overall rationality assumption for the population, to be characterized further down, implies the potential of achieving a stable predictable mass behaviour in the following sense: if some deterministic mass behaviour were to be given, rationality would require that each agent synthesize its individual cost based optimal response as a *tracking* action. Thus the mass trajectory corresponding to rational behaviour would guide the agents to collectively generate the trajectory which, individually, they were assumed to be reacting to in the first place. Indeed, if a mass trajectory with the above fixed point property existed, if it were unique, and furthermore, if each individual had enough information to compute it, then rational agents who were assuming all other agents to be rational would anticipate their collective state of agreement and select a control policy consistent with that state. Thus, in the context of this paper, we make the following rationality assumption: Each agent is rational in the sense that it both (i) optimizes its own cost function, and (ii) assumes that all other agents are being simultaneously rational when evaluating their competitive behaviour. This justifies and motivates the search for mass trajectories with the fixed point property; in fact the resulting situation is seen to be that of a Nash equilibrium holding between any agent and the mass of the other agents.

The central results of this paper consist of the precise characterization of (1) the Nash equilibrium associated with the individual cost functions depending on both the individual and mass behaviour, (2) the consistency (fixed point property) of the mass trajectory under the Nash equilibrium inducing individual feedback controls, and (3) the global attraction property of the mass behaviour in function spaces of policy iterations with respect to such individual optimizing behaviour. This equilibrium then has the rationality and optimality interpretations but we underline that these hypotheses are not employed in the mathematical derivation of the results.

The framework presented in this paper is particularly suitable for optimization of large-scale systems where individuals seek to optimize for their own reward and where it is effectively impossible to achieve global optimality through close coordination between all agents. In this context, the methodology of noncooperative games and state aggregation (particularly stochastic aggregation as presented in Malhamé and Chong (1985)) developed in this paper provides a feasible approach for building simple (decentralized) optimization rules which under appropriate conditions lead to stable population behaviour. Our methodology could potentially provide effective methods for analyzing complex systems arising in socio-economic and engineering areas; see, e.g., Huang, Malhamé and Caines (2004), Baccelli, Hong and Liu (2001).

It is worthwhile noting that the large population limit formulation presented in this paper is relevant to economic problems concerning (mainly static) models with a large number or a continuum of agents; see e.g. Green (1984). However, instead of directly assigning a prior measure in a continuum space for labelling an infinite number of agents, we induce a probability distribution on a parameter space in a natural way via empirical statistics; this approach avoids certain measurability difficulties arising in the direct introduction of dynamics labelled by a continuum (see Judd (1985)). Furthermore, based upon the resulting induced measure, we develop *state aggregation* for the underlying *dynamic* models, and our approach differs from the well-known aggregation techniques initiated by Simon and Ando (1961) based upon time-scales which lead to a form of hierarchical optimization (Sethi and Zhang (1994), Phillips and Kokotovic (1981)).

The paper is organized as follows. In Section 1.2 we introduce the dynamic model. Section 1.3 gives preliminary results on linear tracking. Section 1.4 contains the individual and mass behaviour analysis via a state aggregation procedure. In Section 1.5 we establish the ε-Nash equilibrium property of the decentralized individual control laws. Section 1.6 concludes the paper.

2. The Weakly Coupled Systems

We consider an n dimensional linear stochastic system where the evolution of each state component is described by

$$dz_i = (a_i z_i + b_i u_i)dt + \alpha z^{(n)} dt + \sigma_i dw_i, \qquad 1 \le i \le n, \qquad t \ge 0, \tag{2.1}$$

where $\{w_i, 1 \le i \le n\}$ denotes n independent standard scalar Wiener processes and $z^{(n)} = \frac{1}{n}\sum_{i=1}^{n} z_i$, $\alpha \in \mathbb{R}$. Hence, $z^{(n)}$ may be looked at as a nominal driving term imposed by the population. The initial states $z_i(0)$ are mutually independent and are also independent of $\{w_i, 1 \le i \le n\}$. In addition, $E|z_i(0)|^2 < \infty$ and $b_i \neq 0$. Each state component shall be referred to as the state of the corresponding individual (also to be called an agent or a player).

In this paper we investigate the behaviour of the agents when they interact with each other through specific coupling terms appearing in their cost functions; this is displayed in the following set of *individual* cost functions which shall be used henceforth in the analysis:

$$J_i(u_i, v_i) \stackrel{\triangle}{=} E \int_0^\infty e^{-\rho t}[(z_i - v_i)^2 + ru_i^2]dt. \tag{2.2}$$

For simplicity of analysis we assume in this paper that

$$b_i = b > 0, \qquad 1 \le i \le n.$$

In particular we assume the cost-coupling to be of the following form:

$$v_i = \Phi(z^{(n)}) = \Phi(\frac{1}{n}\sum_{k=1}^{n} z_k),$$

where Φ is a continuous function on $\mathbb{R}$, and we study the large-scale system behaviour in the dynamic noncooperative game framework. Evidently the linking term v_i gives a measure of the average effect of the mass formed by all agents in this type of group tracking problem. Here we assume $\rho, r > 0$ and unless otherwise stated, throughout the paper z_i is described by the dynamics (2.1).

3. The Preliminary Linear Tracking Problem

With the particular set of individual costs (2.2), the first key step in our analysis is to construct a certain deterministic approximation of the aggregate impact of the mass on a given player. In the tracking analysis, we begin by replacing the average driving term $z^{(n)}$ in (2.1) by a deterministic function f. This suggests we introduce the auxiliary dynamics

$$d\hat{z}_i = a_i \hat{z}_i dt + bu_i dt + \alpha f dt + \sigma_i dw_i, \tag{3.1}$$

where f is bounded and continuous on $[0, \infty)$. For distinction, the state variable $\hat{z}_i$ is used in (3.1), and all other terms are specified in a similar manner as in (2.1).

For large n, we intend to approximate the term $v_i \stackrel{\triangle}{=} \Phi(\frac{1}{n}\sum_{k=1}^{n} z_k)$ in Section 1.2 by a *deterministic* continuous function z^* defined on $[0, \infty)$. Here we choose z^* in a more general setting without relating it to the function f introduced above. For a given z^*, we construct the *individual cost* associated with (3.1) as follows:

$$J_i(u_i, z^*) = E \int_0^\infty e^{-\rho t}\{[\hat{z}_i - z^*]^2 + ru_i^2\}dt. \tag{3.2}$$

We shall consider the tracking problem with bounded z^*. For minimization of J_i, the admissible control set is taken as $\mathcal{U}_i \triangleq \{u_i | u_i$ adapted to $\sigma(\hat{z}_i(0), w_i(s), s \le t)$, and $E\int_0^\infty e^{-\rho t}(\hat{z}_i^2 + u_i^2)dt < \infty\}$. Define

$$C_b[0,\infty) \triangleq \{x \in C[0,\infty),\ |x|_\infty < \infty\},$$

where $|x|_\infty = \sup_{t\ge 0}|x(t)|$, for $x \in C[0,\infty)$. Under the norm $|\cdot|_\infty$, $C_b[0,\infty)$ is a Banach space; see Yosida (1980).

Let Π_i be the positive solution to the algebraic Riccati equation

$$\rho\Pi_i = 2a_i\Pi_i - \frac{b^2}{r}\Pi_i^2 + 1. \tag{3.3}$$

It is easy to verify that $-a_i + \frac{b^2\Pi_i}{r} + \frac{\rho}{2} > 0$. Denote

$$\beta_1 = -a_i + \frac{b^2}{r}\Pi_i, \quad \beta_2 = -a_i + \frac{b^2}{r}\Pi_i + \rho. \tag{3.4}$$

Clearly, $\beta_2 > \frac{\rho}{2}$. The proofs of Propositions 3.1 and 3.2 below may be obtained following an algebraic approach as in Bensoussan (1992) (pp. 21-25).

PROPOSITION 3.1 *Assume (i)* $E|\hat{z}_i(0)|^2 < \infty$ *and* $f, z^* \in C_b[0,\infty)$*; (ii)* $\Pi_i > 0$ *is the solution to (3.3) and* $\beta_1 = -a_i + \frac{b^2}{r}\Pi_i > 0$*; and (iii)* $s_i \in C_b[0,\infty)$ *is determined by the differential equation*

$$\rho s_i = \frac{ds_i}{dt} + a_i s_i - \frac{b^2}{r}\Pi_i s_i + \alpha\Pi_i f - z^*. \tag{3.5}$$

Then the control law

$$\widehat{u}_i = -\frac{b}{r}(\Pi_i\hat{z}_i + s_i) \tag{3.6}$$

minimizes $J_i(u_i, z^*)$*, for all* $u_i \in \mathcal{U}_i$. □

PROPOSITION 3.2 *Suppose assumptions (i)-(iii) in Proposition 3.1 hold and* $q \in C_b[0,\infty)$ *satisfies*

$$\rho q = \frac{dq}{dt} - \frac{b^2}{r}s_i^2 + (z^*)^2 + 2\alpha f s_i + \sigma_i^2\Pi_i. \tag{3.7}$$

Then the cost for the control law (3.6) is given by $J_i(\widehat{u}_i, z^*) = \Pi_i E\hat{z}_i^2(0) + 2s(0)E\hat{z}_i(0) + q(0)$. □

Remark. In Proposition 3.1, assumption (i) insures that J_i has a finite minimum attained at some $u_i \in \mathcal{U}_i$. Assumption (ii) means that the resulting closed-loop system has a stable pole. □

Remark. s_i in Proposition 3.1 may be uniquely determined only utilizing its boundedness, and it is unnecessary to specify the initial condition for (3.5) separately. Similarly, after $s_i \in C_b[0, \infty)$ is obtained, q in Proposition 3.2 can be uniquely determined from its boundedness. □

PROPOSITION 3.3 *Under the assumptions of Proposition 3.1, there exists a unique initial condition $s_i(0) \in \mathbb{R}$ such that the associated solution s_i to (3.5) is bounded, i.e., $s_i \in C_b[0, \infty)$. And moreover, for the obtained $s_i \in C_b[0, \infty)$, there is a unique initial condition $q(0) \in \mathbb{R}$ for (3.7) such that the solution $q \in C_b[0, \infty)$.*

Proof. Consider (3.5) for an initial condition $s_i(0)$ which leads to

$$s_i(t) = s_i(0)e^{\beta_2 t} + e^{\beta_2 t} \int_0^t e^{-\beta_2 \tau}[z^*(\tau) - \alpha \Pi_i f(\tau)]d\tau.$$

Since $\beta_2 > 0$ always holds, the integral $\int_0^\infty e^{-\beta_2 \tau}[z^*(\tau) - \alpha \Pi_i f(\tau)]d\tau$ exists and is finite. We take initial condition $s_i(0) = -\int_0^\infty e^{-\beta_2 \tau}[z^*(\tau) - \alpha \Pi_i f(\tau)]d\tau$ which yields

$$s_i(t) = e^{\beta_2 t} \int_t^\infty e^{-\beta_2 \tau}[\alpha \Pi_i f(\tau) - z^*(\tau)]d\tau \in C_b[0, \infty),$$

and it is easily verified that any initial condition other than $s_i(0)$ yields an unbounded solution. Similarly, a unique initial condition $q(0)$ in (3.7) may be determined to give $q \in C_b[0, \infty)$. □

4. Competitive Behaviour and Continuum Mass Behaviour

In the weakly coupled situation with individual costs, each agent is assumed to be rational in the sense that it both optimizes its own cost and its strategy is based upon the assumption that the other agents are rational. In other words each agent believes (i.e., has as a hypothesis in the derivation of its strategy) the other agents are optimizers.

Due to the specific structure of the dynamics and cost, under the rationality assumption it is possible to approximate the driving term $z^{(n)}$ and the linking term $v_i = \Phi(z^{(n)})$ by a purely deterministic process f and $z^* = \Phi(f)$, respectively, and as a result, if a deterministic tracking is employed by the i-th agent, its optimality loss will be negligible in large population conditions. Hence, all agents would tend to adopt such a tracking based control strategy if an approximating f and the associated $z^* = \Phi(f)$ were to be given.

However, we stress that the rationality notion is only used to construct the aggregation procedure, and the main theorems in the paper will be based solely upon their mathematical assumptions.

4.1 State aggregation via large population limit

Assume $f \in C_b[0,\infty)$ is given for approximation of $z^{(n)}$, and $s_i \in C_b[0,\infty)$ is a solution to (3.5) computed with $z^* = \Phi(f)$. For the i-th agent, after applying the optimal tracking based control law (3.6), the closed-loop equation is approximated by

$$dz_i = (a_i - \frac{b^2}{r}\Pi_i)z_i dt - \frac{b^2}{r}s_i dt + \alpha f dt + \sigma_i dw_i, \tag{4.1}$$

where f replaces $z^{(n)}$ in (2.1). Taking expectation on both sides of (4.1) yields

$$\frac{d\overline{z}_i}{dt} = (a_i - \frac{b^2}{r}\Pi_i)\overline{z}_i - \frac{b^2}{r}s_i + \alpha f, \tag{4.2}$$

where $\overline{z}_i(t) = Ez_i(t)$ and the initial condition is $\overline{z}_i|_{t=0} = Ez_i(0)$.

We further define the population average of means (simply called population mean) as $\overline{z}^{(n)} \triangleq \frac{1}{n}\sum_{i=1}^n \overline{z}_i$. Note that in the case all agents have i.i.d. dynamics the evolution of $\overline{z}^{(n)}$ is simply expressed using the dynamics of any $\overline{z}_i$ combined with the initial condition $\overline{z}^{(n)}|_{t=0}$.

So far, the individual reaction is determined in a straightforward manner if a mass effect f is given *a priori*. Here one naturally comes up with the important questions: How is the deterministic process f chosen to approximate the overall influence of all players on the given player? In what way does it capture the dynamic behaviour of the collection of many individuals? Since we wish to have $f \approx \frac{1}{n}\sum_{k=1}^n z_k$, for large n it is plausible to express

$$f = \overline{z}^{(n)}, \qquad z^*(t) = \Phi(\overline{z}^{(n)}(t)). \tag{4.3}$$

As n increases, accuracy of the approximations given in (4.3) is expected to improve. After introducing such an equality relation, a dynamic interaction is built up between the individual and the mass: by averaging over the individual mean trajectories, the pair f and z^* is constructed, in response to which the individuals, in turn, optimize their own objectives. Notice that by taking $f = \overline{z}^{(n)}$, the resulting dynamics (4.2) for $\overline{z}_i$ associated with (2.1) are exact, as long as u_i takes the form (3.6).

Our analysis below will be based upon the observation that the large population limit may be employed to determine the effect of the mass of the population on any given individual, and that the population limit is characterized by an empirical distribution, which is assumed to exist. Specifically, our interest is in the case when a_i, $i \geq 1$, is "adequately randomized" in the sense that the population exhibits certain statistical properties. In this context, the association of the value a_i, $i \geq 1$, and the specific index i plays no essential role, and the more important fact is the frequency of occurrence of a_i on different segments in the range space of the sequence $\{a_i, i \geq 1\}$. Within this setup, we assume that the sequence $\{a_i, i \geq 1\}$, has an empirical distribution function $F(a)$.

For the sequence $\{a_i, i \geq 1\}$, we define the empirical distribution associated with the first n agents

$$F_n(x) = \frac{\sum_{i=1}^n 1_{(a_i < x)}}{n}, \qquad x \in \mathbb{R}.$$

We introduce the following assumption:

(**H1**) There exists a distribution function F on $\mathbb{R}$ such that $F_n \to F$ weakly as $n \to \infty$, i.e., $\lim_{n\to\infty} F_n(x) = F(x)$ if F is continuous at $x \in \mathbb{R}$. □

(**H1'**) There exists a distribution function F on $\mathbb{R}$ such that $F_n \to F$ uniformly as $n \to \infty$, i.e., $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0$. □

Remark. It is obvious that (**H1'**) implies (**H1**). Notice that if the sequence $a_1^\infty \stackrel{\triangle}{=} \{a_i, i \geq 1\}$ is sufficiently "randomized" such that a_1^∞ is generated by independent observations on the same underlying distribution function F, then with probability one (**H1'**) holds by Glivenko–Cantelli theorem; see Chow and Teicher (1997). □

For the Riccati equation (3.3), when the coefficient a is used in place of a_i, we denote the corresponding solution by Π_a. Accordingly, we express $\beta_1(a)$ and $\beta_2(a)$ when a and Π_a are substituted into (3.4). Straightforward calculation gives

$$\Pi_a = (\tfrac{b^2}{r})^{-1} \left[a - \tfrac{\rho}{2} + \sqrt{(a - \tfrac{\rho}{2})^2 + \tfrac{b^2}{r}} \right],$$

$$\beta_1(a) = -\tfrac{\rho}{2} + \sqrt{(a - \tfrac{\rho}{2})^2 + \tfrac{b^2}{r}}, \tag{4.4}$$

$$\beta_2(a) = \tfrac{\rho}{2} + \sqrt{(a - \tfrac{\rho}{2})^2 + \tfrac{b^2}{r}}. \tag{4.5}$$

Example 4.1 For the set of parameters: $a = 1$, $b = 1$, $\alpha = 1$, $\sigma = 0.3$, $\rho = 0.5$, $r = 0.1$, we have $\Pi_a = 0.4$, $\beta_1(a) = 3$, $\beta_2(a) = 3.5$. □

To simplify the aggregation procedure we assume zero mean for initial conditions of all agents, i.e., $Ez_i(0) = 0$, $i \geq 1$. The above analysis suggests we consider the large population limit and introduce the equation system:

$$\rho s_a = \frac{ds_a}{dt} + as_a - \frac{b^2}{r}\Pi_a s_a + \alpha\Pi_a \overline{z} - z^*, \tag{4.6}$$

$$\frac{d\overline{z}_a}{dt} = (a - \frac{b^2}{r}\Pi_a)\overline{z}_a - \frac{b^2}{r}s_a + \alpha\overline{z}, \tag{4.7}$$

$$\overline{z} = \int_{\mathcal{A}} \overline{z}_a dF(a), \tag{4.8}$$

$$z^* = \Phi(\overline{z}). \tag{4.9}$$

In the above, each individual differential equation is indexed by the parameter a. For the same reasons as noted in Proposition 3.3, here it is unnecessary to specify the initial condition for s_a derived from optimal tracking, which shall be determined later in an inherent manner. Equation (4.7) with $\overline{z}_a|_{t=0} = 0$ is based upon (4.2). Hence $\overline{z}_a$ is regarded as the expectation given the parameter a in the individual dynamics. Also, in contrast to the arithmetic average for computing $\overline{z}^{(n)}$ appearing in (4.3), (4.8) is derived by use of the empirical distribution function $F(a)$ for the sequence of parameters $a_i \in \mathcal{A}$, $i \geq 1$, with the range space $\mathcal{A}$. Notice that, had the dynamics of (2.1) been nonlinear the calculation of the mean $\overline{z}_a(t)$ dynamics would have involved an integration with respect to the density generated by an associated Fokker-Planck equation as in Malhamé and Chong (1985). Equation (4.8) describing the stochastic aggregation over parameter space would however remain in the same form as in the linear case.

With a little abuse of terminology, we shall conveniently refer to either $\overline{z}$, or in some cases $\Phi(\overline{z})$, as the mass trajectory.

Remark. In the more general case with non-zero $Ez_i(0)$, we may introduce a joint empirical distribution $F_{a,z}$ for the two dimensional sequence $\{(a_i, Ez_i(0)), i \geq 1\}$. Then the function in (4.7) is to be labelled by both the dynamic parameter a and an associated initial condition, and furthermore, the integration in (4.8) is to be computed with respect to $F_{a,z}$. In this paper we only consider the zero initial mean case in order to avoid notational complication. □

We introduce the assumptions:

(**H2**) The function Φ is Lipschitz continuous on $\mathbb{R}$ with a Lipschitz constant $\gamma > 0$, i.e., $|\Phi(y_1) - \Phi(y_2)| \leq \gamma|y_1 - y_2|$ for all $y_1, y_2 \in \mathbb{R}$. □

(**H3**) $\beta_1(a) > 0$ for all $a \in \mathcal{A}$, and $\int_{\mathcal{A}}[\frac{|\alpha|}{\beta_1(a)} + \frac{b^2(\gamma+|\alpha|\Pi_a)}{r\beta_1(a)\beta_2(a)}]dF(a) < 1$, where $\beta_1(a), \beta_2(a)$ are defined by (4.4)-(4.5), $\mathcal{A}$ is a measurable subset of $\mathbb{R}$ and contains all a_i, $i \geq 1$, and $F(a)$ is the empirical distribution function for $\{a_i, i \geq 1\}$, which is assumed to exist. The constant $\gamma > 0$ is specified in (**H2**) □

(**H4**) All agents have mutually independent initial conditions of zero mean, i.e. $Ez_i(0) = 0$, $i \geq 1$. In addition, $\sup_{i\geq 1}[\sigma_i^2 + Ez_i^2(0)] < \infty$. □

We state a sufficient condition to insure $\beta_1(a) > 0$ for $a \in \mathbb{R}$. The proof is trivial and is omitted.

PROPOSITION 4.2 *If $b^2 > \frac{r\rho^2}{4}$, then $\beta_1(a) > 0$ for all $a \in \mathbb{R}$.* □

Remark. Under (**H3**), we have $-\beta_2(a) < -\beta_1(a) < 0$ where $-\beta_1(a)$ is the stable pole of the closed-loop system for the agent with parameter a. $\beta_1(a)$ measures the stability margin. To avoid triviality for the linking term in the cost, we assume $\gamma > 0$ for Φ in (**H2**). □

The following procedure is used to illustrate the interaction between the individual and the mass. First, given $\overline{z} \in C_b[0,\infty)$, Proposition 3.3 implies that (4.6) has the bounded solution

$$s_a(t) = e^{\beta_2(a)t} \int_t^\infty e^{-\beta_2(a)\tau}[\alpha\Pi_a\overline{z}(\tau) - \Phi(\overline{z}(\tau))]d\tau \stackrel{\triangle}{=} \mathcal{T}_1\overline{z}. \tag{4.10}$$

Then under (**H4**), equations (4.7) and (4.8) correspond to the equations below:

$$\begin{aligned}\overline{z}_a(t) = &\int_0^t e^{-\beta_1(a)(t-s)} \\ &\times \left[\alpha\overline{z}(s) + \frac{b^2}{r}e^{\beta_2(a)s}\int_s^\infty e^{-\beta_2(a)\tau}[\Phi(\overline{z}(\tau)) - \alpha\Pi_a\overline{z}(\tau)]d\tau\right] ds,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\overline{z}(t) = &\int_{\mathcal{A}}\int_0^t e^{-\beta_1(a)(t-s)}\left[\alpha\overline{z}(s) + \frac{b^2}{r}e^{\beta_2(a)s}\right. \\ &\times\left.\left[\int_s^\infty e^{-\beta_2(a)\tau}[\Phi(\overline{z}(\tau)) - \alpha\Pi_a\overline{z}(\tau)]d\tau\right]\right] ds dF(a) \\ &\stackrel{\triangle}{=}(\mathcal{T}\overline{z})(t).\end{aligned} \tag{4.12}$$

Here (4.11) indicates what would be the individual mean trajectory resulting from the optimal tracking of a given mass trajectory. Appendix A contains the proof of the following lemma which establishes that $\mathcal{T}$ defined above is a map from $C_b[0,\infty)$ to itself.

LEMMA 4.3 *Under (**H2**)-(**H3**), we have $\mathcal{T}x \in C_b[0,\infty)$, for any $x \in C_b[0,\infty)$.* □

THEOREM 4.4 *Under (**H2**)-(**H3**), the map $\mathcal{T} : C_b[0,\infty) \to C_b[0,\infty)$ has a unique fixed point which is uniformly Lipschitz continuous on $[0,\infty)$.*

Proof. By Lemma 4.3, $\mathcal{T}$ is a map from the Banach space $C_b[0,\infty)$ to itself. For any $x, y \in C_b[0,\infty)$ we have

$$
\begin{aligned}
&|(\mathcal{T}x - \mathcal{T}y)(t)| \\
\le & |x-y|_\infty \int_{\mathcal{A}} \int_0^t |\alpha| e^{-\beta_1(a)(t-s)} ds dF(a) \\
&+ \frac{b^2 |x-y|_\infty}{r} \int_{\mathcal{A}} \int_0^t \int_s^\infty [\gamma + |\alpha|\Pi_a] e^{-\beta_1(a)(t-s)} e^{-\beta_2(a)(\tau-s)} d\tau ds dF(a) \\
\le & |x-y|_\infty \int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF(a) \\
&\quad + \frac{b^2 |x-y|_\infty}{r} \int_{\mathcal{A}} \int_0^t \frac{\gamma + |\alpha|\Pi_a}{\beta_2(a)} e^{-\beta_1(a)(t-s)} ds dF(a) \\
\le & |x-y|_\infty \int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF(a) + \frac{b^2 |x-y|_\infty}{r} \int_{\mathcal{A}} \frac{\gamma + |\alpha|\Pi_a}{\beta_1(a)\beta_2(a)} dF(a) \\
= & |x-y|_\infty \left[\int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF(a) + \frac{b^2}{r} \int_{\mathcal{A}} \frac{\gamma + |\alpha|\Pi_a}{\beta_1(a)\beta_2(a)} dF(a) \right].
\end{aligned}
$$

Then from (**H3**) it follows that $\mathcal{T}$ is a contraction and therefore has a unique fixed point $\overline{z} \in C_b[0,\infty)$.

From the proof of Lemma 4.3 we see that the fixed point $\overline{z} \in C_b[0,\infty)$ is uniformly Lipschitz continuous on $[0,\infty)$ since for any given $x \in C_b[0,\infty)$, $\mathcal{T}x$ is uniformly Lipschitz continuous on $[0,\infty)$ by (A.1). □

THEOREM 4.5 *Under (***H2***)-(***H4***), the equation system (4.6)-(4.9) admits a unique bounded solution.*

Proof. By Theorem 4.4, we obtain a unique $\overline{z} \in C_b[0,\infty)$ solving $\overline{z} = \mathcal{T}\overline{z}$. Let z^* be computed by (4.9). Then $\overline{z}$ together with z^* leads to a unique bounded solution to (4.6) by Proposition 3.3, and subsequently a unique bounded solution to (4.7). The solution $\overline{z}$ to (4.8) is just equivalently given by (4.12). Uniqueness of the bounded solution to (4.6)-(4.9) is obvious by the unique determination of $\overline{z}$ and hence of $z^* = \Phi(\overline{z})$. □

4.2 The virtual agent, policy iteration and attraction to mass behaviour

We proceed to investigate certain asymptotic properties on the interaction between the individual and the mass, and the formulation shall be interpreted in the large population limit (i.e., an infinite population) context. Corresponding to a large population (deterministic) mass effect

$\overline{z}$, let the dynamics for the individual be given as

$$dz_i = a_i z_i dt + bu_i dt + \alpha\overline{z}dt + \sigma_i dw_i.$$

At this stage, however, we do not relate $\overline{z}$ to the fixed point equation (4.12). Assume each agent is assigned a cost according to (3.2) with $z^* = \Phi(\overline{z})$, i.e.,

$$J_i(u_i, \Phi(\overline{z})) = E\int_0^\infty e^{-\rho t}\{[z_i - \Phi(\overline{z})]^2 + ru_i^2\}ds, \qquad i \geq 1. \tag{4.13}$$

We now introduce a so-called *virtual agent* to represent the mass effect and use $\overline{z} \in C_b[0,\infty)$ to describe the behaviour of the virtual agent. Here the virtual agent acts as a passive player in the sense that $\overline{z}$ appears as an exogenous function of time and $\Phi(\overline{z})$ is to be tracked by the agents.

Then after each selection of the set of individual control laws, a new $\overline{z}$ will be induced as specified below; subsequently, the individual shall consider its optimal policy (over the whole time horizon) to respond to this updated $\overline{z}$. Thus, the interplay between a given individual and the virtual agent representing the mass may be described as a sequence of virtual plays which may be employed by the individual as a calculation device to eventually learn the mass behaviour. In the following policy iteration analysis in function spaces, we take the virtual agent as a *passive leader* and the individual agents as *active followers*.

It is of interest to note that the virtual play described in this section has a resemblance in spirit to the so-called tâtonnement in economic theory which was first proposed by Walras in 1874 and formalized in a modern version in terms of ordinary differential equations by Samuelson in 1947 (for relevant literature, the reader is referred to Mas-Colell, Whinston and Green (1995) (pp. 620-626) and references therein). Specifically, in price tâtonnement, given an initial non-equilibrium price, the economic agents will each dynamically adjust its price in a trial and error process where the ensemble of all excess demands is assumed to be announced to all agents by a certain central planner. Such a process is continuously carried out in fictional time (i.e., with infinitesimal duration of iterations) and is highly informative in illuminating behavioural properties of the (Walrasian) equilibrium price. When the process converges to an equilibrium, it is termed as possessing tâtonnement stability. In contrast, our virtual play here takes a more abstract form since the interaction of agents is specified in policy spaces for feedback controls, and the agents update their strategies via an optimal tracking action in response to an envisaged population effect at each step, which differs from the qualitative adjustment of agents in tâtonnement.

Now, we describe the iterative update of an agent's policy from its *policy space*. For a fixed iteration number $k \geq 0$, suppose that there is *a priori* $\overline{z}^{(k)} \in C_b[0,\infty)$. Then by Proposition 3.1 the optimal control for the i-th agent using the cost (4.13) with respect to $\overline{z} = \overline{z}^{(k)}$ is given as

$$u_i^{(k+1)} = -\frac{b}{r}(\Pi_i z_i + s_i^{(k+1)})$$

where $s_i^{(k+1)} \in C_b[0,\infty)$ is given by

$$\rho s_i^{(k+1)} = \frac{ds_i^{(k+1)}}{dt} + a_i s_i^{(k+1)} - \frac{b^2}{r}\Pi_i s_i^{(k+1)} + \alpha \Pi_i \overline{z}^{(k)} - \Phi(\overline{z}^{(k)}). \quad (4.14)$$

By Proposition 3.3, the unique solution $s_i^{(k+1)} \in C_b[0,\infty)$ to (4.14) may be represented by the map

$$s_i^{(k+1)} = e^{\beta_2(a_i)t} \int_t^\infty e^{-\beta_2(a_i)\tau} [\alpha \Pi_i \overline{z}^{(k)}(\tau) - \Phi(\overline{z}^{(k)}(\tau))] d\tau. \quad (4.15)$$

Subsequently, the control laws $\{u_i^{(k+1)}, i \geq 1\}$ produce a mass trajectory

$$\overline{z}^{(k+1)} = \int_{\mathcal{A}} \overline{z}_a^{(k+1)} dF(a),$$

where

$$\frac{d\overline{z}_a^{(k+1)}}{dt} = -\beta_1(a)\overline{z}_a^{(k+1)} - \frac{b^2}{r} s_a^{(k+1)} + \alpha \overline{z}^{(k)}, \quad (4.16)$$

with initial condition $\overline{z}_a^{(k+1)}|_{t=0} = 0$ by (**H4**). Notice that (4.16) is indexed by the parameter $a \in \mathcal{A}$ instead of all i's. Then the virtual agent's state (as a function) $\overline{z}$ corresponding to $u_i^{(k+1)}$ is updated as $\overline{z}^{(k+1)}$. From the above and using the operator introduced in (4.12), we get the recursion for $\overline{z}^{(k)}$ as

$$\overline{z}^{(k+1)} = \mathcal{T}\overline{z}^{(k)},$$

where $\overline{z}^{(k+1)}|_{t=0} = 0$ for all k.

By the iterative adjustments of the individual strategies in response to the virtual agent, we induce the mass behaviour by a sequence of functions $\overline{z}^{(k)} = \mathcal{T}\overline{z}^{(k-1)} = \mathcal{T}^k \overline{z}^{(0)}$. The next proposition establishes that as the population grows, a statistical mass equilibrium exists and it is globally attracting.

PROPOSITION 4.6 *Under (***H2***)-(***H4***),* $\lim_{k\to\infty} \overline{z}^{(k)} = \overline{z}$ *for any* $\overline{z}^{(0)} \in C_b[0,\infty)$*, where* $\overline{z}$ *is determined by (4.6)-(4.9).*

Proof. This follows as a corollary to Theorem 4.4. □

4.3 Explicit solution with uniform agents

In the case of a system of uniform agents (i.e., $a_i \equiv a$) with a linear function Φ, a solution to the state aggregation equation system may be explicitly calculated. However, the distribution function $F(a)$ degenerates to point mass and (4.8) is no longer required. Since $\overline{z}$ coincides with $\overline{z}_a$, we simply specify it by (4.7) which is the dynamics of the latter. We consider the case $\Phi(z) = \hat{\gamma}(z+\eta)$. The equation system (4.6)-(4.9) specializes to

$$\rho s_a = \frac{ds_a}{dt} + a s_a - \frac{b^2}{r}\Pi_a s_a + \alpha \Pi_a \overline{z} - z^*, \tag{4.17}$$

$$\frac{d\overline{z}}{dt} = (a - \frac{b^2}{r}\Pi_a)\overline{z} + \alpha\overline{z} - \frac{b^2}{r}s_a, \tag{4.18}$$

$$z^* = \Phi(\overline{z}) = \hat{\gamma}(\overline{z}+\eta). \tag{4.19}$$

Here we shall compute a solution with a general initial condition $\overline{z}(0)$ for (4.18), which is not necessarily zero. Setting the derivatives to zero, we write a set of steady state equations as follows

$$\begin{cases} \beta_2(a)s_a(\infty) - \alpha\Pi_a\overline{z}(\infty) + z^*(\infty) = 0 \\ -\frac{b^2}{r}s_a(\infty) + (\alpha - \beta_1(a))\overline{z}(\infty) = 0 \\ \hat{\gamma}\overline{z}(\infty) - z^*(\infty) = -\hat{\gamma}\eta. \end{cases} \tag{4.20}$$

It can be verified that under **(H3)** we have $\Theta \stackrel{\triangle}{=} \beta_2(a)(\beta_1(a)-\alpha) + \frac{b^2}{r}(\alpha\Pi_a - \hat{\gamma}) > 0$, and therefore (4.20) is nonsingular and has a unique solution $(s_a(\infty), \overline{z}(\infty), z^*(\infty))$. Denote

$$\lambda_1 = \frac{\rho+\alpha-\sqrt{(\rho+\alpha)^2+4\Theta}}{2} < 0, \ \ \lambda_2 = \frac{\rho+\alpha+\sqrt{(\rho+\alpha)^2+4\Theta}}{2} > 0. \tag{4.21}$$

Using the same method as in Huang, Caines, and Malhamé (2004c), an explicit solution for the equation system (4.17)-(4.18) may be computed.

PROPOSITION 4.7 *Under* **(H2)**-(**H3**)*, the unique bounded solution* $(\overline{z}, s_a)$ *in (4.17)-(4.18) is given by*

$$\overline{z}(t) = \overline{z}(\infty) + (\overline{z}(0) - \overline{z}(\infty))e^{\lambda_1 t},$$

$$s_a(t) = s_a(\infty) + \frac{\hat{\gamma} - \alpha\Pi_a}{\beta_2 - \lambda_1}(\overline{z}(\infty) - \overline{z}(0))e^{\lambda_1 t},$$

where $\lambda_1 < 0$ *is given by (4.21) and* $\beta_1 = -a + \frac{b^2}{r}\Pi_a$, $\beta_2 = -a + \frac{b^2}{r}\Pi_a + \rho$. □

Notice that in the case (**H4**) is imposed, we need to set $\overline{z}(0) = 0$ in Proposition 4.7.

5. The Decentralized ε-Nash Equilibrium

We continue to consider the system of n agents and rewrite the dynamics in Section 1.2 as follows:

$$dz_i = (a_i z_i + bu_i)dt + \alpha z^{(n)}dt + \sigma_i dw_i, \qquad 1 \le i \le n, \qquad t \ge 0. \quad (5.1)$$

The individual costs of the agents are given by (2.2) with the linking term $v_i = \Phi(\frac{1}{n}\sum_{k=1}^n z_k)$. To indicate the dependence of the cost J_i on u_i and the set of controls of all other agents, we write it as $J_i(u_i, u_{-i})$ where u_{-i} denotes the row $(u_1, \cdots, u_n)$ with u_i deleted, so that

$$J_i(u_i, u_{-i}) \triangleq E\int_0^\infty e^{-\rho t}\{[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2 + ru_i^2\}dt. \quad (5.2)$$

The new notation for the cost should be easily distinguished from $J_i(u_i, v_i)$, $J_i(u_i, z^*)$, etc., which have been introduced earlier. We postpone the specification of the admissible control set for each agent until when we introduce the notion of ε-Nash equilibria in Section 1.5.2. We use u_i^0 to denote the optimal tracking based control law,

$$u_i^0 = -\frac{b}{r}(\Pi_i z_i + s_i), \quad (5.3)$$

where s_i is derived from (4.6)-(4.9) by matching a_i to a, and s_i implicitly depends on $\overline{z}$ therein. We also use u_{-i}^0 to denote $(u_1^0, \cdots, u_n^0)$ with u_i^0 deleted. It should be emphasized that in the following asymptotic analysis the control law u_i^0 for the i-th agent among a population of n agents is constructed using the limit empirical distribution $F(a)$. This gives a conceptually simpler determination of the individual control law without explicitly using the population size.

Recall that in Section 1.4, no boundedness requirement is imposed on $\mathcal{A}$. For the performance analysis of this section, we need to restrict $\{a_i, i \ge 1\}$ to be bounded. We introduce the assumption:

(**H5**) The set $\mathcal{A}$ in (**H3**) is the union of a finite number of disjoint compact intervals and $\widehat{\varepsilon} > 0$ is a constant such that $\beta_1(a) \ge \widehat{\varepsilon}$ for all $a \in \mathcal{A}$. □

Notice that under the positivity assumption of $\beta_1(a)$ in (**H3**), the compactness of $\mathcal{A}$ and continuity of $\beta_1(a)$ ensure that $\widehat{\varepsilon}$ specified above always exists.

Concerning notation in this section, we make the important convention as follows. $\overline{z}_a$, given by (4.7), denotes the individual mean computed in the large population limit context, and $\overline{z}_a^{(n)}$ stands for the mean of the agents with $a_i = a$ in a population of n agents and it is computed using the n dimensional closed-loop dynamics associated with the control laws u_i^0, $1 \le i \le n$. Also, each of s_a, $\overline{z}$ and z^* is computed using (4.6)-(4.9) based upon the large population limit.

5.1 Stability guarantees for closed-loop systems

In order to analyze the closed-loop behaviour when the control law u_i^0 is applied by the i-th agent, we define the diagonal matrix

$$B_n = \begin{pmatrix} -\beta_1(a_1) & & \\ & \cdots & \\ & & -\beta_1(a_n) \end{pmatrix}$$

and denote by $1_{n\times n}$ the $n \times n$ matrix with each entry being one, i.e., $1_{n\times n} = e(n) \times e^T(n)$, where $e^T(n) = [1, \cdots, 1]$. Let

$$\overline{B}_n = B_n + \frac{\alpha}{n} 1_{n\times n}.$$

Hence $\overline{B}_n$ is a real symmetric matrix which has n real eigenvalues. Our further equilibrium analysis relies on the closed-loop stability when the control laws u_i^0, $1 \le i \le n$, are applied. We introduce the following property related to closed-loop stability.

(**P1**) There exist $\mu^* < 0$ and integer $N_0 > 0$ such that for all $n \ge N_0$,

$$\overline{B}_n \le \mu^* I_n$$

where I_n is the $n \times n$ identity matrix. □

In the case $\alpha \le 0$ and $\inf_{i\ge 1} \beta_1(a_i) = \beta^* > 0$, it is easy to verify (**P1**). We give a sufficient condition to validate (**P1**) for the case $\alpha > 0$.

PROPOSITION 5.1 *Assume (i) $\alpha > 0$, (ii) $\beta_1(a_i) \ge \beta^* > 0$ for all $i \ge 1$, and (iii) there exists $N_0 > 0$ such that*

$$\sup_{n\ge N_0} \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha}{\beta_1(a_i)} < 1.$$

*Then (**P1**) holds for all $n \ge N_0$.*

Proof. See Appendix B. □

COROLLARY 5.2 *Assumptions (**H1**)-(**H3**) and (**H5**) imply (**P1**).*

Proof. It suffices to verify condition (iii) in Proposition 5.1 for the case $\alpha > 0$. Under the assumptions in the corollary, we may use a weak convergence argument to show

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha}{\beta_1(a_i)} = \lim_{n\to\infty} \int_{\mathcal{A}} \frac{\alpha}{\beta_1(a)} dF_n(a) = \int_{\mathcal{A}} \frac{\alpha}{\beta_1(a)} dF(a) < 1,$$

where the inequality is implied by (**H3**). Hence there exists N_0 such that condition (iii) holds. □

LEMMA 5.3 *Assuming (***H1***)-(***H5***), we have the estimate*

$$\sup_{t\geq 0} E[\sum_{i=1}^{n}(z_i - Ez_i)(t)]^2 = O(n),$$

where the set of states z_i, $1 \leq i \leq n$, corresponds to the control laws u_i^0, $1 \leq i \leq n$, given by (5.3).

Proof. Consider the system of n agents using the control law u_i^0. Here associated with u_i^0, both s_a and z^* are computed via (4.6)-(4.9) based on the large population limit and are independent of n. By use of the closed-loop dynamics, we express each $z_i(t)$ in terms of the initial condition $z_i(0)$, $z^{(n)}$, s_{a_i} and the Wiener process w_i. It is easy to verify

$$\begin{aligned}\sum_{i=1}^{n}[z_i(t) - Ez_i(t)] =& \sum_{i=1}^{n} e^{-\beta_1(a_i)t} z_i(0) + \sum_{i=1}^{n}\int_0^t e^{-\beta_1(a_i)(t-\tau)}\sigma_i dw_i \\ &+ \frac{1}{n}\sum_{i=1}^{n}\int_0^t \alpha e^{-\beta_1(a_i)(t-\tau)}\sum_{k=1}^{n}[z_k(\tau) - Ez_k(\tau)]d\tau.\end{aligned}$$

By (**H3**), we can take a sufficiently small but fixed $\varepsilon_0 > 0$ such that $(1+\varepsilon_0)[\int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF(a)]^2 < 1 - \varepsilon_0$. Denote $\xi(t) = \sum_{i=1}^{n}[z_i(t) - Ez_i(t)]$, and $\Delta(t) = \int_{\mathcal{A}}\int_0^t |\alpha| e^{-\beta_1(a)(t-\tau)} d\tau dF_n(a)$. By use of the inequality $(y_1 + y_2 + y_3)^2 \leq (1+\varepsilon_0)y_1^2 + 2(1+1/\varepsilon_0)(y_2^2 + y_3^2)$, we obtain

$$\begin{aligned}E\xi^2(t) \leq& (2 + \frac{2}{\varepsilon_0})E\left[(\sum_{i=1}^{n} e^{-\beta_1(a_i)t} z_i(0))^2 + (\sum_{i=1}^{n}\int_0^t e^{-\beta_1(a_i)(t-\tau)}\sigma_i dw_i)^2\right] \\ &+ (1+\varepsilon_0)E\left[\int_0^t \frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\beta_1(a_i)(t-\tau)}\xi(\tau)d\tau\right]^2 \\ \leq& nC + (1+\varepsilon_0)\Delta^2(t)E\left[\int_{\mathcal{A}}\int_0^t \Delta^{-1}(t)|\alpha|e^{-\beta_1(a)(t-\tau)}\xi(\tau)d\tau dF_n(a)\right]^2 \\ \leq& nC + (1+\varepsilon_0)\Delta^2(t)E\int_{\mathcal{A}}\int_0^t \Delta^{-1}(t)|\alpha|e^{-\beta_1(a)(t-\tau)}\xi^2(\tau)d\tau dF_n(a) \qquad (5.4) \\ \leq& nC + \sup_{0\leq\tau\leq t} E\xi^2(\tau)(1+\varepsilon_0)\left[\int_{\mathcal{A}}\frac{|\alpha|}{\beta_1(a)}dF_n\right]^2, \qquad (5.5)\end{aligned}$$

where $C > 0$ is a constant independent of n and t. Here (5.4) follows from Jensen's inequality since for the fixed $t > 0$, $\Delta^{-1}(t)|\alpha|e^{-\beta_1(a)(t-\tau)}d\tau dF_n(a)$ induces a measure on the product space $[0, t] \times \mathcal{A}$ with a total measure of one. Employing the weak convergence of F_n we can show that there exists $N_1 > 0$ such that for all $n \geq N_1$,

$$(1+\varepsilon_0)\left|\left[\int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)}dF_n\right]^2 - \left[\int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)}dF\right]^2\right| \leq \frac{\varepsilon_0}{2}.$$

Hence for all $n \geq N_1$ and any fixed $T > 0$, from (5.5) we have

$$E\xi^2(t) \leq nC + (1 - \frac{\varepsilon_0}{2}) \sup_{0 \leq \tau \leq T} E\xi^2(\tau), \qquad 0 \leq t \leq T,$$

which yields

$$\sup_{0 \leq t \leq T} E\xi^2(t) \leq nC + (1 - \frac{\varepsilon_0}{2}) \sup_{0 \leq t \leq T} E\xi^2(t).$$

And therefore,

$$\sup_{0 \leq t \leq T} E\xi^2(t) \leq \frac{2nC}{\varepsilon_0}.$$

Since $T > 0$ is arbitrary and C is independent of T, the lemma follows. □

In fact, given the control law u_i^0, we can refine the proof of Lemma 5.3 to show for sufficiently large $N_1 > 0$, $\sup_{n \geq N_1} \sup_{t \geq 0, 1 \leq k \leq n} Ez_k^2(t) < \infty$. Thus the tracking based control law u_i^0 (depending on the fixed point aggregation procedure in Section 1.4) is stabilizing for n-agent systems for n sufficiently large.

5.2 The asymptotic equilibrium analysis

Within the context of a population of n agents, for any $1 \leq k \leq n$, the k-th agent's admissible control set $\mathcal{U}_k$ consists of all feedback controls u_k adapted to the σ-algebra $\sigma(z_i(\tau), \tau \leq t, 1 \leq i \leq n)$ (i.e., $u_k(t)$ is a function of $(t, z_1(t), \cdots, z_n(t))$) such that a unique strong solution to the closed-loop system of the n agents exists on $[0, \infty)$. In this setup we give the definition.

DEFINITION 5.4 *A set of controls $u_k \in \mathcal{U}_k, 1 \leq k \leq n$, for n players is called an ε-*Nash equilibrium *with respect to the costs $J_k, 1 \leq k \leq n$, if there exists $\varepsilon \geq 0$ such that for any fixed $1 \leq i \leq n$, we have*

$$J_i(u_i, u_{-i}) \leq J_i(u_i', u_{-i}) + \varepsilon, \tag{5.6}$$

when any alternative control $u_i' \in \mathcal{U}_i$ is applied by the i-th player. □

If $\varepsilon = 0$ in (5.6), then Definition 5.4 specializes to the usual Nash equilibrium (Aubin (1998)).

Remark. The admissible control set $\mathcal{U}_k$ is not decentralized since the k-th agent has perfect information on other agents' states. In effect, such admissible control sets lead to a stronger qualification of the ε-Nash equilibrium property for the decentralized control analyzed in this section. □

Given the distribution function F, $\overline{z} \in C_b[0,\infty)$ and the associated $z^* = \Phi(\overline{z})$, from (4.6)-(4.7) it is seen that both s_a and $\overline{z}_a$ may be explicitly expressed as a function of $a \in \mathcal{A}$. Notice that $\overline{z}$ is determined from (4.12) and is independent of a. We introduce the following property:

(**P2**) Let $\mathcal{A}$ be the set specified in (**H5**). (i) $\sup_{a\in\mathcal{A}} |\overline{z}_a|_\infty < \infty$, and (ii) $\lim_{a'\to a} \sup_t |\overline{z}_a(t) - \overline{z}_{a'}(t)| = 0$ with a vanishing rate depending only on $|a - a'|$, for $a, a' \in \mathcal{A}$. □

The proposition below gives a sufficient condition to insure (**P2**).

Proposition 5.5 *Assume (*__H1__*)-(*__H5__*) holds. Then $\overline{z}_a(t)$ has the property (*__P2__*).*

Proof. See Appendix B. □

We note that for validating property (ii) in (**P2**) in Proposition 5.5, the set $\mathcal{A}$ is not required to be compact in the proof. Now we define

$$\epsilon_n(t) = |\int_{\mathcal{A}} \overline{z}_a(t) dF_n(a) - \int_{\mathcal{A}} \overline{z}_a(t) dF(a)|, \qquad t \geq 0, \tag{5.7}$$

$$\epsilon_n'(t) = |\int_{\mathcal{A}} \overline{z}_a(t) dF_n(a) - \int_{\mathcal{A}} \overline{z}_a^{(n)}(t) dF_n(a)|, \qquad t \geq 0. \tag{5.8}$$

As mentioned earlier, here $\overline{z}_a$ is determined using the large population limit and $\overline{z}_a^{(n)}$ denotes the mean of agents with $a_i = a$ in a system of n agents taking the control laws u_i^0, $1 \leq i \leq n$.

Lemma 5.6 *Under (*__H1__*)-(*__H5__*), we have*

$$\lim_{n\to\infty} \overline{\epsilon}_n \stackrel{\triangle}{=} \lim_{n\to\infty} \sup_{t\geq 0} \epsilon_n(t) = 0,$$

where $\epsilon_n(t)$ is defined by (5.7).

Proof. Letting $I_C = [-C, C]$ for $C > 0$, we have

$$\begin{aligned}\epsilon_n(t) = |&\int_{\mathcal{A}\cap I_C} \overline{z}_a(t) dF_n(a) + \int_{\mathcal{A}\cap(\mathbb{R}\setminus I_C)} \overline{z}_a(t) dF_n(a) \\ &- \int_{\mathcal{A}\cap I_C} \overline{z}_a(t) dF(a) - \int_{\mathcal{A}\cap(\mathbb{R}\setminus I_C)} \overline{z}_a(t) dF(a)| \\ \stackrel{\triangle}{=} &|I_n^{(1)} + I_n^{(2)} - I^{(1)} - I^{(2)}|.\end{aligned}$$

Now for any fixed $\varepsilon > 0$, there exists a sufficiently large constant $C > 0$ such that F is continuous at $a = \pm C$ and $I_C \supset \mathcal{A}$ which leads to

$$|I_n^{(2)}| + |I^{(2)}| = 0,$$

since $\int_{I_C} dF_n(a) = \int_{I_C} dF(a) = 1$. We write

$$\begin{aligned} |I_n^{(1)} - I^{(1)}| &= |\int_{\mathcal{A}\cap I_C} \overline{z}_a(t) dF_n(a) - \int_{\mathcal{A}\cap I_C} \overline{z}_a(t) dF(a)| \\ &= |\int_{I_C} \overline{z}'_a(t) dF_n(a) - \int_{I_C} \overline{z}'_a(t) dF(a)|, \end{aligned}$$

where we make the convention that the domain of $\overline{z}_a(t)$ (as a function of a), if necessary, is extended from $\mathcal{A}$ to $\mathbb{R}$ (hence covering I_C) such that properties (i) and (ii) in (**P2**) still hold after $\mathcal{A}$ is replaced by $\mathbb{R}$. We denote the resulting function by $\overline{z}'_a(t)$ which is identical to $\overline{z}_a(t)$ on $\mathcal{A}$. For instance, in the case $\mathcal{A} = [c_1, c_2]$ with $c_2 < \infty$, we may simply set $\overline{z}'_a(t) = \overline{z}_{c_2}(t)$ when $a > c_2$. Such an extension can deal with the general case when $\mathcal{A}$ consists of a finite number of disjoint bounded and closed subintervals.

Next we combine the equicontinuity of $\overline{z}'_a(t)$ in $a \in I_C$ w.r.t. $t \in [0, \infty)$ insured by (**P2**) and the above extension procedure, continuity of F at $a = \pm C$, and the standard subinterval dividing technique for the proof of Helly-Bray theorem (see Chow and Teicher (1997), pp. 274-275) to conclude that there exists a sufficiently large n_0 such that for all $n \geq n_0$,

$$|I_n^{(1)} - I^{(1)}| = |\int_{I_C} \overline{z}'_a(t) dF_n(a) - \int_{I_C} \overline{z}'_a(t) dF(a)| \leq \frac{\varepsilon}{2},$$

for the arbitrary but fixed ε, and consequently $\lim_{n\to\infty} \sup_{t\geq 0} \epsilon_n(t) = 0$. This completes the proof. □

In the proof of Lemma 5.6, in order to preserve properties (i) and (ii) in (**P2**), we extend $\overline{z}_a(t)$ to $a \notin \mathcal{A}$ in a specific manner and avoid directly using (4.6)-(4.9) to calculate $\overline{z}_a(t)$, $a \notin \mathcal{A}$, even if the equation system may give a well defined $\overline{z}_a(t)$ for some $a \notin \mathcal{A}$. To show the merit of such an extension, we consider a simple scenario as follows. Suppose $\mathcal{A} = [0, 1]$ and F has discontinuities at $a = 0, 1$. Then by choosing $I_C = [-1, 2] \supset \mathcal{A}$ and using the obtained function $\overline{z}'_a$ (which is continuous on $[-1, 2]$), we can insure the applicability of the technique of the Helly-Bray theorem which requires the limit distribution function F to be continuous at the endpoints of the interval of integration of a continuous function. Notice that in this case F is continuous at $a = -1, 2$.

PROPOSITION 5.7 *Suppose (**H1**)-(**H5**) hold. Then we have*

$$\lim_{n\to\infty} \overline{\epsilon}'_n \stackrel{\triangle}{=} \lim_{n\to\infty} \sup_{t\geq 0} \epsilon'_n(t) = 0,$$

where $\epsilon'_n(t)$ *is defined by (5.8).*

Proof. First, using the closed-loop dynamics for $\overline{z}_a^{(n)}$ and $\overline{z}_a$ we get the relation

$$\frac{d(\overline{z}_a^{(n)} - \overline{z}_a)}{dt} = -\beta_1(a)(\overline{z}_a^{(n)} - \overline{z}_a) + \alpha(\overline{z}^{(n)} - \overline{z})$$

with initial condition $(\overline{z}_a^{(n)} - \overline{z}_a)|_{t=0} = 0$. $\overline{z}_a^{(n)}$ is the mean of agents with $a_i = a$, $1 \leq i \leq n$, and $\overline{z}^{(n)} = \frac{1}{n}\sum_{i=1}^n \overline{z}_{a_i}$. Notice that the assumptions here implies (**P1**) for sufficiently large n (see Corollary 5.2) which further insures $|\overline{z}_a^{(n)}|_\infty < \infty$. Here $|x|_\infty = \sup_{t\geq 0}|x(t)|$ for $x \in C_b[0,\infty)$. Hence it follows that

$$|\overline{z}_a^{(n)} - \overline{z}_a|_\infty \leq \frac{|\alpha|}{\beta_1(a)}|\overline{z}^{(n)} - \overline{z}|_\infty < \infty \tag{5.9}$$

which further leads to

$$|\frac{1}{n}\sum_{i=1}^n \overline{z}_{a_i}^{(n)} - \overline{z} + \overline{z} - \frac{1}{n}\sum_{i=1}^n \overline{z}_{a_i}|_\infty \leq |\overline{z}^{(n)} - \overline{z}|_\infty \frac{1}{n}\sum_{i=1}^n \frac{|\alpha|}{\beta_1(a_i)}.$$

Hence we have

$$\begin{aligned} |\overline{z}^{(n)} - \overline{z}|_\infty \leq |\overline{z}^{(n)} - \overline{z}|_\infty \int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF_n(a) \\ + \left| \int_{\mathcal{A}} \overline{z}_a dF(a) - \int_{\mathcal{A}} \overline{z}_a dF_n(a) \right|_\infty . \end{aligned} \tag{5.10}$$

On the other hand, for sufficiently large n_0, we may use the continuity and boundedness of $\frac{1}{\beta_1(a)}$ together with the weak convergence of F_n to get $\sup_{n\geq n_0}\int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF_n(a) < 1$ resulting from (**H3**), and moreover, by the uniform boundedness and equicontinuity of $\overline{z}_a$ as shown by Proposition 5.5, we can prove $|\int_{\mathcal{A}} \overline{z}_a dF(a) - \int_{\mathcal{A}} \overline{z}_a dF_n(a)|_\infty = o(1)$ by Lemma 5.6. Hence by (5.10) we conclude that $|\overline{z}^{(n)} - \overline{z}|_\infty = o(1)$ as $n \to \infty$. Finally the proof follows from $\overline{\epsilon}'_n \leq |\overline{z}^{(n)} - \overline{z}|_\infty \int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF_n(a) = o(1)$. □

Lemma 5.8 *Under (**H1**)-(**H5**), for $z^* \in C_b[0,\infty)$ determined by (4.6)-(4.9), we have*

$$E\int_0^\infty e^{-\rho t}\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2_{(u_i^0,u_{-i}^0)} dt = O(\gamma^2(\overline{\epsilon}_n + \overline{\epsilon}'_n)^2 + \frac{\gamma^2}{n}), \tag{5.11}$$

where $\overline{\epsilon}_n$ and $\overline{\epsilon}'_n$ are given in (5.7)-(5.8), and the set of states z_k, $1 \le k \le n$, is associated with u_k^0 given by (5.3). □

Proof. In the proof we shall omit the control u_k^0 associated with z_k in various places. Obviously we have

$$\frac{1}{n}\sum_{k=1}^n Ez_k = \int_{\mathcal{A}} \overline{z}_a^{(n)} dF_n(a).$$

Setting

$$\Psi \triangleq |z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)| \le \gamma \left| \int_{a\in\mathcal{A}} \overline{z}_a dF(a) - \int_{a\in\mathcal{A}} \overline{z}_a dF_n(a) + \int_{a\in\mathcal{A}} \overline{z}_a dF_n(a) - \frac{1}{n}\sum_{k=1}^n z_k \right|$$

where the inequality follows from (**H2**), we obtain

$$\begin{aligned} E\Psi^2(t) \le& \gamma^2 E\Big[(\int_{a\in\mathcal{A}} \overline{z}_a dF(a) - \int_{a\in\mathcal{A}} \overline{z}_a dF_n(a)) \\ &+ (\int_{\mathcal{A}} \overline{z}_a dF_n(a) - \frac{1}{n}\sum_{k=1}^n Ez_k) + (\frac{1}{n}\sum_{k=1}^n Ez_k - \frac{1}{n}\sum_{k=1}^n z_k)\Big]^2 \\ \le& 2\gamma^2(\overline{\epsilon}_n + \overline{\epsilon}'_n)^2 + 2\gamma^2 E\left[\frac{1}{n}\sum_{k=1}^n (z_k - Ez_k)\right]^2 \\ \le& 2\gamma^2(\overline{\epsilon}_n + \overline{\epsilon}'_n)^2 + O(\frac{\gamma^2}{n}), \end{aligned}$$

where the upper bound given as the last term holds uniformly w.r.t. $t \ge 0$ by virtue of Lemma 5.3, and therefore (5.11) follows. □

Lemma 5.9 *Letting $0 < T < \infty$, for any Lebesgue measurable function $x : [0,T] \to \mathbb{R}$ such that $\int_0^T x_t^2 dt < \infty$ and for any $\delta \le \rho$, we have*

$$\int_0^T e^{-\rho t}\left[x_t^2 - \delta\int_0^t x_s^2 ds\right] dt \ge 0. \tag{5.12}$$

Proof. Since both x_t^2 and $\int_0^t x_s^2 ds$ (as functions of t) have finite integral on $[0,T]$, we may split the integrand in (5.12) to compute

$$\begin{aligned} & \int_0^T e^{-\rho t} x_t^2 dt - \delta \int_0^T \int_0^t e^{-\rho t} x_s^2 ds dt \\ = & \int_0^T e^{-\rho t} x_t^2 dt - \delta \int_0^T \int_s^T e^{-\rho t} x_s^2 dt ds \\ \geq & \int_0^T e^{-\rho t} x_t^2 dt - \frac{\delta}{\rho} \int_0^T e^{-\rho s} x_s^2 ds \geq 0, \end{aligned}$$

where we exchange the order of integration in the double integral by Fubini's theorem. □

For the main results in Theorems 5.10 and 5.11, u_i^0 is the optimal tracking based control law for the i-th player given by (5.3) for which s_i and the associated reference tracking trajectory z^* are computed using (4.6)-(4.9) for the large population limit. Thus both s_i and z^* are independent of the population size.

THEOREM 5.10 *Under (**H1**)-(**H5**), we have*

$$|J_i(u_i^0, u_{-i}^0) - J_i(u_i^0, z^*)| = O(\overline{\epsilon}_n + \overline{\epsilon}'_n + \frac{1}{\sqrt{n}}),$$

as $n \to \infty$, where $J_i(u_i^0, z^*)$ is the cost with respect to z^* by setting $f = \overline{z}$ in Section 1.3 (or equivalently, replacing $z^{(n)}$ in (5.1) by $\overline{z}$), $J_i(u_i^0, u_{-i}^0)$ is determined by (5.2), $\overline{\epsilon}_n$, $\overline{\epsilon}'_n$ and u_i^0 are the same as in Lemma 5.8. □

Here the same initial condition $z_i(0)$ is used for computing $J_i(u_i^0, u_{-i}^0)$ and $J_i(u_i^0, z^*)$. The proof is done by a similar decomposition technique as in proving Theorem 5.11 below and is postponed until after the proof of the latter.

In Theorem 5.11 we need to consider the perturbation in the control of a given agent. We point out that when the control laws change from (u_i^0, u_{-i}^0) to (u_i, u_{-i}^0) for the system of n agents, a change will accordingly take place for each of the n state components since z_k, $k \neq i$, is coupled with $\frac{1}{n} z_i$ even if the set of control laws u_{-i}^0 remains the same. Here we use u_{-i}^0 to denote the row $(u_1^0, \cdots, u_n^0)$ with u_i^0 deleted.

THEOREM 5.11 *Under (**H1**)-(**H5**), the set of controls $u_i^0, 1 \leq i \leq n$, for the n players is an ε-Nash equilibrium with respect to the costs $J_i(u_i, u_{-i})$, $1 \leq i \leq n$, i.e.,*

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \leq \inf_{u_i} J_i(u_i, u_{-i}^0) \tag{5.13}$$

$$\leq J_i(u_i^0, u_{-i}^0) \tag{5.14}$$

where $0 < \varepsilon = O(\bar{\epsilon}_n + \bar{\epsilon}'_n + \frac{1}{\sqrt{n}})$ *(and hence* $\varepsilon \to 0$*) as* $n \to \infty$*, and* $u_i \in \mathcal{U}_i$ *is any alternative control which depends on* $(t, z_1, \cdots, z_n)$*.*

Proof. The inequality (5.14) is obviously true. We prove the inequality (5.13). We consider all full state dependent $u_i \in \mathcal{U}_i$ satisfying

$$J_i(u_i, u_{-i}^0) \le J_i(u_i^0, u_{-i}^0). \tag{5.15}$$

In the remaining part of the proof, (u_i, u_{-i}^0) appearing in each place is assumed to satisfy (5.15). We shall use $C > 0$ to denote a generic constant which is independent of n and the index of the agents, and its value may vary in different places.

We first estimate the RHS of (5.15). Invoking the stability property of the closed-loop of the n agents all adopting u_i^0, $1 \le i \le n$, and using a similar method as in Lemma 5.3, we can show that there exists C (independent of n and i), such that $E\int_0^\infty e^{-\rho t} z_i^2(t)|_{(u_i^0, u_{-i}^0)} dt < C$ and

$$E\int_0^\infty e^{-\rho t}\left[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2_{(u_i^0, u_{-i}^0)} dt \le C.$$

Furthermore, since $\mathcal{A}$ is compact, for $u_i^0 = -\frac{b}{r}(\Pi_i z_i + s_i)$, we can find C such that $\Pi_i + |s_i| \le C$. It readily follows that the RHS of (5.15) is bounded by C.

Hence for (u_i, u_{-i}^0) satisfying (5.15), we can find a fixed constant C independent of n such that

$$J_i(u_i, u_{-i}^0) = E\int_0^\infty e^{-\rho t}\left\{\left[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2_{(u_i, u_{-i}^0)} + r u_i^2\right\} dt \le C. \tag{5.16}$$

Now, for (u_i, u_{-i}^0) satisfying (5.15) and hence (5.16), we may express $z_k, k \ne i, 1 \le k \le n$, in terms of their initial conditions $z_k(0)$, $k \ne i$, the Wiener integral, as well as $\frac{1}{n} z_i$ which acts as an input in the closed-loop dynamics of the $n-1$ agents. In addition, similar to establishing (**P1**), we can show that for large n, a mean square stability holds for the $n-1$ agents when $\frac{1}{n} z_i$ is removed, and that the closed-loop (symmetric) gain matrix has a maximum eigenvalue less than $\frac{1}{2}\mu^*$, where $\mu^* < 0$ is determined in (**P1**).

We may write $\frac{1}{n}\sum_{k=1}^n z_k = \Delta + \frac{1}{n} z_i + \frac{1}{n}\int_0^t f_n(t-s) z_i(s) ds$, where Δ depends only on the initial conditions of z_k, $k \ne i$, and the Wiener processes, $E\Delta^2$ is bounded by a fixed constant independent of n and t, and $\sup_n |f_n(s)| \leq ce^{\mu^* s/2}$ with $c > 0$ for $s \geq 0$. Then using Lipschitz continuity of Φ and basic estimates, we obtain from (5.16)

$$E\int_0^\infty e^{-\rho t}\left\{z_i(t) - \Phi\left[\frac{1}{n}z_i(t) + \frac{1}{n}\int_0^t f_n(t-s)z_i(s)ds\right]\right\}^2_{(u_i,u^0_{-i})} dt \leq C. \tag{5.17}$$

Fixing $0 < T < \infty$, by tedious but elementary estimates we have

$$\begin{aligned}
&E\int_0^T e^{-\rho t}\left\{\left[z_i(t) - \Phi\left(\frac{z_i(t)}{n} + \frac{1}{n}\int_0^t f_n(t-s)z_i(s)ds\right)\right]^2 - \frac{z_i^2(t)}{8}\right\}_{(u_i,u^0_{-i})} dt \\
&\geq E\int_0^T e^{-\rho t}\left\{\frac{3}{4}z_i^2(t) - 6\gamma^2\left[\frac{z_i(t)}{n} + \frac{1}{n}\int_0^t f_n(t-s)z_i(s)ds\right]^2\right. \\
&\qquad \left. -6\Phi^2(0) - \frac{z_i^2(t)}{8}\right\} dt \\
&\geq \int_0^T e^{-\rho t}\left\{\frac{1}{2}z_i^2(t) - \frac{12\gamma^2}{n^2}\left[\int_0^t f_n(t-s)z_i(s)ds\right]^2\right\} dt - C \\
&\geq \int_0^T e^{-\rho t}\left[\frac{1}{2}z_i^2(t) - \frac{12\gamma^2}{n^2}\int_0^t f_n^2(t-s)ds\int_0^t z_i^2(s)ds\right] dt - C \\
&\geq E\int_0^T e^{-\rho t}\left[\frac{1}{2}z_i^2(t) - \frac{12\gamma^2 c^2}{n^2|\mu^*|}\int_0^t z_i^2(s)ds\right] dt - C.
\end{aligned} \tag{5.18}$$

By Lemma 5.9, for all sufficiently large n, we have

$$\int_0^T e^{-\rho t}\left[\frac{1}{2}z_i^2(t) - \frac{12\gamma^2 c^2}{n^2|\mu^*|}\int_0^t z_i^2(s)ds\right]_{(u_i,u^0_{-i})} dt \geq 0. \tag{5.19}$$

Notice that for almost all sample paths, z_i is continuous on $[0, T]$ so that the integral in (5.19) is well defined. Hence by (5.18) and (5.19), we have

$$\begin{aligned}
&E\int_0^T e^{-\rho t}\frac{z_i^2(t)}{8}\Big|_{(u_i,u^0_{-i})} dt \leq C + E\int_0^\infty e^{-\rho t} \\
&\qquad \times\left[z_i(t) - \Phi\left(\frac{1}{n}z_i(t) + \frac{1}{n}\int_0^t f_n(t-s)z_i(s)ds\right)\right]^2_{(u_i,u^0_{-i})} dt \leq C
\end{aligned}$$

where the last inequality follows from (5.17) and C is independent of T.

Subsequently, for (u_i, u^0_{-i}) satisfying (5.15), we assert that there exists $C > 0$ independent of n such that

$$E\int_0^\infty e^{-\rho t}z_i^2|_{(u_i,u^0_{-i})}dt + E\int_0^\infty e^{-\rho t}(z_i - z^*)^2|_{(u_i,u^0_{-i})}dt \leq C, \tag{5.20}$$

where z^* is determined from (4.6)-(4.9) as in Lemma 5.8.

We compare $\Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i,u_{-i}^0)}$ with $\Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i^0,u_{-i}^0)}$ by use of the $n-1$ dimensional closed-loop dynamics for z_k, $k \neq i$, and after basic estimates using (5.20), we may obtain

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2_{(u_i,u_{-i}^0)} dt \\
&= E\int_0^\infty e^{-\rho t}\Bigg[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i^0,u_{-i}^0)} \\
&\quad +\Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i^0,u_{-i}^0)} - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i,u_{-i}^0)}\Bigg]^2 dt \\
&= E\int_0^\infty e^{-\rho t}\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)|_{(u_i^0,u_{-i}^0)}\right]^2 dt + O(\frac{\gamma}{n}). \qquad (5.21)
\end{aligned}$$

Here and hereafter in the proof, unless otherwise indicated, the state process is always associated with the control (u_i, u_{-i}^0). For notational brevity, in the following we omit the associated control without causing confusion. Now, on the other hand we have

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}\left\{[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2_{(u_i,u_{-i}^0)} + ru_i^2\right\} dt \\
=&E\int_0^\infty e^{-\rho t}\left\{[(z_i - z^*) + (z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k))]^2 + ru_i^2\right\} dt \\
=&E\int_0^\infty e^{-\rho t}[(z_i - z^*)^2 + ru_i^2]dt + E\int_0^\infty e^{-\rho t}\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2 dt \\
&+ 2E\int_0^\infty e^{-\rho t}(z_i - z^*)\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right] dt \triangleq I_1 + I_2 + I_3. \qquad (5.22)
\end{aligned}$$

Approximating $z^{(n)}$ by $\overline{z}$ (with u_i for z_i), after careful estimates we have

$$\begin{aligned}
I_1 &\geq J_i(u_i^0, z^*) - O(\overline{\epsilon}_n + \overline{\epsilon}_n' + \frac{1}{\sqrt{n}}) \\
&\geq J_i(u_i^0, u_{-i}^0) - O(\overline{\epsilon}_n + \overline{\epsilon}_n' + \frac{1}{\sqrt{n}}), \qquad (5.23)
\end{aligned}$$

$$I_2 = O(\gamma^2(\overline{\epsilon}_n + \overline{\epsilon}_n')^2 + \frac{\gamma + \gamma^2}{n}), \qquad (5.24)$$

where (5.23) follows from Theorem 5.10, and (5.24) follows from Lemma 5.8 and (5.21). Here we deliberately use $J_i(u_i^0, z^*)$ to denote the optimal tracking cost specified in Section 1.3 with $f = \overline{z}$ in the dynamics. Moreover, by Schwarz inequality and (5.20) we have

$$\begin{aligned} |I_3| &\le 2\int_0^\infty e^{-\rho t}[E(z_i - z^*)^2]^{\frac{1}{2}} \left\{ E[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2 \right\}^{\frac{1}{2}} dt \\ &\le 2\left[\int_0^\infty e^{-\rho t}E(z_i - z^*)^2 dt\right]^{\frac{1}{2}} \left\{\int_0^\infty e^{-\rho t}E[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2 dt\right\}^{\frac{1}{2}} \\ &= O(\sqrt{I_2}) = O(\overline{\epsilon}_n + \overline{\epsilon}'_n + \frac{1}{\sqrt{n}}). \end{aligned} \tag{5.25}$$

Hence it follows that there exists $c > 0$ such that

$$J_i(u_i, u_i^0) \ge J_i(u_i^0, u_{-i}^0) - c(\overline{\epsilon}_n + \overline{\epsilon}'_n + \frac{1}{\sqrt{n}}),$$

where c is independent of n. This completes the proof. □

Proof of Theorem 5.10. As in (5.22) we make the decomposition

$$\begin{aligned} & J_i(u_i^0, u_{-i}^0) \\ =& E\int_0^\infty e^{-\rho t}\left\{[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2_{(u_i^0, u_{-i}^0)} + r(u_i^0)^2\right\} dt \\ =& E\int_0^\infty e^{-\rho t}\left\{[(z_i - z^*) + (z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2_{(u_i^0, u_{-i}^0)} + r(u_i^0)^2\right\} dt \\ =& E\int_0^\infty e^{-\rho t}[(z_i - z^*)^2 + r(u_i^0)^2]dt + E\int_0^\infty e^{-\rho t}\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]^2_{(u_i^0, u_{-i}^0)} dt \\ & + 2E\int_0^\infty e^{-\rho t}(z_i - z^*)\left[z^* - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)\right]_{(u_i^0, u_{-i}^0)} dt \\ \stackrel{\triangle}{=}& J_i(u_i^0, u_{-i}^0, z^*) + I'_2 + I'_3. \end{aligned} \tag{5.26}$$

Finally, similar to (5.24) and (5.25), we apply Schwarz inequality and Lemma 5.8 to obtain

$$|I'_2 + I'_3| = O(\overline{\epsilon}_n + \overline{\epsilon}'_n + \frac{1}{\sqrt{n}}),$$

and we can further show $|J_i(u_i^0, u_{-i}^0, z^*) - J_i(u_i^0, z^*)| = O(\overline{\epsilon}_n + \overline{\epsilon}'_n + \frac{1}{\sqrt{n}})$. This completes the proof. □

It should be noted that the proof of Theorem 5.10 does not depend on Theorem 5.11.

6. Conclusions and Future Research

In this paper we study the individual and mass behaviour in large-population weakly coupled dynamic systems with non-uniform agents. In the framework of noncooperative games, we employ a state aggregation technique to develop decentralized control laws for the agents. The resulting set of individual control laws has an ε-Nash equilibrium property, and furthermore, an attraction property of the mass bebaviour is illustrated.

The further investigation of statistical mechanics methods for such weakly coupled systems is of interest. Also, it is of interest to study decentralized optimization in a system configuration where the number of agents changes from time to time. This kind of model is well motivated in many economic and engineering scenarios; see e.g. Liu and Passino (2004), Baccelli, Hong and Liu (2001). In general, the resulting analysis requires appropriately aggregating a more randomized mass effect due to the time varying population, and introducing cost measures for *active agents* as well.

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Appendix A

Proof of Lemma 4.3. For any $x \in C_b[0,\infty)$, we have

$$
\begin{aligned}
|(\mathcal{T}x)(t)| \le& \int_{\mathcal{A}}\int_0^t e^{-\beta_1(a)(t-s)}|\alpha x|_\infty ds dF(a) + \frac{b^2}{r}\int_{\mathcal{A}}\int_0^t\int_s^\infty e^{-\beta_1(a)(t-s)} \\
&\times e^{-\beta_2(a)(\tau-s)}\left[\sup_{\tau\in\mathbb{R}}|\Phi(x(\tau))| + |\alpha x|_\infty \Pi_a\right] d\tau ds dF(a) \\
\le& |x|_\infty \int_{\mathcal{A}} \frac{|\alpha|}{\beta_1(a)} dF(a) + \big[|\Phi(0)| + \gamma|x|_\infty\big]\frac{b^2}{r}\int_{\mathcal{A}}\frac{dF(a)}{\beta_1(a)\beta_2(a)} \\
&+ \frac{b^2|x|_\infty}{r}\int_{\mathcal{A}}\frac{|\alpha|\Pi_a}{\beta_1(a)\beta_2(a)}dF(a) < \infty.
\end{aligned}
$$

Hence for a given $x \in C_b[0,\infty)$, by **(H3)** we have $\sup_{t\ge 0}|(\mathcal{T}x)(t)| < \infty$. We now show continuity of $\mathcal{T}x$ on $[0,\infty)$. Assuming $0 \le t_1 < t_2 < \infty$, we have

$$
\begin{aligned}
&(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1) \\
=& \int_{\mathcal{A}} \int_0^{t_2} \int_s^{\infty} \frac{b^2}{r} e^{-\beta_1(a)(t_2-s)} e^{-\beta_2(a)(\tau-s)} \Phi(x(\tau)) d\tau ds dF(a) \\
&- \int_{\mathcal{A}} \int_0^{t_1} \int_s^{\infty} \frac{b^2}{r} e^{-\beta_1(a)(t_1-s)} e^{-\beta_2(a)(\tau-s)} \Phi(x(\tau)) d\tau ds dF(a) \\
&+ \int_{\mathcal{A}} \int_0^{t_2} \int_s^{\infty} \frac{b^2}{r} e^{-\beta_1(a)(t_2-s)} e^{-\beta_2(a)(\tau-s)} (-\alpha)\Pi_a x(\tau) d\tau ds dF(a) \\
&- \int_{\mathcal{A}} \int_0^{t_1} \int_s^{\infty} \frac{b^2}{r} e^{-\beta_1(a)(t_1-s)} e^{\beta_2(a)(\tau-s)} (-\alpha)\Pi_a x(\tau) d\tau ds dF(a) \\
&+ \int_{\mathcal{A}} \int_0^{t_2} e^{-\beta_1(a)(t_2-s)} \alpha x(s) ds dF(a) \\
&- \int_{\mathcal{A}} \int_0^{t_1} e^{-\beta_1(a)(t_1-s)} \alpha x(s) ds dF(a) \stackrel{\triangle}{=} I_1 - I_2 + I_3 - I_4 + I_5 - I_6
\end{aligned}
$$

where the terms I_i, $1 \le i \le 6$ are each determined in an obvious manner.

In the following analysis, we will repeatedly use the fact $|e^{-d_1} - e^{-d_2}| \le e^{-d_1}|d_1 - d_2|$ for $0 \le d_1 \le d_2$. We have the estimate

$$
\begin{aligned}
I_1 - I_2 =& \frac{b^2}{r} \int_{\mathcal{A}} \int_0^{t_1} \int_s^{\infty} [e^{-\beta_1(a)(t_2-s)} - e^{-\beta_1(a)(t_1-s)}] e^{-\beta_2(a)(\tau-s)} \\
&\times \Phi(x(\tau)) d\tau ds dF(a) \\
&+ \frac{b^2}{r} \int_{\mathcal{A}} \int_{t_1}^{t_2} \int_s^{\infty} e^{-\beta_1(a)(t_2-s)} e^{-\beta_2(a)(\tau-s)} \Phi(x(\tau)) d\tau ds dF(a) \\
\stackrel{\triangle}{=}& \Delta_1 + \Delta_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
|\Delta_1| \le& \frac{b^2}{r} [|\Phi(0)| + \gamma |x|_\infty] \int_{\mathcal{A}} \int_0^{t_1} \int_s^{\infty} e^{-\beta_1(a)(t_1-s)} \\
&\times \beta_1(a)|t_2 - t_1| e^{-\beta_2(a)(\tau-s)} d\tau ds dF(a) \\
=& \frac{b^2}{r} [|\Phi(0)| + \gamma |x|_\infty] \int_{\mathcal{A}} \int_0^{t_1} e^{-\beta_1(a)(t_1-s)} \beta_1(a)|t_2 - t_1| \frac{1}{\beta_2(a)} ds dF(a) \\
=& \frac{b^2}{r} [|\Phi(0)| + \gamma |x|_\infty] |t_2 - t_1| \int_{\mathcal{A}} [1 - e^{-\beta_1(a)t_1}] \frac{1}{\beta_2(a)} dF(a) \\
\le& \frac{b^2}{r} [|\Phi(0)| + \gamma |x|_\infty] |t_2 - t_1| \int_{\mathcal{A}} \frac{1}{\beta_2(a)} dF(a)
\end{aligned}
$$

where $\beta_2(a) \geq \frac{\varrho}{2} + \frac{|b|}{\sqrt{r}}$ for all $a \in \mathbb{R}$, and

$$|\Delta_2| \leq \frac{b^2}{r}[|\Phi(0)| + \gamma|x|_\infty] \int_{\mathcal{A}} \int_{t_1}^{t_2} \int_s^\infty e^{-\beta_1(a)(t_2-s)} e^{-\beta_2(a)(\tau-s)} d\tau ds dF(a)$$
$$\leq \frac{b^2}{r}[|\Phi(0)| + \gamma|x|_\infty] \int_{\mathcal{A}} \int_{t_1}^{t_2} e^{-\beta_1(a)(t_2-s)} \frac{1}{\beta_2(a)} ds dF(a)$$
$$\leq \frac{b^2}{r}[|\Phi(0)| + \gamma|x|_\infty] \int_{\mathcal{A}} [1 - e^{-\beta_1(a)(t_2-t_1)}] \frac{1}{\beta_1(a)\beta_2(a)} dF(a)$$
$$\leq \frac{b^2}{r}[|\Phi(0)| + \gamma|x|_\infty]|t_2 - t_1| \int_{\mathcal{A}} \frac{1}{\beta_2(a)} dF(a).$$

Hence,

$$|I_1 - I_2| \leq \frac{2b^2}{r}[|\Phi(0)| + \gamma|x|_\infty]|t_2 - t_1| \int_{\mathcal{A}} \frac{1}{\beta_2(a)} dF(a).$$

Similarly we have

$$|I_3 - I_4| \leq \frac{2b^2|\alpha x|_\infty}{r}|t_2 - t_1| \int_{\mathcal{A}} \frac{\Pi_a}{\beta_2(a)} dF(a)$$

where the integral is finite since $\frac{\Pi_a}{\beta_2(a)}$ is bounded for $a \in \mathbb{R}$. Furthermore, we have

$$|I_5 - I_6| \leq 2|\alpha x|_\infty |t_2 - t_1|.$$

Hence it follows from that

$$|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)| \leq |I_1 - I_2| + |I_3 - I_4| + |I_5 - I_6|$$
$$\leq \frac{2b^2}{r}[|\Phi(0)| + \gamma|x|_\infty]|t_2 - t_1| \int_{\mathcal{A}} \frac{1}{\beta_2(a)} dF(a)$$
$$+ \frac{2b^2|\alpha x|_\infty}{r}|t_2 - t_1| \int_{\mathcal{A}} \frac{\Pi_a}{\beta_2(a)} dF(a) + 2|\alpha x|_\infty |t_2 - t_1|. \tag{A.1}$$

Then the lemma follows. □

Appendix B

Proof of Proposition 5.1. Let us denote by $\lambda_{max}(\overline{B}_n)$ the largest eigenvalue of the real symmetric matrix $\overline{B}_n$. Define the set $S = \{x \in \mathbb{R}^n : |x| = (\sum_{i=1}^n x_i^2)^{1/2} = 1\}$. Then we have

$$\lambda_{max}(\overline{B}_n) = \sup_S x^T \overline{B}_n x \overset{\triangle}{=} \sup_S \Lambda(x).$$

It is easy to show that

$$\Lambda(x) = -\sum_{i=1}^{n} \beta_1(a_i)x_i^2 + \frac{\alpha}{n}(\sum_{i=1}^{n} x_i)^2.$$

Assuming the supremum of Λ on S is attained at y, we can obtain the necessary condition for y by the Lagrangian multiplier method and we assert that there exists $\mu \in \mathbb{R}$ such that

$$2\beta_1(a_i)y_i - \frac{2\alpha}{n}\sum_{i=1}^{n} y_j + 2\mu y_i = 0, \qquad 1 \le i \le n, \tag{B.1}$$

$$\sum_{i=1}^{n} y_i^2 = 1, \tag{B.2}$$

where the first equation is obtained by the necessary condition for the supremum of the function $\Lambda(x) + \mu(\sum_{i=1}^{n} x_i^2 - 1)$ with $x \in \mathbb{R}^n$.

Let $S^+ = \{x \in S : x_i \ge 0, 1 \le i \le n\}$. For any $x \in S$, we denote $\widetilde{x} = (|x_1|, \cdots, |x_n|)^T$. Clearly we have $\Lambda(\widetilde{x}) \ge \Lambda(x)$ and it is impossible to attain the supremum at x which has both strictly positive and strictly negative entries. Thus we have $\sup_S \Lambda(x) = \sup_{S^+} \Lambda(x)$. Now it suffices to determine the supremum by solving (B.1) and (B.2) under the additional constraint $y_i \ge 0$, $1 \le i \le n$, i.e, $y \in S^+$ (then accordingly, $-y$ also attains the supremum by symmetry).

Since $y_i \ge 0$ and $\alpha > 0$, it necessarily follows that $\beta_1(a_i) + \mu > 0$ by (B.1). Furthermore, by (B.1) we may introduce an undetermined constant $c > 0$ such for each $1 \le i \le n$,

$$y_i = \frac{c}{\beta_1(a_i) + \mu}. \tag{B.3}$$

Substituting (B.3) into (B.2), we get

$$c^2 \sum_{i=1}^{n} \frac{1}{[\beta_1(a_i) + \mu]^2} = 1 \tag{B.4}$$

which yields

$$0 < c = (\sum_{i=1}^{n} \frac{1}{[\beta_1(a_i) + \mu]^2})^{-1/2}. \tag{B.5}$$

Combining (B.5) with (B.1) gives

$$(\beta_1(a_i) + \mu)\frac{c}{\beta_1(a_i) + \mu} = \frac{c\alpha}{n}\sum_{i=1}^{n} \frac{1}{\beta_1(a_i) + \mu}.$$

This yields

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\beta_1(a_i)+\mu}=1. \tag{B.6}$$

Further, by making use of (B.3), (B.6) and then (B.4) we compute

$$\sup_S \Lambda(x) = -\sum_{i=1}^{n}\beta_1(a_i)\frac{c^2}{[\beta_1(a_i)+\mu]^2}+\frac{\alpha}{n}(\sum_{i=1}^{n}\frac{c}{\beta_1(a_i)+\mu})^2=\mu. \tag{B.7}$$

Hence the largest eigenvalue of $\overline{B}_n$ is given by $\lambda_{max}(\overline{B}_n)=\mu$ which satisfies (B.6). Now for a fixed n, let $G_n(\nu)=\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\beta_1(a_i)+\nu}$, $\nu\in(-\beta_{1,n},\infty)$, where $\beta_{1,n}=\inf_{1\le i\le n}\beta_1(a_i)$. Obviously $G_n(\nu)$ is strictly monotone and $\lim_{\nu\to-\beta_{1,n}}G_n(\nu)=\infty$, $G_n(\infty)=0$. Therefore there is a unique μ^* satisfying (B.6) on $(-\beta_{1,n},\infty)$.

Recalling conditions (ii) and (iii), we see this shows there exists a fixed $\mu^*<0$ which may be taken as satisfying $\mu^*>-\beta^*$, such that for all $n\ge N_0$, $\mu\in[\mu^*,\infty)$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\mu+\beta_1(a_i)}<1$$

which implies that $\mu<\mu^*$ where μ is given by (B.6). This completes the proof. □

Proof of Proposition 5.5. Taking $a,a'\in\mathcal{A}$, by use of (4.11) we obtain

$$\begin{aligned}|\overline{z}_a(t)-\overline{z}_{a'}(t)| \le & \Big|\int_0^t e^{-\beta_1(a)(t-s)}\alpha\overline{z}(s)ds-\int_0^t e^{-\beta_1(a')(t-s)}\alpha\overline{z}(s)ds\Big| \\ & +\Big|\frac{b^2}{r}\int_0^t e^{-\beta_1(a)(t-s)}\int_s^\infty e^{-\beta_2(a)(\tau-s)}\Phi(\overline{z}(\tau))d\tau ds \\ & -\frac{b^2}{r}\int_0^t e^{-\beta_1(a')(t-s)}\int_s^\infty e^{-\beta_2(a')(\tau-s)}\Phi(\overline{z}(\tau))d\tau ds\Big| \\ & +\Big|\frac{b^2}{r}\int_0^t e^{-\beta_1(a)(t-s)}\int_s^\infty e^{-\beta_2(a)(\tau-s)}\alpha\Pi_a\overline{z}(\tau)d\tau ds \\ & -\frac{b^2}{r}\int_0^t e^{-\beta_1(a')(t-s)}\int_s^\infty e^{-\beta_2(a')(\tau-s)}\alpha\Pi_{a'}\overline{z}(\tau)d\tau ds\Big| \\ \triangleq & \Delta_1+\Delta_2+\Delta_3.\end{aligned}$$

By direct calculation we have the estimates

$$\Delta_1 \le \frac{|\alpha\overline{z}|_\infty |\beta_1(a)-\beta_1(a')|}{\min\{\beta_1^2(a),\beta_1^2(a')\}},$$

$$\Delta_2 \le \frac{b^2}{r}(|\Phi(0)|+\gamma|\overline{z}|_\infty) \times \left[\frac{|\beta_2(a)-\beta_2(a')|}{\beta_1(a)\min\{\beta_2^2(a),\beta_2^2(a')\}} + \frac{|\beta_1(a)-\beta_1(a')|}{\beta_2(a')\min\{\beta_1^2(a),\beta_1^2(a')\}}\right]$$

In order to estimate Δ_3, we write

$$\begin{aligned}\Delta_3 \le \Big|&\frac{b^2}{r}\int_0^t e^{-\beta_1(a)(t-s)}\int_s^\infty e^{-\beta_2(a)(\tau-s)}\alpha\Pi_a\overline{z}(\tau)d\tau ds \\ &- \frac{b^2}{r}\int_0^t e^{-\beta_1(a')(t-s)}\int_s^\infty e^{-\beta_2(a)(\tau-s)}\alpha\Pi_a\overline{z}(\tau)d\tau ds\Big| \\ &+\Big|\frac{b^2}{r}\int_0^t e^{-\beta_1(a')(t-s)}\int_s^\infty e^{-\beta_2(a)(\tau-s)}\alpha\Pi_a\overline{z}(\tau)d\tau ds \\ &- \frac{b^2}{r}\int_0^t e^{-\beta_1(a')(t-s)}\int_s^\infty e^{-\beta_2(a')(\tau-s)}\alpha\Pi_{a'}\overline{z}(\tau)d\tau ds\Big| \triangleq \Delta_{31}+\Delta_{32}.\end{aligned}$$

We have

$$\Delta_{31} \le \frac{b^2}{r}|\alpha\overline{z}|_\infty \frac{\Pi_a}{\beta_2(a)}\frac{|\beta_1(a)-\beta_1(a')|}{\min\{\beta_1^2(a),\beta_1^2(a')\}} \tag{B.8}$$

where it is obvious that $\sup_{a\in\mathbb{R}}\frac{\Pi_a}{\beta_2(a)}<\infty$. Since

$$\begin{aligned}&\Big|\int_s^\infty [e^{-\beta_2(a)(\tau-s)}\Pi_a - e^{-\beta_2(a')(\tau-s)}\Pi_{a'}]d\tau\Big| \\ &\le \int_s^\infty e^{-\beta_2(a)(\tau-s)}|\Pi_a-\Pi_{a'}|d\tau + \int_s^\infty \left|e^{-\beta_2(a)(\tau-s)}-e^{-\beta_2(a')(\tau-s)}\right|\Pi_{a'}d\tau \\ &\le \frac{|\Pi_a-\Pi_{a'}|}{\beta_2(a)} + \frac{\Pi_{a'}|\beta_2(a)-\beta_2(a')|}{\min\{\beta_2^2(a),\beta_2^2(a')\}},\end{aligned}$$

it follows that

$$\Delta_{32} \le \frac{b^2|\alpha\overline{z}|_\infty}{r}\left[\frac{|\Pi_a-\Pi_{a'}|}{\beta_1(a')\beta_2(a)} + \frac{\Pi_{a'}|\beta_2(a)-\beta_2(a')|}{\beta_1(a')\min\{\beta_2^2(a),\beta_2^2(a')\}}\right]$$

where $\sup_{a'\in\mathcal{A}}\frac{\Pi_{a'}}{|\beta_1(a')|}<\infty$.

Since $\beta_2(a)>\beta_1(a)>\widehat{\varepsilon}>0$ for any $a\in\mathcal{A}$, we conclude that there exists a constant C independent of a, a' and t such that

$$|\overline{z}_a(t)-\overline{z}_{a'}(t)| \le C[|\beta_1(a)-\beta_1(a')|+|\beta_2(a)-\beta_2(a')|+|\Pi_a-\Pi_{a'}|]. \tag{B.9}$$

It is straightforward to further show $\sup_{t\in\mathbb{R}_+, a\in\mathcal{A}} |\overline{z}_a(t)| < \infty$, and then the proposition follows from (B.9) combined with the global Lipschitz continuity for each of $\beta_1(a), \beta_2(a), \Pi_a$ as a function of a on $\mathbb{R}$. □

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